Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 1001-1008

A STUDY ON L_p SPACE FOR $P = \infty$

R. RAMYA, P. PRIYA, R. SUBRAMANI and N. VIJAYASEETHA

Department of Mathematics
Dhanalakshmi Srinivasan College of Arts
and Science For Women (Autonomous)
Perambalur, India
E-mail: ramya25071987@gmail.com

Abstract

We obtain Fourier transform by limiting processes of Fourier series. Since it was first used by French mathematician jean Baptist Fourier (1768-1830) in a manuscript submitted to the institute of France in 1807. He said that Fourier transform is a mathematical procedure which transforms a function from time domain to frequency domain. Fourier transform is a magical mathematical tool that decompresses only function into the sum of sinusoidal basis function.

Introduction

Real analysis is the branch of mathematical analysis that studies the behavior of real numbers, sequences and series of real numbers and real functions. Some particular properties of real-valued sequences and functions that real analysis studies include convergence, limits, continuity, differentiability and integrability.

If L^P spaces are functions spaces defined using a natural generalization of the $P-norm$ for finite-dimensional vector spaces. They are sometimes called Lebesgue spaces named after Henri Lebesgue (Don ford and Schwartz 1958) although according to the Bourbaki group (Bourbaki 1987) they were first introduced by frigyes Riesz (Riesz 1910) L^P spaces form an important class of Banach spaces in functional analysis and of topological vector spaces. Lebesgue spaces are also used in the theoretical discussion of problems in physics, statistics, finance, engineering and other discipline.

2020 Mathematics Subject Classification: 42B05.
Keywords: Linear space, Hilbert space, Lebesgue, Norm, Convergence, Limit.
Received November 2, 2021; Accepted November 15, 2021

Definition. A set is a well defined collection of objects. The object of a set are called the elements or members of that set and their membership to define by certain conditions. The sets are usually denoted by the capital letters of English alphabets say A, B, …, Y Z.

Example. The collection of the letter $a, b, c, d \ldots$

Definition. Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$

(a) xRx, for every $x \in A$ (Reflexivity)

(b) $xRy \Rightarrow yRx$ (symmetry)

(c) xRy and $yRz \Rightarrow xRz$ (Transitivity)

Definition. Let A and B be two sets then a rule or correspondence which associates each element of A to a unique element of B, is called a function or mapping from set A to set B.

Example. $f : A \to B$ then f is a function from A to B If $1 \leq P < \infty$ and E is a measurable set such that $m(E) < \infty$ then show that $L^{\infty}(E) \subset L^{P}(E)$, for each P also show that.

L^p SPACE FOR $P = \infty$

Definition. A non-negatie real number M is said to be an essential bound for the real-values measurable function f defined on $E[a, b]$ such that $m(E) > 0$, if

$$| f(x) | \leq M, \text{ a. e on } E$$

Definition. The class of all the essentially bounded measurable functions defined on E is defined as L^{∞} space. Thus

$$L^{\infty}(E) = \{f : ess \sup | f | < \infty\}$$

Obviously,

(i) every bounded function on E is in $L^{\infty}(E)$

(ii) $ess\ \sup|\, af + bg\,| \le |\, a\,|\, ess\ Sup|\, g\,|$

(iii) this space $L^{\infty}(E)$ is also a linear space over R.

Theorem. *Show that* (L^P, d) *is a metric space.*

Proof. In order to prove that d is a metric on L^P,

We are to prove that

(i) $d(f, g) \ge 0$

(ii) $d(f, g) = 0$ iff $f = g$

(iii) $d(f, g) = d(g, f)$

(iv) $d(f, g) \le d(f, h) + d(h, g), \forall f, g, h \in L^P$.

Let $f, g \in L^P[a, b]$ be arbitrary

(i) $|\, f - g\,|_P = \left(\int_a^b |\, f - g\,|^P dx\right)^{1/p} \ge 0$

$$d(f, g) = \|\, f - g\,\|_P \ge 0$$

(ii) $d(f, g) = 0 \Leftrightarrow \|\, f - g\,\|_P = 0$

$$\Leftrightarrow \int_a^b |\, f - g\,|^P dx = 0$$

$$\Leftrightarrow |\, f - g\,|^P = 0 \quad \text{a, e in } [a, b]$$

$$\Leftrightarrow f = g \quad \text{a, e in } [a, b]$$

(iii) $|\, f - g\,| = |\, g - f\,|$

$$|\, f - g\,|^P = |\, g - f\,|^P$$

$$\left(\int_a^b |\, f - g\,|^p\right)^{1/p} = \left(\int_a^b |\, g - f\,|^P\right)^{1/p}$$

$$\| f - g \|^{P} = \| g - f \|_{P}$$

$$d(f, g) = d(g, f)$$

(iv) let $f, g, h \in L^{P}$.

Then we have

$d(f, g) = \| f - g \|_{P} = \| (f - h) + (h - g) \|_{P}$

$\leq \| f - h \|_{P} + \| h - g \|_{P}$ (using Minkowski's inequality)

$$= d(f, h) + d(h, g)$$

i.e.) $d(f, g) \leq d(f, h) + d(h, g)$

Then function 'd' is a metric on L^{P} space and hence (L^{P}, d) is a metric space.

