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FUZZY STACK AND ITS APPLICATIONS

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Abstract

The paper deals with an introduction and study of fuzzy stack as generalization of fuzzy grill [1]. The basic intent, among other things, is to determine the extent to which the results in [9] obtained via fuzzy grills, can be extended to our setting of fuzzy stacks, and also to address the digression by suitable counterexamples. Two operators viz. $\Phi_{\mathcal{S}}$ and $\psi_{\mathcal{S}}$, where $\mathcal{S}$ is a fuzzy stack on a fuzzy topological space X, are considered on X and it turns out that these operators ultimately give rise to a strong generalized fuzzy topology on X instead of fuzzy topology, latter being the case with similar operators in terms of a fuzzy grill.

1. Introduction

It is well known that Zadeh [13] was the initiator of the celebrated notion of fuzzy sets in 1965. Fuzzy topology is just a kind of topology developed in terms of fuzzy sets, which originated from a paper of Chang [2] in 1968. The concept of grills on a general topological space was first introduced by Choquet [4] in 1947. In fuzzy setting, the concepts of fuzzy grills and fuzzy stacks on fuzzy topological spaces were initiated by Azad [1] in 1981, basically for the study of a fuzzy basic proximity and related ideas. Many other mathematicians (e.g. see [3, 5, 12]) utilized fuzzy grills as well for investigations of different others fuzzy topological aspects.

In [9], the authors defined and studied an induced fuzzy topology on the ambient set, associated rather naturally to the existing fuzzy topology on the set and a fuzzy grill on it.

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In 2012, W. K. Min and Y. K. Kim [7] introduced an operator defined by a stack on a given topological space and investigated some basic properties. Certain recent endeavours by some mathematicians are being noticed towards extension of these works to more generalized frameworks for dealing with many topological concepts more effectively.

In what follows in this paper, simply X, shall stand for a fuzzy topological space (fts, in short) X endowed with a fuzzy topology Fuzzy grill, Fuzzy stack, operators τ, as defined by Chang [2], I^X denotes the set of all fuzzy sets on a nonempty set X, where I stands for [0; 1]. For a fuzzy set V in X, the set $\{x \in X : V(x > 0)\}$ is called the support of V and is denoted by supp V. The constant fuzzy sets taking on the constant values 0 Fuzzy Stack and its applications and 1 respectively on X shall be denoted by 0_X and 1_X [13].

For fuzzy sets A and B, the notation AqB stands for the fact that A is quasi-coincident (q-coincident) with B, $A(y) + B(y) > 1$ for certain $y \in X$ [8]. The negation of this statement is denoted by $A\overline{q}B$.

A fuzzy singleton or a fuzzy point [8] with support y and value $\alpha (0 < \alpha \leq 1)$ is denoted by y_α. According to prevalent definitions, we have for two fuzzy sets P, R in $X, (P \leq R \Leftrightarrow P(y) \leq R(y) \forall y \in X)$ and $y_\alpha \leq P \Leftrightarrow \alpha \leq P(y)$, where y_α is a fuzzy singleton in X.

A non-zero fuzzy set B is such that supp $B \neq \phi$, i.e., $B \neq 0_X$. A fuzzy set A is said to be a q-nbd of a fuzzy set B [8] if for some fuzzy open set V, BqV and $V \leq A$. A q-nbd A of B is called an open q-nbd of B if A is fuzzy open. For a given fuzzy singleton y_α, $Q(y_\alpha)$ will denoted the set of all open q-nbds of y_α. Given an fts X and a fuzzy set P in an fts X, $cl(P)$, $\text{int}(P)$, $1_X - P$ (or sometimes as $1 - P$) respectively denote the closure, interior and fuzzy complement of P respectively in X.

In this article, the idea of fuzzy stack is initiated as a generalization of fuzzy grill and study the concept at length. Our primary objective is, inter-alia, to see as to how far the results obtained in terms of fuzzy grill, can be extended to our present setting of fuzzy stacks. In the next section, we will recall three operators on the set I^X and find a few elementary properties concerning them as prerequisites.

In Section 3, we introduce fuzzy stack and define an operator $\Phi_{\mathcal{S}}$ by means of a stack $\mathcal{S}$ in the same way as is done in terms of a fuzzy grill. We achieve some generalizations of the corresponding results in [9], the digressions are also justified by suitable examples.

2. Some Operators on I^X

Here, we recall the definitions of three operators on the set I^X of all fuzzy sets on a set X and derive a few results concerning them as pre-requisites.

