Advances and Applications in Mathematical Sciences
Volume 19, Issue 12, Otober 2020, Pages 1213-1226

A NOTE ON THE DERIVATION OF POISSON BRACKET

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Abstract

The Poisson 2-form of the Poisson manifold is introduced. We describe the locally and globally hamiltonians vectors fields on symplectic manifold and on Poisson manifold in terms of derivations.

1. Introduction

We recall that a Poisson structure on a smooth manifold M is due to the existence of a bracket $\{\,,\,\}$ on the algebra $C^{\infty}(M)$ of smooth functions on M such that the pair $(C^{\infty}(M), \{\,,\,\})$ is a real Lie algebra such that, for any $f \in C^{\infty}(M)$ the map

$$ad(f) : C^{\infty}(M) \to C^{\infty}(M),\ g \mapsto \{f, g\}$$

is a derivation of commutative algebra i.e.

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$$

2010 Mathematics Subject Classification: 53D17, 17B66.
Keywords: Poisson manifolds, Poisson 2-form, hamiltonians vectors fields, Poisson vectors fields.
Received July 1, 2019; Accepted May 15, 20

for $f, g, h \in C^{\infty}(M)$. The bracket $\{,\}$ is called the Poisson bracket, the Lie algebra $(C^{\infty}(M), \{,\})$ is called the Poisson and M is a Poisson manifold [3],[6].

Let $\mathfrak{X}(M)$ be the $C^{\infty}(M)$-module of smooth vector fields on M. A smooth manifold M is called a symplectic manifold if there is on M a nondegenerate skew-symmetric 2-form

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^{\infty}(M)$$

such that

$$d\omega = 0$$

where d is the exterior differentiation operator.

Hence, the linear mapping

$$\omega^{b} : \mathfrak{X}(M) \to \Lambda^{1}(M), \; X \to i_X\omega$$

given by

$$(i_X\omega)(Y) = \omega(X, Y)$$

is an isomorphism of $C^{\infty}(M)$-modules.

To every function $f \in C^{\infty}(M)$ one associates the Hamiltonian vector field X_f which is the unique vector field satisfying $i_{X_f}\omega = df$. One says then that f is a Hamiltonian function, corresponding to this vector field.

Let (M, ω) be a symplectic manifold and $f, g \in C^{\infty}(M)$. The bracket of f and g such that

$$\{f, g\} = \omega(X_f, X_g)$$

gives $C^{\infty}(M)$ the structure of a Poisson algebra [3]. Thus, every symplectic manifold is a Poisson manifold.

2. The Poisson 2-form

Let $\Omega_{\mathbb{R}}[C^{\infty}(M)]$ be the $C^{\infty}(M)$- module of Kälher differentials of $C^{\infty}(M)$ i.e. the set

$$\Omega_{\mathbb{R}}[C^{\infty}(M)] = \frac{I}{I^2}$$

where I is the $C^{\infty}(M)$- submodule of $C^{\infty}(M) \underset{\mathbb{R}}{\otimes} C^{\infty}(M)$ generated by the elements of the form $f \otimes 1_{C^{\infty}(M)} - 1_{C^{\infty}(M^A)} \otimes f$ with $f \in C^{\infty}(M)$.

The linear map

$$\delta_M : C^{\infty}(M) \to \Omega_{\mathbb{R}}[C^{\infty}(M)], f \mapsto \overline{f \otimes 1_{C^{\infty}(M)} - 1_{C^{\infty}(M)} \otimes f}$$

is the canonical derivation which the image of δ_M generates the $C^{\infty}(M)$- module $\Omega_{\mathbb{R}}[C^{\infty}(M)]$, i.e. for $x \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$x = \sum_{i \in I : finite} f_i \cdot \delta_M(g_i),$$

with $f_i, g_i \in C^{\infty}(M)$ for any $i \in I$ [1], [4].

The pair $(\Omega_{\mathbb{R}}[C^{\infty}(M)], \delta_M)$ satisfies the following universal property: for every $C^{\infty}(M)$- module $\mathcal{M}$ and for every derivation

$$\varphi : C^{\infty}(M) \to \mathcal{M},$$

there exists a unique $C^{\infty}(M)$- linear map

$$\widetilde{\varphi} : \Omega_{\mathbb{R}}[C^{\infty}(M)] \to \mathcal{M}$$

such that

$$\widetilde{\varphi} \circ \delta_M = \varphi.$$

Thus, the linear mapping

$$Hom_{C^\infty(M)}(\Omega_{\mathbb{R}}[C^\infty(M)], \mathcal{M}) \to Der_{\mathbb{R}}(C^\infty(M), \mathcal{M}) \mapsto \psi \circ \delta_M$$

