Advances and Applications in Mathematical Sciences
Volume 21, Issue 11, September 2022, Pages 6545-6555

CONNECTED CERTIFIED DOMINATION NUMBER OF CERTAIN GRAPHS

AZHAM ILYASS[1] and VISHWAJEET S. GOSWAMI[2]

[1,2]Department of Mathematics
Chandigarh University, Mohali
140413, Punjab, India
E-mail: aazamlone22@gmail.com

Abstract

The concept of connected certified domination is introduced in this paper. A γ-set D of a graph $Z = (V_Z, E_Z)$ is called a certified dominating set (γ_{cer}-set) of Z if $|N(u) \cap (V_Z \setminus D)|$ is either 0 or at least 2 for all $u \in D$, and it is supposed to be a connected dominating set if its induced subgraph is connected. Specifically, γ_{cer}-set D_c of graph Z is called a connected certified dominating set (γ_{cer}^c-set) if the subgraph induced by D_c, that is, $Z[D_c]$ is connected. The cardinality of a minimum γ_{cer}^c-set is called the connected certified domination number (CCDN) of Z denoted by $\gamma_{cer}^c(Z)$. Herein, we are representing the CCDN of some graphs. We then characterize certain classes of graphs with $\gamma = \gamma_{cer}^c$.

1. Introduction

In our research work, we consider finite, undirected graphs not having a loop or several edges. We direct the reader to [1] for any terminology or notation not specifically described here.

Since Euler gave a solution to the famous Konigsberg seven Bridge Problem over three centuries ago, graph theory has evolved into an appealing and significant topic of mathematics that has been significantly developed, as demonstrated by hundreds of articles. In 1735, Leonhard Euler, a prominent Swiss mathematician, fixed the controversial Konigsberg bridge issue, which

2020 Mathematics Subject Classification: 05C05.
Keywords: Dominating set, certified domination number, connected certified domination number.
[2]Corresponding author; E-mail: vishwajeet.e9858@cumail.in
Received February 12, 2022; Accepted April 16, 2022

had baffled researchers for years, and his optimization method paved the way for the study of graph theory, an entirely new branch of mathematics. Mathematicians have developed and expanded the domain of graph theory during the last three decades by inventing new and crucial concepts. A graph's primary structure, which consists of points termed as vertices along with lines connecting pairs of points called as edges, makes it simple to use and transforms it into a powerful tool for addressing a wide range of mathematical problems in fields like topology, geometry, and number theory. In fact, it may be used to analyze any network that has a collection of items, some of them are connected. A graph is a natural model for such networks, with vertices defining the items and edges expressing the relationships between the things. The flexibility of graph theory is one of the primary reasons why it has piqued the interest of academics in fields like engineering, communication networks, transportation and logistics, biology optimization, complex network, etc. Caccetta [2], as well as Caccetta and Vijayan [3], addressed the applications of graph theory. Specifically, [4]-[7], including pioneering works by [8], [9], have written volumes on graph theory and various applications of graph theory.

In the last three decades, the Domination theory of graphs has been extensively studied and developed by mathematicians as a major research domain of graph theory. Domination has its origin in 1862 when [10] investigated the issue of identifying the smallest number of queens required to cover an $n \times n$ chessboard. Watkins [11] surveys the evolution and significant expansion of this fruitful field of domination theory from chessboard issues quite effectively. Around 1960 Berge [8] and Ore [9] started the mathematical exploration of domination theory in graphs. [1], [12], [13] provide a thorough examination of the rationale and applications of graph domination, as well as a complete consideration of a wide range of domination characteristics. The literature on domination has been excellently surveyed by [14]. There is a plethora of material on domination theory; we recommend readers outstanding books [1], [13] on domination-related parameters.

Suppose that we're given a group of X officials and a group of Y civilians. There must be an official $v \in X$ for each civil $u \in Y$ who can attend u, and every time any such v is attending u, there must also be another civil $w \in Y$

that observes v, i.e. w must act as a kind of witness, to sidestep any mismanagement from v. In the case of a certain social network, what is the smallest number of connected officials necessary to ensure such a service?

The aforementioned issue motivates us to propose the concept of connected certified domination.