Definition 2.1. ([9, 6]) Given two fuzzy sets P and Q in X, $(P+Q)$, $(P-Q)$ and the Lukasiewicz conjunction $P * Q$ are defined to be the following fuzzy sets:

(i) $$(P+Q)(y) = \begin{cases} P(y)+Q(y), & \text{if } P(y)+Q(y) \leq 1 \\ 1, & \text{if } P(y)+Q(y) > 1 \end{cases}$$

(ii) $$(P-Q)(y) = \begin{cases} P(y)-Q(y), & \text{if } P(y) > Q(y) \\ 0, & \text{if } P(y) \leq Q(y) \end{cases}$$

(iii) $$(P*Q)(y) = \begin{cases} P(y)+Q(y)-1 & \text{if } P(y)+Q(y) > 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $y \in X$.

Lemma 2.2. *For any three fuzzy sets* P, Q *and* R *in an fts* X,

(i) $P*(Q \vee R) = (P*Q) \vee (P*R)$.

(ii) $P \leq Q \Rightarrow P*R \leq Q*R$.

Proof. We omit the straightforward proof. □

Lemma 2.3. *For any three fuzzy sets* P, Q *and* R *in* X,

$P*(Q*R) = (P*Q)*Q = (P*Q)*R$.

Proof. For any $y \in X$,

$$(P * (Q * R))(y) = \begin{cases} (P * (Q + R - 1))(y), & \text{if } Q(y) + R(y) > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} P(y) + Q(y) + R(y) - 2, & \text{if } P(y) + Q(y) + R(y) > 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} ((P + R - 1) * Q)(y), & \text{if } P(y) + Q(y) + R(y) > 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= ((P * R) * Q)(y).$$

Thus $P * (Q * R) = (P * R) * Q$.

The above calculations establish $P * (Q * R) = (P * Q) * R$ as well. □

The following result will be used in the sequel.

Lemma 2.4. ([9]) *If* P, Q *are fuzzy sets in an fts* X, *then* $P - (P - Q) \leq Q$.

Lemma 2.5. *If* P, Q, U, V *are fuzzy sets in an fts X, then*

$$(P \vee Q) * (U \wedge V) \leq (P * U) \vee (Q * V), \ldots, (\alpha).$$

Proof. Let us denote the left-hand side and right hand side of (α) by L and R respectively.

Let $z \in X$ be arbitrary. Without loss of generality we can assume that

$(P \vee Q)(z) + (U \wedge V)(z) > 1$. We first see that $[(P * U) \vee (Q * V)](z)$ cannot be 0. For, otherwise $P(z) + U(z) \leq 1$ and $Q(z) + V(z) \leq 1$, which give $(P \vee Q)(z) + (U \wedge V)(z) \leq 1$, a contradiction.

Now, two cases may arise:

(i) $P(z) + U(z) \leq 1$ and $Q(z) + V(z) > 1$

(ii) $P(z) + U(z) > 1$ and $Q(z) + V(z) \leq 1$.

We deal with case (i) only, case (ii) being quite similar. For this case, the right hand side of $(\alpha) = Q(z) + V(z) - 1$.

Subcase (i): $P(z) \leq Q(z)$ and $U(z) \leq V(z)$.

We have, $L(z) = Q(z) + U(z) - 1$ or $0 \leq Q(z) + V(z) - 1 = R(z)$.

Subcase (ii): $P(z) \leq Q(z)$ and $U(z) > V(z)$.

Then $L(z) = Q(z) + V(z) - 1 = R(z)$.

Subcase (iii): $P(z) > Q(z)$ and $U(z) \leq V(z)$.

Here, $L(z) = P(z) + U(z) - 1 \leq 0 \Rightarrow L = 0_X$.

Subcase (iv): $P(z) > Q(z)$ and $U(z) > V(z)$.

Here we see that $L(z) = P(z) + U(z) > Q(z) + V(z)$. But $P(z) + U(z) \leq 1$ while $Q(z) + V(z) > 1$ for the case under consideration.

Hence this subcase is not tenable.

3. Fuzzy Stack and Operator $\Phi_{\mathcal{S}}$

Definition 3.1. ([1]) A fuzzy stack $\mathcal{S}$ is a non-empty collection of fuzzy sets in an fts X such that

(i) $0_X \notin \mathcal{S}$

(ii) If P, Q are fuzzy sets on X, $P \in \mathcal{S}$ and $P \leq Q \Rightarrow Q \in \mathcal{S}$.

If, in addition, the following condition (iii) is satisfied, then $\mathcal{S}$ is called a fuzzy grill [1] on X, where

(iii) If $P, Q \in I^X$ and $P \vee Q \in \mathcal{S} \Rightarrow P \in \mathcal{S}$ or $Q \in \mathcal{S}$.