Theorem. *If* L^{P} *space is a normed linear space.*

Proof. Let $f, g \in L^{P}[a, b]$ and $c \in R$ be arbitrary

(1) $f, g \in L^{P}[a, b] \Rightarrow f + g \in L^{P}[a, b]$

(2) $f \in L^{P}[a, b]$ and $c \in R \Rightarrow cf \in L^{P}[a, b]$

For $f \in L^{P} \in [a, b] \Rightarrow f$ is measurable over $[a, b]$

$\Rightarrow cf$ is measureable over $[a, b]$

$\Rightarrow L^{P}[a, b]$ is a linear space (c is a constant)

(3) $| f(x) | \geq 0, \forall x \in [a, b]$

$$\left(\int_{a}^{b} | f(x) |^{P} \right)^{1/p} \geq 0$$

$$\| f \|_{P} \geq 0$$

$$\| f \|_{P} = 0 \Leftrightarrow f = 0 \text{ a.e}$$

In particular, $\| f \|_P = 0 \Leftrightarrow f = 0$ a.e

$$(4)\ \| cf \|_P = \left(\int_a^b | cf |^P \right)^{1/p}$$

$$= | c | \left(\int_a^b | f |^P \right)^{1/P}$$

$$= | c | \| f \|_P$$

(5) $\| f + g\|_P \leq \| f \|_P + \| g \|_P$.

Hence L^P – Space is a normal linear space

Theorem. *If f is a bounded measurable function defined on* $[a, b]$ *then for given* $\varepsilon > 0$ *there exists a continuous function g on* $[a, b]$ *such that*

$$\| f - g \|_2 < \varepsilon.$$

Proof. Let

$$F(x) = \int_0^x f(x)dt \text{ where } x \in [a, b]$$

Then

$$| F(x + h) - F(x) | = \left| \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right|$$

$$= \left| \int_x^{x-h} f(x)dt \right|$$

$$\leq \int_x^{x+h} | f(t) | dt$$

Where $| f(x) | \leq M, \forall x \in [a, b]$

Taking $h < \delta$ and $Mh < \varepsilon$

We get

$| x + h - x | < \delta \Rightarrow | F(x + h) - F(x) | < \varepsilon$

$\Rightarrow F(x)$ is continuous on $[a, b]$

Let

$$G_n(x) = n\int_x^{x+h} f(t)dt : x \in [a, b] \text{ and } n \in N;$$

Then

$G_n(x) = n[F(x + 1/n) \cdot F(x)]$

($f(x)$ is continuous on $[a, b] \Rightarrow G_n(x)$ is continuous on $[a, b] \forall n$)

Again, since

$F(x) = \int_a^x f(t)dt,\ x \in [a, b]$

$F'(x) = f(x)$ a.e in $[a, b]$

Now,

$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} \frac{F(x + (1/n)) - F(x)}{1/n}$

$= \lim_{n\to\infty} \frac{F(x + h) - F(x)}{h},\ h = \frac{1}{n}$

$= F'(x) = f(x)$ a.e in $[a, b]$

And hence

$\lim_{n\to\infty} | G_n(x) - f(x) |^2 = 0$

Also

$| G_n(x) | \leq \left| \int_x^{x+(1/n)} f(t)dt \right|$

$\int_x^{x+(1/n)} | f(t) | dt = M.$

Hence $| G_n(x) | \leq M, \forall n \in N$ and $\forall x \in [a, b]$

$[G_n(x) - f(x)]^2 \leq (M + M)^2 = 4m^2,\ x \in [a, b].$

On applying Lebesgue bounded convergence theorem, we get

$$\lim_{n\to\infty} \int_a^b (G_n - f)^2 = \int_a^b \lim \{G_n - f\}^2 = 0$$

$$\Rightarrow \lim_{n\to\infty} (G_n - f)^2 = 0$$

$$\Rightarrow \lim_{n\to\infty} \| G_n - f \|_2 = 0$$

$$\Rightarrow \lim_{n\to\infty} \| f - G_n \| = 0$$

$\Rightarrow$ for given $\varepsilon > 0$, there exist $n_0 \in N$, such that $n \geq n_0$

$$\Rightarrow \| f - G_n \|_2 < \in$$

Particularly for

$$\Rightarrow \| f - G_{n0} \|_2 < \in \Rightarrow \| f - g \|_2 < \in$$

Thus there exists a continuous function $G_{n0}(x) = g(x)$

$$= n_0 \int^{x+(1/n_0)} f(t)dt,\ x \in [a, b].$$

Which satisfies the given condition.

Conclusion

In this desertion we discussed about L^P spaces to find out the value of P and Holomorphic Fourier transforms used in some theorem. With this method the space of measurable functions for which the P^{th} power of the absolute value is Lebesgue integrable where functions which agree almost everywhere are identified. L^P Spaces are used in theoretical discussion of problems in statistics and others disciplines.

The Holomorphic Fourier transforms of a function of time is a complex-valued function of frequency whose (absolute value) represents the amount of that frequency present in original function. The Fourier transform of function in L^P requires the study if distribution. The Fourier transforms is a total that breaks a wave form (a function or signa) into an alternate representation characterized by sines and cosines

References

[1] Rudin Walter, Real and Complex Analysis (3rd Edition) Newyork: MC Graw Hill, (1987).

[2] Reisz, Frigyes, untersuchungen uber system integrierbarer Function, Mathematische Annalen 69(4) (1910), 449-497.

[3] R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis 3rd Edition John Wiley and Sons, (2000).

[4] Schwartz, Laurent Transformation de Laplace des Distribution, Comm Sem Math, Univ Lund [Medd, Lunds Univ, Math Sem], (1952).

[5] Charlamboss Airprantes and Owen Burkinshow, Problems in Real Analysis, Academic Press, Elsevier, (1998).

[6] Dunford, Nelson, Schwartz, Jacob T linear operators, Volume I, Wiley inter science, (1958).

[7] J. K. Hunter and B. Nachtergaele, Applied analysis world scientific Publishing Singapore, (2001).