The theory of certified domination was presented by [15] in 2020. Considering a graph $Z = (V_Z, E_Z)$, a set D is known to be a γ-set if the neighborhood of D is the whole of V_Z i.e., $N_Z[D] = V_Z$. The cardinality of a minimal γ-set in Z is named as the domination number of Z and denoted by $\gamma(Z)$. γ-set D is said to be a γ_{cer}-set if $|N(u) \cap (V_Z \setminus D)|$ is either 0 or at least 2 for all $u \in D$. A γ_{cer}-set D_c is called as γ_{cer}^c-set if its induced subgraph is connected. We shall address the cardinality of a minimal γ_{cer}^c-set to be called as CCDN of Z which may be denoted by $\gamma_{cer}^c(Z)$. In section 2, of this paper, we show the exact value of CCDN of some graphs. In section 3 we characterize graphs with $\gamma = \gamma_{cer}^c$.

1.1 Definitions and Notations

Z is said to be a trivial or empty graph if it has only one vertex and no edges, and it is said to be a nontrivial or nonempty graph if it contains a minimum of one edge. For $u \epsilon V(Z)$, if $\deg(u) = 0$ then u is referred to as an isolated vertex and if $\deg(u) = 1$ then u is termed as a leaf. A vertex of degree $n - 1$ in a graph Z of order n is called a universal vertex. The minimal and maximal degree of Z is represented by $\delta(Z)$ and $\Delta(Z)$. Order of a graph $Z = (V_Z, E_Z)$ is $|V(Z)|$ and size of a graph Z is $|E(Z)|$ [16].

The Cartesian product of the graph Z_1 and graph Z_2 with a set of vertices $V(Z_1) \times V(Z_2)$ is denoted by $Z_1 \blacksquare Z_2$ and any two arbitrary vertices (a, b) and (a', b') in $V(Z_1) \times V(Z_2)$ are neighbors in $Z_1 \blacksquare Z_2$ if either $a = a'$ and $bb' \epsilon E(Z_2)$, or $b = b'$ and $aa' \epsilon E(Z_1)$ [16].

The lexicographic product of the graph Z_1 and graph Z_2 with a set of vertices is $V(Z_1) \times V(Z_2)$ is denoted by $Z_1 * Z_2$ and two arbitrary vertices

(r, s) and (r', s') in $V(Z_1) \times V(Z_2)$ are adjacent in $Z_1 * Z_2$ iff r is adjacent to r' in Z_1 or $r = r'$, and s is adjacent to s' in Z_2 [16].

For the definition of a complete graph, bipartite graph, fan graph, star graph, cyclic graphs, and other graphs used, we refer the reader to [4], [16].

2. CCDN of Some Graphs

In this part, we evaluate the CCDN for some graphs. From the definition, it is clear that for any graph Z of order n, $\gamma^c_{Cer} \leq n$ and $\gamma^c_{Cer} \neq n - 1$.

Observation 2.1. If $K_{m,n}$ is a complete bipartite graph, then

$$\gamma^c_{Cer}(K_{m,n}) = 2, \text{ for } 3 \leq m \leq n.$$

Observation 2.2. Let $Z_1 \cong C_n \blacksquare P_3$ and $Z_2 \cong P_3 \blacksquare P_n$, then

1. $\gamma^c_{Cer}(Z_1) = n,\ n \geq 3.$

2. $\gamma^c_{Cer}(Z_2) = n,\ \forall\, n \geq 1.$

Observation 2.3.

(1) CCDN of Star graph is $\gamma^c_{Cer}(K_{1,n-1}) = 1,$ for $n \geq 2.$

(2) CCDN of the Wheel graph is $\gamma^c_{Cer}(W_n) = 1.$

(3) CCDN of double Star graph is $\gamma^c_{Cer}(D_{r,s}) = 2,$ where $r, s \geq 2.$

(4) CCDN of Double Wheel graph is $\gamma^c_{Cer}(DW_n) = 1,\ n \geq 2.$

Observation 2.4. Let Z be a lexicographic product of path graphs P_m and P_n, then $\gamma^c_{Cer}(P_m * P_n) = \{m - 2;$ for $m, n \geq 3\}.$

Observation 2.5. Let Z be a graph and $|Z| \geq 3,$ then $\gamma^c_{Cer}(Z) = 1$ if and only if Z has a universal vertex.

Observation 2.6. If Z is the Cartesian product of star graphs then $\gamma^c_{Cer}(K_{1,n-1} \blacksquare K_{1,n-1}) = 1.$

Proposition 2.1. *If K_n is an n-vertex complete graph, then* $\gamma^c_{Cer}(K_n) = \begin{cases} 1 & if\ n = 1\ or\ n \geq 3, \\ 2 & if\ n = 2. \end{cases}$

Proof. Let $Z \cong K_n$ be a complete graph with $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.