Remark 3.2. It is clear that a fuzzy grill G on X is a fuzzy stack on X. But the converse may not be true as is shown below:

Example 3.3. Let $X = \{x_1, x_2\}$ and $\tau = \{0_X, 1_X, P, Q\}$ where $P(x_1) = 0.3$ and $P(x_2) = Q(x_1) = Q(x_2) = 0.6$. Clearly, (X, τ) is an fts. Let $\mathcal{G}_1$ contain 1_X and all those fuzzy sets G_1 on X for which $0.3 \leq G_1(x_1) \leq 1$ and $0.2 \leq G_1(x_2) \leq 1$.

Now consider the fuzzy sets A and B in X given by

$A(x_1) = 0.5$, $A(x_2) = 0$ and $B(x_2) = 0$, $B(x_2) = 0.4$. We see that $A \vee B \in \mathcal{G}_1$ but neither $A \in \mathcal{G}_1$ n or $B \in \mathcal{G}_1$.

Thus $\mathcal{G}_1$ is not a fuzzy grill but is obviously a fuzzy stack on X.

An fts (X, τ) endowed with a fuzzy stack S will be called an sfts, to be denoted by $(X, \tau, \mathcal{S})$.

As proposed in the introduction, we now define an operator $\Phi_{\mathcal{S}}$ in an sfts as follows.

Definition 3.4. For an sfts $(X, \tau, \mathcal{S})$, we define an operator $\Phi_{\mathcal{S}} : I^X \to I^X$ given by $\Phi_{\mathcal{S}}(P) = \vee\{y_\alpha : P * U \in \mathcal{S}, \forall U \in Q(y_\alpha)\}$.

Some properties of the operator $\Phi_{\mathcal{S}}$ are stated in the following.

Theorem 3.5. *In an sfts* $(X, \tau, \mathcal{S})$, *the following results hold*:

(i) $\Phi_{\mathcal{S}}(0_X) = 0_X$, $\Phi_{\mathcal{S}}(1_X) = 1_X$.

(ii) If $P, Q \in I^X$, $P \leq Q \Rightarrow \Phi_{\mathcal{S}}(P) \leq \Phi_{\mathcal{S}}(Q)$.

(iii) *Given two fuzzy stacks* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *in fts* (X, τ) *with* $\mathcal{S}_1 \leq \mathcal{S}_2$, *then* $\Phi_{\mathcal{S}_1}(P) \leq \Phi_{\mathcal{S}_2}(P)$, *for any* $P \in I^X$.

(iv) $\Phi_{\mathcal{S}}(P) \leq cl(P)$, *for any* $P \in I^X$.

(v) $\Phi_{\mathcal{S}}(P)$ *is closed, for any* $P \in I^X$.

(vi) $\Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(P)) \leq \Phi_{\mathcal{S}}(P)$ *for any* $P \in I^X$, *where equality does not hold, in general*.

(vii) *For any* $P \in I^X$, $P \notin \Phi_{\mathcal{S}}(P) = 0_X$.

Proof. (i) It is obvious.

(ii) Let $y_\alpha \leq \Phi_{\mathcal{S}}(P)$. Then $U \in Q(y_\alpha) \Rightarrow P * U \in \mathcal{S} \Rightarrow Q * U \in \mathcal{S}$ (as $P \leq Q \Rightarrow P * U \leq Q * U$).

Hence $y_\alpha \in \Phi_{\mathcal{S}}(Q)$.

(iii) $y_\alpha \in \Phi_{\mathcal{S}_1}(P) \Rightarrow$ for all $U \in Q(y_\alpha)$, $P * U \in \mathcal{S}_1 \leq \mathcal{S}_2 \Rightarrow y_\alpha \leq \Phi_{\mathcal{S}_2}(P)$.

(iv) Let $y_\alpha \not\leq cl(P)$; then there exists $U \in Q(y_\alpha)$ such that $U\overline{q}A$. Then $U(z) + P(z) \leq 1, \forall z \in X,$ so that $(U * P)(z) = 0, \forall z \in X.$ i.e., $U * P = 0_X \notin \mathcal{S}$. This gives $y_\alpha \not\leq \Phi_{\mathcal{S}}(P)$.

(v) Let $y_\alpha \leq cl\,(\Phi_{\mathcal{S}}(P))$ and $U \in Q(y_\alpha)$, gives $\Phi_{\mathcal{S}}(P)qU$, so that $\exists z \in X$ such that $\Phi_{\mathcal{S}}(P)(z) + U(z) > 1$. Let $[\Phi_{\mathcal{S}}(P)](z) = t$. Then $z_t \leq \Phi_{\mathcal{S}}(P)$ and $U \in Q(z_t)$, which give $P * U \in \mathcal{S}$.

(vi) From (iv), we have $\Phi_{\mathcal{S}}(P) \leq cl(P)$ and from (v), (v), $cl(\Phi_{\mathcal{S}}(P)) = \Phi_{\mathcal{S}}(P)$. Thus $\Phi_{\mathcal{S}}[\Phi_{\mathcal{S}}(P)] \leq cl(\Phi_{\mathcal{S}}(P)) = \Phi_{\mathcal{S}}(P)$. That the inclusion cannot, in general, be reversed is shown in the following example.

