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INSPECTING INTEGER SOLUTIONS FOR AN EXPONENTIAL DIOPHANTINE EQUATION $p^x + (p+2)^y = z^2$

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Abstract

In this script, an exponential Diophantine equation in three variables $p^x + (p+2)^y = z^2$ where p is a prime number given by $p = 4n+1$ for persuaded possibility of $n \in R$ the set of all real numbers is explored for the presence of integers solutions or the concerned equation has no solution under the assumption that the sum of the exponents x and y is 1, 2 and 3.

1. Introduction

A Diophantine equation is one in which the only possible solutions are integers. Number theory includes the study of Diophantine equations. Many scholars have been working on the solution of the Diophantine equation of the form $p^x + q^y = z^2$ in recent years, where p and q are separate primes

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and x, y, and z are non-negative integers. Asthana and Singh demonstrated in [1] that the Diophantine equation $3^x + 13^y = z^2$ with non-negative integers x, y, and z has only four solutions $(x, y, z) = (1, 0, 2), (1, 1, 4), (3, 2, 14)$ and $(5, 1, 16)$. In [2], Burshte in Nechemia showed a few months ago that the Diophantine equation $p^x + (p+4)^y = z^2$ has no solution, where x, y, and z are positive integers and p, $p+4$ are primes with $p > 3$. For an extension analysis one may refer [1-4, 6, 8-10]. In this paper, an exponential Diophantine equation in three unknowns $p^x + (p+2)^y = z^2$ where p is a prime number stated by $p = 4n+1$ for specific opportunity of $n \in R$, the set of all real numbers is analysed for the occurrence of solutions in integers or this equation has no solution under the hypothesis that $x + y$ is 1, 2 and 3.

2. Progression of Investigations

An exponential equation for determining whether an integer solution occurs or not is considered as

$$p^x + (p+2)^y = z^2 \tag{1}$$

where p is a prime number of the form $p = 4n+1$ for certain $n \in R$, the set of all real numbers.

Let us seek solutions for (1) by assuming the sum of the exponents is less than or equal to 3.

The opportunities of the above statement are illustrated below.

(i) $x = 0,\ y = 1$ and $x = 1,\ y = 0$

(ii) $x = 0,\ y = 2,\ x = 1,\ y = 1$ and $x = 0,\ y = 2$

(iii) $x = 0,\ y = 3,\ x = 1,\ y = 2,\ x = 1,\ y = 1$ and $x = 3,\ y = 0$.

Below is a full description of how to analyse each of the nine scenarios.

Case (i). Assume $x = 0$ and $y = 1$

These two selected values of x and y diminish (1) to the subsequent equation comprising two-variables p and z

$$p + 3 = z^2 \tag{2}$$

$$\Rightarrow 4n + 4 = z^2$$

Even though the unique choice $n = 0$ reveals an integer value for z as $z = \pm 2$, the same value of n provides that $p = 1$ which is not a prime number.

Hence, there exists no solutions in integer.

Case (ii). Let $x = 1,\ y = 0$

These two predilections of x and y moderate (1) to the equation as specified below.

$$p + 1 = z^2 \tag{3}$$

$$\Rightarrow 4n + 2 = z^2$$

Note that when $n = \frac{1}{2},\ z = \pm 2$

Therefore, $p = 3$ As a result, the possible integer solution is $(p, x, y, z) = (3, 1, 0, \pm 2)$

Case (iii). Let $x = 0,\ y = 2$

These suppositions reduce (1) to the quadratic equation with two unknowns as shown below.

$$1 + (p + 2)^2 = z^2$$

$$\Rightarrow (p + 2)^2 = z^2 - 1 \tag{4}$$

The foregoing postulation always false since the square of an integer minus one can never be a square.

The conclusion is neither (4) nor (1) has a solution.

Case (iv). Deliberate $x = 1$ and $y = 1$

Using these assumptions, the alterative form of (1) is offered by

$$2p + 2 = z^2 \tag{5}$$

$$8n + 4 = z^2$$

As in case (i), the options $n = 4$ and $n = 220$ delivers $z = \pm 6$ and $z = \pm 42$ respectively.

Also, these two values of n affords the equivalent chances of p as $p = 17$ and $p = 881$

Then, the original equation can be changed into $17^x + 19^y = z^2$ and $881^x + 883^y = z^2$.

Consequently, the four integer solutions to (5) are exemplified by

$$(p, x, y, z) = \{(17, 1, 1, \pm 6), (881, 1, 1, \pm 42)\}$$

Case (v). Elect $x = 2, y = 0$

These elected values of x and y shortened (1) to the resulting equation

$$p^2 + 1 = z^2$$

$$\Rightarrow (4n + 1)^2 = z^2 - 1. \tag{6}$$

The proclamation quantified above does not hold according to the same explanation given in case (iii).

It is determined that there is no solution to (6) and hence to (1).

Case (vi). Adopt $x = 0, y = 3$

As an outcome of the choices made above, (1) is reduced to a three-degree equation

$$1 + (p + 2)^3 = z^2 \tag{7}$$

$$\Rightarrow (4n + 3)^3 = z^2 - 1$$

$$\Rightarrow r^3 = z^2 - 1$$

where

$$r = 4n + 3 \tag{8}$$

This is practicable only if $(r, z) = (2, \pm 3)$

The equivalent value of n corresponding to the above value of r is

$$n = -\frac{1}{4}$$

Also, this choice of n reveals that $p = 0$ which is not prime number.