is an isomorphism of $C^\infty(M)$- modules [1]. In particular,

$$(\Omega_{\mathbb{R}}[C^\infty(M)])^* \simeq Der_{\mathbb{R}}[C^\infty(M)].$$

When $(M, \{\,,\})$ is a Poisson manifold, the map

$$ad : C^\infty(M) \to Der_{\mathbb{R}}[C^\infty(M)], f \mapsto ad(f)$$

is a derivation. Let

$$\widetilde{ad} : \Omega_{\mathbb{R}}[C^\infty(M)] \to Der_{\mathbb{R}}[C^\infty(M)]$$

be a unique $C^\infty(M)$- linear map such that

$$\widetilde{ad} \circ \delta_M = ad.$$

Consider the canonical isomorphism

$$\sigma_M : \Omega_{\mathbb{R}}[C^\infty(M)]^* \to Der_{\mathbb{R}}[C^\infty(M)]^*, \widetilde{ad} \mapsto \widetilde{ad} \circ \delta_M.$$

The $C^\infty(M)$- linear map

$$-(\sigma_M^{-1} \circ \widetilde{ad}) : \Omega_{\mathbb{R}}[C^\infty(M)] \overset{\widetilde{ad}}{\to} Der_{\mathbb{R}}[C^\infty(M)] \overset{\sigma_M^{-1}}{\to} \Omega_{\mathbb{R}}[C^\infty(M)]^*$$

induces a skew-symmetric $C^\infty(M)$- bilinear form

$$\omega_M : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \to C^\infty(M), (x, y) \mapsto -[(\sigma_M^{-1} \circ \widetilde{ad})(x)](y).$$

Theorem 1. *The manifold M is a Poisson manifold if and only if there exists a skew-symmetric* 2*-form*

$$\omega_M : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \to C^\infty(M)$$

such that for any f and g in $C^\infty(M)$,

$$\{f, g\} = -\omega_M[\delta_M(f), \delta_M(g)]$$

defines a structure of Lie algebra over $C^\infty(M)$.

In this case, we say that ω_M is the Poisson 2-form of the Poisson manifold M and we denote (M, ω_M) the Poisson manifold of Poisson 2-form ω_M.

If (M, ω_M) is a Poisson manifold, then for any $f \in C^\infty(M)$ and $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have

$$[\widetilde{ad}(x)](f) = -\omega_M(x, \delta_M(f))$$

Indeed,

$$\begin{aligned}[\widetilde{ad}(x)](f) &= [(\sigma_M^{-1} \circ \widetilde{ad})(x)](\delta_M(f)) \\ &= -\omega_M(x, \delta_M(f)).\end{aligned}$$

For $D \in Der_{\mathbb{R}}[C^\infty(M)]$, the map

$$\mathfrak{L}_D = i_D \circ \delta_M^1 + \delta_M \circ \widetilde{D} : \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)]) \to \Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$$

is called Lie derivative with respect to a derivative D.

For any $D \in Der_{\mathbb{R}}[C^\infty(M)]$, $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$ and $f \in C^\infty(M)$, then

$$\mathfrak{L}_{f \cdot D}(x) = f \cdot \mathfrak{L}_D(x) + \widetilde{D}(x) \cdot \delta_M(f),$$

$$\mathfrak{L}_D(fx) = D(f) \cdot x + f \cdot \mathfrak{L}_D(x)$$

and

$$\mathfrak{L}_{\widetilde{ad}[\delta_M(f)]}[\delta_M(g)] = \delta_M\{f, g\}.$$

3. Hamiltonian Vector Fields when M is a Poisson Manifold

For $p \in \mathbb{N}$, we denote $\mathfrak{L}_{sks}^p[C^\infty(M), C^\infty(M)]$ the $C^\infty(M)$- module of skew-symmetric multilinear forms of degree p from $C^\infty(M)$ into $C^\infty(M)$. For $p = 0$, we have

$$\mathfrak{L}_{sks}^0[C^\infty(M), C^\infty(M)] = C^\infty(M).$$

We denote

$$\mathfrak{L}_{sks}[C^{\infty}(M), C^{\infty}(M)] \to \overset{n}{\underset{p=0}{\oplus}} \mathfrak{L}_{sks}[C^{\infty}(M), C^{\infty}(M)]$$

the algebra of skew-symmetric multilinear forms. Let

$$d_{ad} : \mathfrak{L}_{sks}[C^{\infty}(M), C^{\infty}(M)] \to \mathfrak{L}_{sks}[C^{\infty}(M), C^{\infty}(M)]$$