By definition, $\forall\ v \in V(Z)$ where $|Z| = n$ and if $\deg(v) = n - 1$ then Z is called as a complete graph.

Now, for $n = 1$ and $n = 2$ it is obvious that $\gamma^c_{Cer}(K_1) = 1$ and $\gamma^c_{Cer}(K_2) = 2$.

For $n \geq 3$, let $V(Z) = \{u_1, u_2, u_3, \ldots, u_n\}$ be the vertex set of Z.

Since Z is a complete graph and every vertex of a complete graph is a universal vertex,

$\therefore \deg(u_i) = n - 1, 1 \leq i \leq n$.

$\Rightarrow$ Each vertex of graph Z is dominating every other vertex of Z.

$\Rightarrow D_s$ of Z is $\{u_i\}$, where $i = 1, 2, \ldots, n$.

Since, for $n \geq 3$, u_i is dominating at least 2 vertices.

$\therefore \gamma_{cer}$-set of Z is $\{u_i\}$. In addition, we know that a graph with one vertex is a connected graph.

$\Rightarrow \gamma^c_{cer}$-set of Z is also $\{u_i\}$.

Hence, for $n \geq 3 \gamma^c_{cer}(K_n) = 1$.

Examples. CCDN of K_3 and K_5 is given in the figures below:

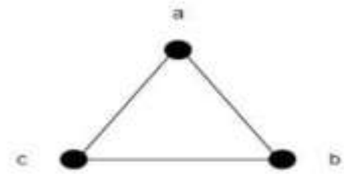

Figure 2.1.

γ_{cer}^{c}-set of $K_3 = \{a\}$ and $\gamma_{Cer}^{c}(K_3) = 1$.

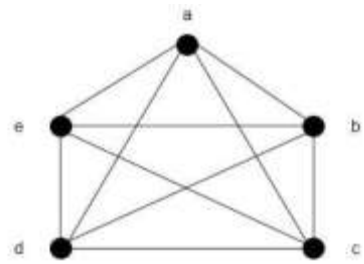

Figure 2.2.

γ_{cer}^{c}-set of $K_5 = \{a\}$ and $\gamma_{Cer}^{c}(K_5) = 1$.

Proposition 2.2. *Let Z be a fan graph $F_{p,q}$, then the CCDN of fan graph is*

$$\gamma_{cer}^{c}(F_{p,q}) = \begin{cases} 1 & \text{if } \ p \geq 1,\ q = 2,\ 3 \\ 1 & \text{if } \ p = 1,\ q \geq 2 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $Z \cong F_{p,q}$ be a fan graph on $p + q$ vertices and consider D be a minimum γ_{cer}^{c}-set of Z. By definition, a fan graph is $F_{p,q} = K_p + P_q$, where K_p is the graph with p isolated nodes; P_q is the path on q nodes.

Case I. When $p \geq 1,\ q = 2,\ 3$.

(a) For $q = 2$.

Let $Z \cong F_{p,2}$ be a fan graph on $p + 2$ vertices with $2p + 1$ edges and let

D_c be a minimum CCDS of graph Z. By definition of the fan graph, the graph $Z = \overline{K_p} + P_2$ where $\overline{K_p}$ is the empty graph on p nodes and P_2 is the path graph on 2 nodes.

Let $X(P_2) = \{x, y\}$. There are 2 nodes available in path P_2 of the fan graph, so if we choose any one vertex from path P_2, then all the other vertices of Z are dominated by our chosen vertex, and the graph with one vertex is always connected. So we will get a minimum γ_{cer}^{c}-set and its cardinality is the CCDN of graph Z.

Hence the γ_{cer}^{c}-set D_c of $Z = \{x\}$ or $\{y\}$.

Therefore, the CCDN of the fan graph $F_{p,2}$ is 1.

i.e., $\gamma_{cer}^{c}(F_{p,2}) = 1$.

(b) For $q = 3$.

Similarly, for the fan graph, $F_{p,3} = \overline{K_p} + P_3$ on $p+3$ vertices with $3p+1$ edges, the CCDN is 1.

i.e., $\gamma_{cer}^{c}(F_{p,3}) = 1$.

Hence, from the above two cases, we conclude that,

$\gamma_{cer}^{c}(F_{p,q}) = 1$ when $p \geq 1$, $q = 2, 3$.

Case II. When, $p = 1$, $q \geq 2$, in this case $|V_{\overline{K_p}}| = 1$ and $|V_{P_q}| \geq 2$.