As a consequence of the result, (1) does not possess an integer solution however z is an integer.

Case (vii). Consider $x = 1,\ y = 2$

The replacement of such values of x and y translate (1) to the equation in terms of n and z as

$$16n^2 + 28n + 10 = z^2 \tag{9}$$

which can be revised by

$$(2u + 7)(2u - 7) = (2z + 2\sqrt{10})(2z - 2\sqrt{10}) \text{ where } u = 4n + \frac{7}{2} \tag{10}$$

Consider the fractional form of (10) as

$$\frac{2u + 7}{2z + 2\sqrt{10}} \frac{2z - 2\sqrt{10}}{2u - 7} = \frac{a}{b}, \neq 0$$

Covert the above equation into double equations and resolving it by the method of cross-multiplication, it is attained by

$$u = \frac{4ab\sqrt{10 - 7}(a^2 + b^2)}{(b^2 + a^2)} \Rightarrow n = \frac{2ab\sqrt{10 -} 7b^2}{4(b^2 + a^2)} \tag{11}$$

$$z = \frac{\sqrt{10}(a^2 + b^2) - 7ab}{(b^2 + a^2)} \tag{12}$$

Note that, z is not an integer for any selections of a and b.

Therefore, (9) and henceforth (1) does not have any solution in integer.

Remark. An additional form of (10) is taken as

$$\frac{2u+7}{2z+2\sqrt{10}} = \frac{2z-2\sqrt{10}}{2u-7} = \frac{c}{b},\ d \neq 0$$

Following the exact method as explained in the previous case, it is calculated by

$$u = \frac{7d^2 + 7c^2 - 4\sqrt{10}cd}{2(c^2 + d^2)} \Rightarrow n = \frac{7d^2 + 2\sqrt{10}cd}{4(c^2 + d^2)} \tag{13}$$

$$n = \frac{7cd + \sqrt{10}c^2 + \sqrt{10}d^2}{(c^2 + d^2)} \tag{14}$$

Here also for any values of c and d, the value of z is not an integer.

Hence, (13) and therefore (1) does not consume any solution in integer.

Case (viii). Choose $x = 2,\ y = 1$

These options condense (1) to an equation with two unknowns of degree two as

$$16n^2 + 12n + 4 = z^2 \tag{15}$$

which can be factorized as

$$(2v+3)(2v-3) = (2z-4)(2z-4) \text{ where } v = 4n + \frac{3}{2} \tag{16}$$

Contemplate (16) in an ensuing fraction form

$$\frac{2v+3}{2z+4} = \frac{2z-4}{2v-3} = \frac{e}{f},\ f \neq 0$$

Repeating the same process as explained earlier, it is received by

$$v = \frac{8ef - 3e^2 - 3f^2}{(f^2 - e^2)} \Rightarrow n = \frac{4ef - 3f^2}{4(f^2 - e^2)} \tag{17}$$

$$z = \frac{2e^2 + 2f^2 - 3ef}{4(f^2 - e^2)} \tag{18}$$

A keen observation from (17) and (18) is the comparable values of n and z

are $n = 0$ and $z = \pm 2$ for each of the two pairs $(e, f) = (3k, 4k)$ and Likewise, this choice of n exposes that $p = 1$ which is not a prime number. Consequently, (1) cannot have an integer solution.

Remark. Intend (16) in the resultant ratio form

$$\frac{2v+3}{2z-4} = \frac{2z+4}{2v-3} = \frac{h}{i},\ i \neq 0$$

Repetition of the prior clarifications established that

$$v = \frac{3i^2 + 3h^2 - 8ih}{2(h^2 - i^2)} \Rightarrow n = \frac{3i^2 + 4ih}{4(h^2 - i^2)} \tag{19}$$

$$z = \frac{3ih + 2h^2 + 2i^2}{2(h^2 - i^2)} \tag{20}$$

A deep notification from (19) and (20) is the values of n and z are $n = 0$ and $z = \pm 2$ when $(h, i) = (3k, 4k)$ and $(h, i) = (k, 0)$.

Moreover, this option of n gives $p = 1$ which is not a prime number.

Subsequently, it is impossible to find an integer solution to (1).

Case (ix). Assume $x = 3,\ y = 0$

Replication of the choices made above minimized (1) into

$$(4n+1)^3 + 1 = z^2$$

$$\Rightarrow r^3 = z^2 - 1$$

where

$$(r, z) = (2, \pm 3) \tag{21}$$

The only feasible solution of (21) is pointed out by $(r, z) = (2, \pm 3)$

The corresponding value of n for the above-mentioned value of r is

$$n = \frac{1}{4}$$

Further, the above n provides that $p = 2$ which is an even prime number. Hence, an integer solution to (1) is represented by $(p, x, y, z) = (2, 3, 0 \pm 3)$.

3. Conclusion

In this paper, it is proved that an exponential equation $p^x = (p+2)^y = z^2$ in three variables where $p = 4n+1$ for suitable $n \in R$ has limited number of integer solutions indicated by $(p, x, y, z) = \{(3, 1, 0, \pm 2)(17, 1, 1, \pm 6), (881, 1, 1, \pm 42), (2, 3, 0, \pm 3)\}$ when $x + y$ is 1, 2 and 3. In this way, one search integer solutions for various Exponential equations with base as any other prime numbers and $x + y > 3$.

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