(vii) Let $y_\alpha \leq \Phi_{\mathcal{S}}(P)$. Then for every $U \in Q(y_\alpha)$, $P * U \in \mathcal{S}$. Clearly $P \in \mathcal{S}$, we have contradiction. Thus $\Phi_{\mathcal{S}}(P) = 0_X$. □

Example 3.6. Let $X = \{x_1, x_2\}$ and $\tau = \{0_X, 1_X, A\}$, where $A(x_1) = 0.6$ and $A(x_2) = 0.8$. Then (X, τ) is an fts. Let $\mathcal{S}$ be the fuzzy stack consisting of all fuzzy sets G on X such that $0.5 \leq G(x_1) \leq 1$ and $0.6 \leq G(x_2) \leq 1$.

Now let P be a fuzzy set in X such that $P(x_1) = 0.8$, $P(x_2) = 0.6$. Then $\Phi_{\mathcal{S}}(P) = C$, where $C(x_1) = 0.4$, $C(x_2) = 0.2$. Again $\Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(P)) = \Phi_{\mathcal{S}}(C) = 0_X$. So, $\Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(P)) \neq \Phi_{\mathcal{S}}(P)$.

Remark 3.7. It follows from (ii) and (iii) of the above Theorem 3.5 that the operator $\Phi_{\mathcal{S}}$ increases with the increase of the fuzzy stack $\mathcal{S}$ and also of the fuzzy set on which $\Phi_{\mathcal{S}}$ is operated. It is thus clear that if P, Q are fuzzy sets in an sfts $(X, \tau, \mathcal{S})$, $\Phi_{\mathcal{S}}(P) \vee \Phi_{\mathcal{S}}(Q) \leq (P \vee Q)$, where equality may not prevail, as is shown by the example below.

Example 3.8. Let $X = \{x_1, x_2\}$ be endowed with the fuzzy topology $\tau = \{0_X, 1_X, A, B\}$, where $A(x_1) = 0.3$ and $A(x_2) = B(x_1) = B(x_2) = 0.6$. Let $\mathcal{S}$ be a fuzzy stack consisting of all those fuzzy sets G on X for which $0.3 \leq G(x_1) \leq 1$ and $0.2 \leq G(x_2) \leq 1$.

Suppose C and D are two fuzzy sets in X such that $C(x_1) = 0.5$, $C(x_2) = 0$ and $D(x_1) = 0$, $D(x_2) = 0.4$. We easily check that $\Phi_{\mathcal{S}}(C) = \Phi_{\mathcal{S}}(D) = 0_X$, where-as $\Phi_{\mathcal{S}}(C \vee D) = E$ where $E(x_1) = 0.4$ and $E(x_2) = 0.4$. Thus $\Phi_{\mathcal{S}}(C \vee D) \neq \Phi_{\mathcal{S}}(C) \vee \Phi_{\mathcal{S}}(D)$.

Theorem 3.9. *If for some fuzzy point y_α in X and some $U \in Q(y_\alpha)$, $U * P \notin \mathcal{S}$ holds, then $U * \Phi_{\mathcal{S}}(P) \notin \mathcal{S}$ and in particular, $U * \Phi_{\mathcal{S}}(P) = 0_X$.*

Proof. Let $U \in Q(y_\alpha)$ with $U * P \notin \mathcal{S}$. Assume that $U * \Phi_{\mathcal{S}}(P) \notin \mathcal{S}$; then $U * \Phi_{\mathcal{S}}(P) \neq 0_X$, so that $\exists x \in X$ such that $\Phi_{\mathcal{S}}(P)(x) + U(x) > 1$. Let $[\Phi_{\mathcal{S}}(P)](x) = \beta$. Thus $x_\beta \leq \Phi_{\mathcal{S}}(P)$ and $U \in Q(x_\beta)$, so that $P * U \in \mathcal{S}$, contradiction. Thus $U * \Phi_{\mathcal{S}}(P) \notin \mathcal{S}$.