$\therefore$ Every vertex of the path $P_{q, q \geq 2}$ is connected to the single vertex of $\overline{K_p}$.

$\Rightarrow |\gamma_{Cer}^{c} - \text{set}|$ of $F_{p,q}$ is 1, and hence $\gamma_{Cer}^{c}(F_{p,q}) = 1$.

Case III. When $p \geq 2$, $q \geq 4$.

Let $V(P_q) = \{u_1, u_2, \ldots, u_q\}$.

There are q nodes available in the path P_q of fan graph, so that if we choose any two adjacent vertices u_i and u_j from the path P_q with $q \geq 4$, then all the other vertices of $\overline{K_p}$ with $p \geq 2$ will be adjacent to u_i and u_j and all the remaining vertices of the graph $F_{p,q}$ are dominated by our chosen vertices. And in all the cases u_i and u_j will dominate at least 2 vertices in $V_Z \setminus D$. So, we will get a minimum γ_{cer}^c-set and its cardinality is the CCDN of the graph.

Hence γ_{cer}^c-set D_c of $F_{p,q}$ $\forall$ $p \geq 2, q \geq 4$ is $= \{u_i, u_j\}$ where $4 \leq i, j \leq q$ and $i \neq j$.

$\therefore$ the CCDN of graph $F_{p,q}$ is 2.

i.e. $\gamma_{Cer}^c(F_{p,q}) = 2$ $\forall$ $p \geq 2$, $q \geq 4$.

Example. CCDN of fan graph $F_{2,4}$ and $F_{3,4}$ is given in the figures below:

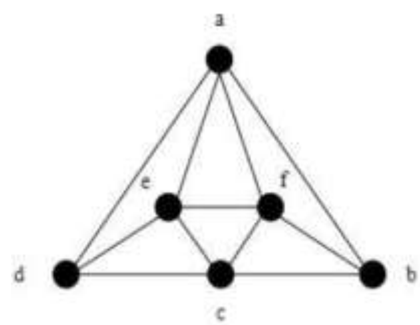

Figure 2.3. Fan graph $F_{2,4}$.

γ_{cer}^c-set of $F_{2,4}$ $= \{e, f\}$ and $\gamma_{Cer}^c(F_{2,4}) = 2$

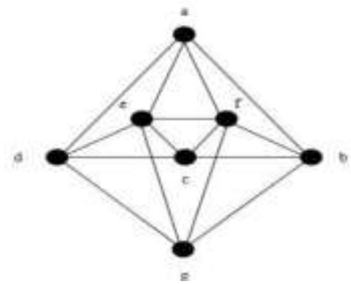

Figure 2.4. Fan graph $F_{3,4}$.

γ_{cer}^{c}-set of $F_{3,4} = \{e, q\}$ and $\gamma_{Cer}^{c}(F_{3,4}) = 2$.

3. Graphs with $\gamma = \gamma_{cer}^{c}$

Graphs with $\gamma = \gamma_{cer}$ is already investigated by Dettlaff et al. in [17]. In this part, we look at the fundamental properties that ensure $\gamma(Z)$ and $\gamma_{cer}^{c}(Z)$ are equal for any connected graph Z. In the first part of this section, we start with the following prerequisite conditions for the equality of $\gamma(Z)$ and $\gamma_{cer}^{c}(Z)$ of a graph Z. In the second portion, we discuss some graphs with equal $\gamma = \gamma_{cer}^{c}$.

Observation 3.1. If a graph Z has a unique minimal connected dominating set (γ_c-set), then $\gamma(Z) = \gamma_{cer}^{c}(Z)$.

Proposition 3.1. *Consider a connected graph Z with $|Z| \geq 3$, then $\gamma(Z) = \gamma_{cer}^{c}(Z)$ iff Z has a γ_c-set D_c such that $|N(u) \cap (V_Z \setminus D_c)|$ is either* 0 *or at least* 2 *for all $u \in D_c$ in $V_Z - D_c$.*

Proof. Suppose that $\gamma(Z) = \gamma_{cer}^{c}(Z)$. Let D_c be γ_{cer}^{c}-set of Z. Then $\gamma(Z) = \gamma_{cer}^{c}(Z)$ implies that D_c is also a γ_c-set of Z. Let u_1 be any vertex of D_c. Since u_1 is not a degree zero vertex of D_c, the set $N(u_1) \cap (V_Z - D_c)$ is non-empty (otherwise $D_c - \{u_1\}$ would be a smaller γ-set of Z). As a result, because D_c is a γ_{cer}^{c}-set of Z, we have $|N(u_1) \cap (V_Z - D_c)| \geq 2$.