be the operator of cohomology associated with the adjoint representation

$$ad : C^{\infty}(M) \to Der[C^{\infty}(M)].$$

Thus, for any $\eta \in \mathfrak{L}^{p}_{sks}[C^{\infty}(M), C^{\infty}(M)]$ and for any $f_1, f_2, \ldots, f_{p+1} \in C^{\infty}(M)$,

$$\begin{aligned} d_{ad}\eta)(f_1, f_2, \ldots, f_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} ad(f_i)[\eta(f_1, f_2, \ldots, \hat{f}_i, \ldots, f_{p+1})] \\ &+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \eta(\{f_i, f_j\}, f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_{p+1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i-1} \{f_i, \eta(f_1, f_2, \ldots, \hat{f}_i, \ldots, f_{p+1})\} \\ &+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \eta(\{f_i, f_j\}, f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_{p+1}) \end{aligned}$$

where $\hat{f}_i$ means that the term f_i is omitted.

Poisson cohomology $H_{pois}(M)$ can be computed using a subcomplex of the Eilenberg-Chevalley complex of the Lie algebra $(C^{\infty}(M), \{\,,\})$.

$H^0_{Pois}(M)$ is the space $Cas(M)$ of Casimir functions i.e.

$$\begin{aligned} H^0_{Pois}(M) &= \{f \in C^{\infty}(M)\, tel\, que\, \{f, g\} = 0\, \forall g \in C^{\infty}(M)\} \\ &= \ker(f \mapsto X_f). \end{aligned}$$

Definition 1. Let M be a Poisson manifold with bracket $\{,\}$. A vector field X on M is locally hamiltonian if X is closed for the cohomology associated with the adjoint representation

$$ad : C^{\infty}(M) \to Der_{\mathbb{R}}[C^{\infty}(M)]$$

i.e. $d_{ad}^{1}X = 0$, where

$$d_{ad}^{1}X : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$$

$$(f, g) \mapsto (d_{ad}^{1}X)(f, g).$$

Proposition 2. *When M is a Poisson manifold with bracket $\{,\}$ then a locally hamiltonian vector field is the derivation of the Poisson algebra $C^{\infty}(M)$.*

Proof. Let X be a locally hamiltonian vector field on a Poisson manifold M i.e.

$$d_{ad}^{1}X = 0.$$

For any f and $g \in C^{\infty}(M)$,

$$(d_{ad}^{1}X)(f, g) = 0$$

i.e.

$$\begin{aligned} 0 &= (d_{ad}^{1}X)(f, g) \\ &= ad(f)[X(g)] - ad(g)[X(f)] - X(\{f, g\}) \\ &= \{f, X(g)\} - \{g, X(f)\} - X(\{f, g\}) \end{aligned}$$

i.e.

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}.$$

Thus, X is a derivation of the Poisson algebra $C^{\infty}(M)$.

Definition 2. When M is a Poisson manifold with bracket $\{,\}$ a vector field X is globally hamiltonian if X is exact for the cohomology associated with the adjoint representation

$$ad : C^{\infty}(M) \to Der[C^{\infty}(M)]$$

i.e. there exists $f \in C^{\infty}(M)$ such that $X = d_{ad}(f)$.

Proposition 3. *Let M be a Poisson manifold with bracket* $\{,\}$. *Then a globally hamiltonian vector field is the derivation interior of the Poisson algebra* $C^{\infty}(M)$.

Proof. Let X be a globally hamiltonian vector field on a Poisson manifold M i.e. there exists $f \in C^{\infty}(M)$ such that

$$X = d^{0}_{ad} f.$$

For any $g \in C^{\infty}(M)$,

$$\begin{aligned} X(g) &= (d^{0}_{ad} f)(g) \\ &= ad(g)(f) \\ &= \{g, f\} \\ &= -\{f, g\} \\ &= -ad(f)(g). \end{aligned}$$

i.e. $X = -ad(f)$. Thus, X is the derivation interior of the Poisson algebra $C^{\infty}(M)$.

When (M, ω) is a symplectic manifold, we recall that a vector field X on M is locally Hamiltonian if the form $i_X \omega$ is closed for the de Rham cohomology i.e. X is a derivation of the Lie algebra $C^{\infty}(M)$ induced by the structure of Poisson defined by the symplectic manifold (M, ω). Thus, for any f and g in $C^{\infty}(M)$,

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}.$$

$X \in \mathfrak{X}(M)$ is globally hamiltonian if there exists $f \in C^{\infty}(M)$ such that $i_X \omega = df$ i.e. the 1-form $i_X \omega$ is d-exact [1] et[2].