For the second part, let us suppose $U * \Phi_{\mathcal{S}}(P) \neq 0_X$ for some $U \in Q(y_\alpha)$. Then there exists a fuzzy point $x_\beta \leq U * \Phi_{\mathcal{S}}(P)$ and so by an argument as above, $U * P \in \mathcal{S}$. Thus $U * \Phi_{\mathcal{S}}(P) = 0_X$. □

Theorem 3.10. *For any fuzzy set A in X, $\Phi_{\mathcal{S}}(A \vee \Phi_{\mathcal{S}}(A)) = \Phi_{\mathcal{S}}(A)$.*

Proof. It is obvious that $\Phi_{\mathcal{S}}(A) \leq \Phi_{\mathcal{S}}(A \vee \Phi_{\mathcal{S}}(A))$. Let $x_\alpha \not\leq \Phi_{\mathcal{S}}(A)$; then there exists an open q-nbd U of x_α such that $U * A \notin \mathcal{S}$. Hence $U * \Phi_{\mathcal{S}}(A) = 0_X$ and $U * (A \vee \Phi_{\mathcal{S}}(A)) = (U * A) \vee (U * \Phi_{\mathcal{S}}(A)) = U * A \notin \mathcal{S}$ (using Theorem 3.9 and Lemma 2.2). So, $x_\alpha \not\leq \Phi_{\mathcal{S}}(A \vee \Phi_{\mathcal{S}}(A))$. □

4. Operator $\Psi_{\mathcal{S}}$

In this section, we consider a new operator $\Psi_{\mathcal{S}}$, defined in terms of $\Phi_{\mathcal{S}}$, in an sfts $(X, \tau, \mathcal{S})$. It is shown in [9] that if $\mathcal{S}$ is a fuzzy grill on X, then the corresponding operator $\Psi_{\mathcal{S}}$ gives rise to a fuzzy topology $\tau_{\mathcal{S}}$ on X, finer that τ. But in the context of our setting with a fuzzy stack $\mathcal{S}$, it will be seen that the induced structure $\tau_{\mathcal{S}}$ is a strong generalized fuzzy topology (i.e., a structure which satisfies all the axioms of a fuzzy topology except that it is not necessarily closed under finite intersections) [10].

Definition 4.1. For an sfts $(X, \tau, \mathcal{S})$, we define an operator $\Psi_{\mathcal{S}} : I^X \to I^X$ given by $\Psi_{\mathcal{S}}(P) = P \vee \Phi_{\mathcal{S}}(P)$ for $P \in I^X$.

Theorem 4.2. *Let $\mathcal{S}$ be a fuzzy stack on an fts X. Then the following conditions hold*:

(i) $\Psi_{\mathcal{S}}(0_X) = 0_X$.

(ii) $P \leq \Psi_{\mathcal{S}}(P)$, *for every fuzzy set P in X, moreover* $\Psi_{\mathcal{S}}(1_X) = 1_X$.

(iii) P, Q *are fuzzy sets in X with* $P \leq Q \Rightarrow \Psi_{\mathcal{S}}(P) \leq \Psi_{\mathcal{S}}(Q)$.

(iv) $\Psi_{\mathcal{S}}(\Psi_{\mathcal{S}}(P)) = \Psi_{\mathcal{S}}(P)$.

Proof. (i), (ii) and (iii) follow from the corresponding results of Theorem 3.5.

(iv) $\Psi_{\mathcal{S}}(\Psi_{\mathcal{S}}(P)) = \Psi_{\mathcal{S}}(P \vee \Phi_{\mathcal{S}}(P)$

$= (P \vee \Phi_{\mathcal{S}}(P)) \vee \Phi_{\mathcal{S}}(P \vee \Phi_{\mathcal{S}}(P))$, by definition of $\Psi_{\mathcal{S}}(P)$

$= (P \vee \Phi_{\mathcal{S}}(P)) \vee \Phi_{\mathcal{S}}(P)$ (using Theorem 3.10)

$= P \vee \Phi_{\mathcal{S}}(P)$

$= \Psi_{\mathcal{S}}(P)$.

Definition 4.3. For an sfts $(X, \tau, \mathcal{S})$, we define a collection $\tau_{\mathcal{S}}$ of fuzzy sets given by $\tau_{\mathcal{S}} = \{U \in I^X : \Psi_{\mathcal{S}}(1_X - U) = 1_X - U\}$.

Theorem 4.4. *Let $\mathcal{S}$ be a fuzzy stack on an fts* X. *Then*

(i) $0_X, 1_X \in \tau_{\mathcal{S}}$.

(ii) *If* $U_\alpha \in \tau_{\mathcal{S}}$ *for* $\alpha \in J$, *then* $\vee U_\alpha \in \tau_{\mathcal{S}}$.

Proof. (i) Since $\Psi_{\mathcal{S}}(1_X) = 1_X$ and $\Psi_{\mathcal{S}}(0_X) = 0_X$, both 0_X, and 1_X are in $\tau_{\mathcal{S}}$.

(ii) Let $U_\alpha \in \tau_{\mathcal{S}}$ for $\alpha \in J$,

Then $1_X - \vee U_\alpha = \wedge(1_X - U_\alpha) \leq (1_X - U_\alpha), \forall \alpha \in J$

$= \Psi_{\mathcal{S}}(1_X - \vee U_\alpha) \leq \Psi_{\mathcal{S}}(1_X - U_\alpha) = (1_X - U_\alpha), \forall \alpha \in J$

$\Rightarrow \Psi_{\mathcal{S}}(1_X - \vee U_\alpha) \leq \wedge(1_X - \vee U_\alpha) = 1_X - \vee U_\alpha$ and hence $\vee U_\alpha \in \tau_{\mathcal{S}}$.