Conversely, if D_c is a γ_c-set of Z and $|N(u_1) \cap (V_Z - D_c)| \geq 2$ the D_c is also γ^c_{cer}-set of Z. And hence $\gamma^c_{cer}(Z) \leq |D_c| = \gamma(Z) \leq \gamma^c_{cer}(Z)$.

$\therefore \gamma(Z) = \gamma^c_{cer}(Z)$.

Corollary. *If Z as a graph has a unique minimal γ_c-set D_c, such that $|N(u) \cap (V_Z \setminus D_c)|$ is either 0 or at least 2 for all $u \in D_c$ in $V_Z - D_c$, then $\gamma(Z) = \gamma^c_{cer}(Z)$.*

Examples of graphs with $\gamma = \gamma^c_{Cer}$.

1. If Z is a complete graph K_n then $\gamma(K_n) = \gamma^c_{cer}(K_n) = 1$.

2. If Z is a star graph $K_{1, n-1}$ then $\gamma(K_{1, n-1}) = \gamma^c_{cer}(K_{1, n-1}) = 1$. Also if Z is the Cartesian product of star graphs then $\gamma(K_{1, n-1} \blacksquare K_{1, n-1}) = \gamma^c_{cer}(K_{1, n-1} \blacksquare K_{1, n-1}) = 1$.

3. If Z is a wheel graph then $\gamma(W_n) = \gamma^c_{cer}(W_n) = 1$.

4. If Z is double star graph $D(r, s)$ then $\gamma(D(r, s)) = \gamma^c_{cer}(D(r, s)) = 2$.

5. If Z is a double wheel graph then $\gamma(DW_n) = \gamma^c_{cer}(DW_n) = 1$.

4. Conclusion

Since thousands of articles have been published in the area of domination in graphs over the years, our study cannot claim to fully cover the new model; rather, it should be viewed as a little addition to the field. In this research work, we have discussed an innovative parameter of domination in graphs called “Connected certified domination”. We have presented CCDN of some graphs and graphs with $\gamma = \gamma^c_{cer}$. This concept can be further applied to a wide range of graphs. We have also observed some graphs with $\gamma_{cer} = \gamma^c_{cer}$ and $\gamma = \gamma_{cer} = \gamma^c_{cer}$.

References

[1] T. W. Haynes, S. Hedetniemi and Peter Slater, Domination in graphs: advanced topics, Routledge 2 (2017).

[2] L. Caccetta, Vulnerability of communication networks, Networks 14(1) (1984), 141-146.

[3] L. Caccetta and K. Vijayan, Applications of graph theory, Ars Combinatoria (1987), 21-27.

[4] C. Berge, Graphs and hypergraphs, 1973.

[5] J. L. Gross, J. Yellen and M. Anderson, Graph Theory and Its Applications, 2018.

[6] West Douglas Brent, Introduction to graph theory, Upper Saddle River: Prentice hall, 2 2001.

[7] L.-H. Hsu and C.-K. Lin, Graph Theory and Interconnection Networks, CRC Press, Boca Raton (2008), pp. 720.

[8] C. Berge, Theory of Graphs and its Applications, Methuen, London, 1962.

[9] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ., 38 (Amer. Math. Soc., Providence, RI), 1962.

[10] C. F. de Jaenisch, Applications de l'Analyse mathematique an Jen des Echecs, Petrograd, 1862.

[11] John J. Watkins, Across the board: the mathematics of chessboard problems, Princeton University Press (2004), pp. 246.

[12] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, Networks 7(3) (1977), 247-261.

[13] T. W. Haynes, S. Hedetniemi and P. Slater, Fundamentals of Domination in Graphs, CRC Press, 2013.

[14] T. Haynes, S. Hedetniemi and M. Henning, Topics in Domination in Graphs, Springer, 2020.

[15] M. Dettlaff, M. Lemańska, J. Topp, R. Ziemann and P. Żyliński, Certified domination, AKCE International Journal of Graphs and Combinatorics 17(01) (2020), 86-97.

[16] R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Second Edition, Heidelberg, Springer, 2012.

[17] M. Dettlaff, M. Lemańska, M. Miotk, J. Topp, R. L. Ziemann and P. Zyliński, Graphs with equal domination and certified domination numbers, Opuscula Mathematica 39(6) (2019), 815-827.