Proposition 4. *A vector field on M globally hamiltonian is a derivation interior of the Lie algebra* $C^{\infty}(M)$ *induced by the structure of Poisson defined by the symplectic manifold* (M, ω).

Proof. Let X be a globally hamiltonian vector field, there exists $f \in C^{\infty}(M)$ such that $i_X \omega = df$. For any $Y \in \mathfrak{X}(M)$,

$$df(Y) = \omega(X, Y).$$

In particular, for any g in $C^{\infty}(M)$, we have

$$\omega(X, X_g) = X_g(f)$$

i.e.

$$X(g) = \{g, f\}$$

$$= -ad(f)(g)$$

i.e. $X = -ad(f)$. Thus, X is the derivation interior of the Poisson algebra $C^{\infty}(M)$.

When (M, ω) is a symplectic manifold, we recall that a vector field X on M is called symplectic if $\mathfrak{L}_X \omega = 0$. Here, $\mathfrak{L}_X$ denotes the Lie derivative in the direction of X.

Proposition 5. *Any locally hamiltonian vector field is symplectic.*

Proof. Let X be a locally hamiltonian vector field. For any f and g in $C^{\infty}(M)$,

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}.$$

For any Y and $Z \in \mathfrak{X}(M)$,

$$(\mathfrak{L}_X \omega)(Y, Z) = X(\omega(Y, Z)) - \omega([X, Y], Z) - \omega(Y, [X, Z]).$$

In particular, for any f and g in $C^{\infty}(M)$, we have

$$(\mathfrak{L}_X \omega)(Y_f, Z_g) = X(\omega(Y_f, Z_g)) - \omega([X, Y_f], Z_g) - \omega(Y_f, [X, Z_g])$$

$$= X(\{f, g\}) - \omega(Y_{X(f)}, Z_g) - \omega(Y_f, Z_{X(g)})$$

$$= X(\{f, g\}) - \{X(f), g\} - \{f, X(g)\}$$

$$= 0.$$

4. Poisson Vector Fields

For any derivation $D : C^{\infty}(M) \to C^{\infty}(M)$, the Lie derivative with respect to D is the map

$$\mathfrak{L}_D : \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)]) \to \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)])$$

such that for $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)])$ and $x_1, \ldots, x_p \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$(\mathfrak{L}_D\eta)(x_1, \ldots, x_p) = D[\eta(x_1, \ldots, x_p)]$$

$$-\sum_{i=1}^{p} \eta(x_1, \ldots, [x, x_i], x_{i+1}, \ldots, x_p).$$

A vector field X on a Poisson manifold (M, ω_M) is called a Poisson vector field if the Lie derivative of ω_M with respect to X vanishes i.e. $\mathfrak{L}_X\omega_M = 0$.

Proposition 6. *Let (M, ω_M) be a Poisson manifold. Then, all globally hamiltonian vector fields are Poisson vector fields.*

Proof. Let X be a globally hamiltonian vector field on M, then there exists $f \in C^{\infty}(M)$ such that $X = -ad(f)$. For any g and $h \in C^{\infty}(M)$

$$\mathfrak{L}_X\omega_M = (\mathfrak{L}_{-ad(f)}\omega_M)([\delta_M(g), \delta_M(h)])$$

$$= -ad(f)(\omega_M[\delta_M(g), \delta_M(h)])$$

$$-(\omega_M[\mathfrak{L}_{-ad(f)}\delta_M(g), \delta_M(h)])$$

$$-(\omega_M[\delta_M(g), \mathfrak{L}_{-ad(f)}, \delta_M(h)])$$

$$= ad(f)(\{g, h\}) + \omega_M[\delta_M\{f, g\}, \delta_M(h)]$$

$$+\omega_M[\delta_M(g), \delta_M\{f, h\}]$$

$$= \{f, \{g, h\}\} - \{\{f, g\}, h\} - \{g, \{f, h\}\}$$

$$= \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$$

Proposition 7. *Let (M, ω_M) be a Poisson manifold. Then, all locally hamiltonian vector fields are Poisson vector fields.*

Proof. Let X be a locally hamiltonian vector field on a Poisson manifold M with bracket $\{,\}$. For any $g, h \in C^{\infty}(M)$,

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}.$$

$$\begin{aligned}\mathfrak{L}_X\omega_M([\delta_M(f), \delta_M(g)]) &= X(\omega_M[\delta_M(f), \delta_M(g)]) \\ &-(\omega_M[\mathfrak{L}_X\delta_M(f), \delta_M(g)]) \\ &-(\omega_M[\delta_M(f), \mathfrak{L}_X\delta_M(g)]) \\ &= X(-\{f, g\}) - \omega_M[\delta_M(X(f)), \delta_M(g)]) \\ &-\omega_M[\delta_M(f), \delta_M(X(g))] \\ &= -X(\{f, g\}) + \{f, X(g)\} + \{X(f), g\} \\ &= 0.\end{aligned}$$