Remark 4.5. For two elements of $\tau_{\mathcal{S}}$, the intersection may not be an element of $\tau_{\mathcal{S}}$ as is shown in the next example.

Example 4.6. Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.4$ and $A(b) = 0.6$ Then (X, τ) is an fts. Let $\mathcal{S}$ be a fuzzy stack consisting of all fuzzy sets V on X for which $0.4 \leq V(a) \leq 1$ and $0.4 \leq V(b) \leq 1$.

Now consider the fuzzy sets P and Q in X such that $P(a) = 0.8, P(b) = 0.5$ and $Q(a) = 0.6, Q(b) = 0.9$. Then $\Phi_{\mathcal{S}}(1_X - P) = 0_X$ and $\Phi_{\mathcal{S}}(1_X - Q) = 0_X$. This implies $\Psi_{\mathcal{S}}(1_X - P) = 1_X - P$ and $\Psi_{\mathcal{S}}(1_X - Q) = 1_X - Q$, i.e., $P, Q \in \tau_{\mathcal{S}}$. But for $P \wedge Q = C$ where $C(a) = 0.6, C(b) = 0.5, \Phi_{\mathcal{S}}(1_X - (A \wedge B)) = D$, where $D(a) = 0.6, D(b) = 4$.

Now $\Psi_{\mathcal{S}}(1_X - (P \wedge Q)) = (1_X - (P \wedge Q)) \vee \Phi_{\mathcal{S}}(1_X - (P \wedge Q)) = (1_X - (P \wedge Q)) \vee D = E \neq (1_X - (P \wedge Q))$, where $E(a) = 0.6, E(b) = 0.5$.

Thus $P \wedge Q \notin \tau_{\mathcal{S}}$.

Remark 4.7. It now follows that for an sfts $(X, \tau, \mathcal{S}), \tau_{\mathcal{S}}$ is a strong generalized fuzzy topology but not necessarily a fuzzy topology on X.

Let $\mathcal{S}$ be a fuzzy stack on an fts X. Then the elements of $\tau_{\mathcal{S}}$ are said to be $\tau_{\mathcal{S}}$-open. If the complement of a member A of I^X is $\tau_{\mathcal{S}}$-open, then A is said to be $\tau_{\mathcal{S}}$-closed.

Definition 4.8. For an sfts $(X, \tau, \mathcal{S})$ and $P \in I^X$, we define the operators $\tau_{\mathcal{S}}\text{-} cl : I^X \to I^X$ as the following:

$\tau_{\mathcal{S}}\text{-}\operatorname{int}(P) = \vee \{U \leq I^X : U \leq P, U \in \tau_{\mathcal{S}}\}$

$\tau_{\mathcal{S}}\text{-} cl(P) = \wedge \{F \leq I^X : P \leq F, 1_X - F \in \tau_{\mathcal{S}}\}.$

Theorem 4.9. *Let $\mathcal{S}$ be a fuzzy stack on an fts X. For $P, Q \in I^X$, the following conditions are satisfied*:

(i) $\tau_{\mathcal{S}}\text{-int}\,(0_X) = 0_X$.

(ii) $\tau_{\mathcal{S}}\text{-int}\,(P) \leq P$.

(iii) *P is $\tau_{\mathcal{S}}$-open if and only if $P = \tau_{\mathcal{S}}\text{-int}\,(P)$.*

(iv) *If $P \leq Q$, then $\tau_{\mathcal{S}}\text{- int}\,(P) \leq \tau_{\mathcal{S}}\text{-int}\,(Q)$.*

(v) $\tau_{\mathcal{S}}\text{-int}\,(\tau_{\mathcal{S}}\text{-int}\,(P)) = \tau_{\mathcal{S}}\text{-int}\,(\mathrm{P})$.

Proof. (i) and (ii) are obvious.

(iii) Clear from the definition of $\tau_{\mathcal{S}}\text{- int}\,(P)$ and Theorem 4.4.

(iv) Since $P \leq Q \Rightarrow \tau_{\mathcal{S}}\text{-int}\,(P) \leq P \leq Q$ and $\tau_{\mathcal{S}}\text{-int}(P)$ is $\tau_{\mathcal{S}}$-open (by (iii)), by definition, $\tau_{\mathcal{S}}\text{-int}\,(P) \leq \tau_{\mathcal{S}}\text{-int}\,(Q)$.

(v) Since $\tau_{\mathcal{S}}\text{-int}\,(P)$ is a $\tau_{\mathcal{S}}$-open (by (iii)), $\tau_{\mathcal{S}}\text{-int}\,(\tau_{\mathcal{S}}\text{-int}\,(P)) = \tau_{\mathcal{S}}\text{-int}\,(P)$. Dually, we have: □

Theorem 4.10. *Let $(X, \tau, \mathcal{S})$ be an sfts. For $A, B \leq I^X$, the following hold*:

(i) $\tau_{\mathcal{S}}\text{-cl}\,(1^X) = 1^X$.

(ii) $A \leq \tau_{\mathcal{S}}\text{-cl}\,(A)$.

(iii) *A is $\tau_{\mathcal{S}}$-closed if and only if $\tau_{\mathcal{S}}\text{-}cl\,(A) = A$.*

(iv) *If $A \leq B$, then $\tau_{\mathcal{S}}\text{-}cl\,(A) \leq \tau_{\mathcal{S}}\text{-}cl\,(B)$.*

(v) $\tau_{\mathcal{S}}\text{-}cl\,(\tau_{\mathcal{S}}\text{-}cl\,(A)) = \tau_{\mathcal{S}}\text{-}cl\,(A)$.

(vi) $\tau_{\mathcal{S}}\text{-}cl\,(1_X - A) = 1_X - \tau_{\mathcal{S}}\text{-int}\,(A)$.

Proof. The proof follows in a straightforward manner. □

Theorem 4.11. *Let $\mathcal{S}$ be a fuzzy stack on an fts X. For $A \in I^X$, A is $\tau_{\mathcal{S}}$-closed iff $\Psi_{\mathcal{S}}(A) = A$.*

Proof. A is $\tau_{\mathcal{S}}$-closed $1_X - A$ is $\tau_{\mathcal{S}}$-open iff $\Psi_{\mathcal{S}}(1_X - (1_X - A)) = 1_X - (1_X - A)$, i.e., $\Psi_{\mathcal{S}}(A) = A$. □

Theorem 4.12. *Let $\mathcal{S}$ be a fuzzy stack on an fts X. Then the following conditions are satisfied*:

(i) $\tau_{\mathcal{S}}$-cl $(P) = \Psi_{\mathcal{S}}(P)$ *for* $P \in I^X$.

(ii) *If* $P \notin \mathcal{S}$, *then* $1_X - P \in \tau_{\mathcal{S}}$ *for* $P \in I^X$.

(iii) $\Phi_{\mathcal{S}}(P)$ *is* $\tau_{\mathcal{S}}$*-closed for* $P \in I^X$.

Proof. (i) By Theorems 4.11 and 4.2(iv), $\Psi_{\mathcal{S}}(P)$ is $\tau_{\mathcal{S}}$-closed. From $P \le \Psi_{\mathcal{S}}(P)$, it follows that $P \le \tau_{\mathcal{S}}\text{-}cl\,(P) \le \Psi_{\mathcal{S}}(P)$. Again, since $P \le \tau_{\mathcal{S}}\text{-}cl\,(P)$, from Theorems 4.2, 4.9 and 4.10, it follows that $\Psi_{\mathcal{S}}(P) \le \Psi_{\mathcal{S}}(\tau_{\mathcal{S}}\text{-}cl\,(P)) = \tau_{\mathcal{S}}\text{-}cl\,(P)$. Consequently, $\tau_{\mathcal{S}}\text{-}cl\,(P) = \Psi_{\mathcal{S}}(P)$.

(ii) If $P \notin \mathcal{S}$, then by Theorem 3.5 (vii), we have $\Phi_{\mathcal{S}}(P) = 0_X$. So $\Psi_{\mathcal{S}}(1_X - (1_X - P)) = \Psi_{\mathcal{S}}(P) = P \vee \Phi_{\mathcal{S}}(P) = P = 1_X - (1_X - P)$. Thus $1_X - P \in \tau_{\mathcal{S}}$.

(iii) For $P \in I^X$, from Theorem 3.5 (vi), $\Psi_{\mathcal{S}}(\Phi_{\mathcal{S}}(P)) = \Phi_{\mathcal{S}}(P) \vee \Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(P)) = \Phi_{\mathcal{S}}(P)$. By Theorem 4.11, we have $\Phi_{\mathcal{S}}(P)$ is $\tau_{\mathcal{S}}$-closed. □

Theorem 4.13. *Let $(X, \tau, \mathcal{S})$ be an sfts. A fuzzy open base $\mathcal{B}_{\mathcal{S}}$ of $\tau_{\mathcal{S}}$ is given by* $\mathcal{B}_{\mathcal{S}} = \{U - A : U \in \tau \text{ and } A \notin \mathcal{S}\}$.