$H^1_{pois}(M)$ is the quotient of the space of Poisson vector fields by the hamiltonians vectors fields. Let μ be a volume form on an orientable manifold M and let X be a vector fields on M [1], [7]. The divergence operator of X with respect to μ is the map $div_{\mu}X$ such that

$$(div_{\mu}X)\mu = \mathfrak{L}_X\mu.$$

When (M, ω_M) is a Poisson manifold, equipped with a volume form μ, for any X and $Y \in \mathfrak{X}(M)$, for any f in $C^{\infty}(M)$, we have

$$div_{\mu}([X, Y]) = X(div_{\mu}Y) - Y(div_{\mu}X),$$

$$div_{\mu}(fX) = X(f) + f div_{\mu}X,$$

and for $f > 0$,

$$div_{f\mu}X = X(\log f) + div_{\mu}X.$$

Proposition 8. *Let (M, ω_M, μ) be a Poisson manifold, equipped with a* *volume form* μ. *Then, the map*

$$X_\mu : C^\infty(M) \to C^\infty(M),\ f \mapsto div_\mu X_f$$

is a Poisson vector field. Moreover, this Poisson vector field X_μ *called modular vector field is a Poisson* 1-*cocycle for the cohomology associated with the adjoint representation.*

Proof. For any f and $g \in C^\infty(M)$,

$$\begin{aligned} X_\mu(fg) &= div_\mu(X_{fg}) \\ &= div_\mu(fX_g + gX_f) \\ &= X_g(f) + fX_\mu(g) + X_f(g) + gX_\mu(f) \\ &= \{g, f\} + fX_\mu(g) + \{f, g\} + gX_\mu(f) \\ &= fX_\mu(g) + gX_\mu(f). \end{aligned}$$

$$\begin{aligned} \mathfrak{L}_{X_\mu}\omega_M([\delta_M(f), \delta_M(g)]) &= X_\mu(\omega_M([\delta_M(f), \delta_M(g)]) \\ &-(\omega_M[\mathfrak{L}_{X_\mu}\delta_M(f), \delta_M(g)]) \\ &-(\omega_M[\delta_M(f), \mathfrak{L}_{X_\mu}\delta_M(g)]) \\ &= X_\mu(-\{f, g\}) - \omega_M[\delta_M(X_\mu(f)), \delta_M(g)] \\ &-(\omega_M[\delta_M(f), \delta_M(X_\mu(g))] \\ &= -X_\mu(\{f, g\}) + \{f, X_\mu(g)\} + \{X_\mu(f), g\} \\ &= -(div_\mu([X_f, X_g])) + \{f, X_\mu(g)\} + \{X_\mu(f), g\} \\ &= -(X_f(X_\mu(g)) - X_g(X_\mu(f))) + \{f, X_\mu(g)\} + \{X_\mu(f), g\} \\ &= -(\{f, X_\mu(g)\} + \{X_\mu(f), g\}) + \{f, X_\mu(g)\} + \{X_\mu(f), g\} \\ &= 0. \end{aligned}$$

Thus, X_μ is a Poisson vector field.

$$(d^1_{ad}X_\mu)(f, g) = ad(f)[X_\mu(g)] - ad(g)[X_\mu(f)] - X_\mu(\{f, g\})$$

$$= \{f, X_\mu(g)\} - \{g, X_\mu(f)\} - X_\mu(\{f, g\})$$

$$= 0.$$

Hence, X_μ is a Poisson 1-cocycle.

Proposition 9. *The Poisson cohomology class of the modular vector field* X_μ *is independent of the choice of volume form* μ.

Proof. For any $f > 0$ and $g \in C^\infty(M)$

$$X_{f\mu}(g) = div_{f\mu}(X_g)$$

$$= X_\mu(g) + X_g(\log f)$$

$$= X_\mu(g) - X_{\log f}(g)$$

$$X_{f\mu} - X_\mu = -X_{\log f}$$

i.e., the vector fields $X_{f\mu}$ and X_μ differ by a Hamiltonian vector field, i.e., there is $h \in C^\infty(M)$ such that

$$-X_{\log f} = d^0_{ad}(h).$$

Thus, $X_{f\mu} - X_\mu = d^0_{ad}(h)$ is a Poisson 1-coboundary i.e., $[X_{f\mu}] = [X_\mu]$. The Poisson cohomology class of the modular form is therefore independent of the chosen volume form.

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