Proof. To ensure that the members of $\mathcal{B}_{\mathcal{S}}$ are $\tau_{\mathcal{S}}$-open, let $V \in \mathcal{B}_{\mathcal{S}}$. Then for some $U \in \tau$ and $A \notin \mathcal{S}, V = U - A$. We only prove that $\Phi_{\mathcal{S}}(1_X - V) \le 1_X - V$ (which in turn, will prove $\Psi_{\mathcal{S}}(1_X - V) \le 1_X - V$, i.e., $(1_X - V)$ is $\tau_{\mathcal{S}}$-closed). On the contrary, let, for some fuzzy point x_α, one have

$x_\alpha \le \Phi_{\mathcal{S}}(1_X - V), \ldots, (i)$

but $x_\alpha \not\le (1_X - V), \ldots, (ii)$

Now for any $C \in Q(x_\alpha)$, a routine check gives $C * (1_X - V) = C - V, \ldots (iii)$ and also in view of (i) we have $C * (1_X - V) \in \mathcal{S}$.

Consequently, $C - V = C - (U - A) \in \mathcal{S}$ for each $C \in Q(x_\alpha), \ldots, (iv)$

Again, $x_\alpha \not\leq (1_X - V)$ (by (ii))) $\Rightarrow \alpha > 1 - V(x) = 1 - (U - A)(x)$.

We claim that $U(x) > A(x)$. For otherwise, $1 - (U - A)(x) = 1 - 0 < \alpha$, contradiction.

Thus $\alpha > 1 - U(x) + A(x)$, so that $\alpha + U(x) > 1 + A(x) > 1$.

Hence $U \in Q(x_\alpha)$ which in virtue of (iv) gives $U - (U - A) \in \mathcal{S}$. Now since by Lemma 2.4, $(U - (U - A)) \leq A$, we have $A \in \mathcal{S}$, which contradicts the assumption that $A \notin \mathcal{S}$. Hence $\Phi_{\mathcal{S}}(1_X - V) \leq 1_X - V$, proving our contention.

It remains to show that given any fuzzy point x_α and an open q-nbd V of x_α in $(X, \tau_{\mathcal{S}})$, $\exists B_1 \in \mathcal{B}_{\mathcal{S}}$ such that $x_\alpha q\, B_1 \leq V$. As V is a q-nbd of x_α in $(X, \tau_{\mathcal{S}})$, there exists a $B \in \tau_{\mathcal{S}}$ with $x_\alpha q\, B$ and $B \leq V$.

Clearly $\Phi_{\mathcal{S}}(1_X - B) \leq (1_X - B)$ and hence $x_\alpha \not\leq \Phi_{\mathcal{S}}(1_X - B)$. Then for some open q-nbd W of x_α in $(X, \tau, \mathcal{S})$ we have $(1_X - B) * W \notin \mathcal{S}$. Then $W - B \notin \mathcal{S}$. (using (iii)).

If we now take $B_1 = W - (W - B)$, then $B_1 \in \mathcal{B}_{\mathcal{S}}$, and by Lemma 2.4, $W - (W - B) \leq B \leq V$.

Finally, we need to show that $x_\alpha q B_1$. In fact, if $W(x) \leq B(x)$ then $\alpha + W(x) - (W - B)(x) = \alpha + W(x) > 1$; and, if $W(x) > B(x)$ then $\alpha + W(x) - (W - B)(x) = \alpha + W(x) - [W(x) - B(x)] = \alpha + B(x) > 1$. □

Thus in any case $x_\alpha q\, B_1$ and we are through.

Corollary 4.14. *Let* $(X, \tau, \mathcal{S})$ *be an sfts. If* $P \in \tau_{\mathcal{S}}$, $P = \vee(U_\alpha - A_\alpha)$ *for* $U_\alpha \in \tau$ *and* $A_\alpha \notin \mathcal{S}\,(\alpha \in \Lambda$*- some indexing set*$)$.

Theorem 4.15. *Let* $(X, \tau, \mathcal{S})$ *be an sfts. Then* $\tau \leq \tau_{\mathcal{S}}$.

Proof. Clear. □

Definition 4.16. Corresponding to a given fuzzy set P in an sfts $(X, \tau, \mathcal{S})$, a fuzzy set $\widetilde{P}$ is defined by $\widetilde{P} = \{y_\alpha \leq P : y_\alpha \not\leq \Phi_{\mathcal{S}}(P)\}$.

Remark 4.17. It follows at once from the above definition that for any $P \in I^X$,

(i) $y_\alpha \not\leq \widetilde{P}$ iff $y_\alpha \leq \Phi_{\mathcal{S}}(P)$.

(ii) $\widetilde{P} \wedge \Phi_{\mathcal{S}}(P) = 0_X$.

The following theorem shows that a given fuzzy set can be decomposed via the above operator.

Theorem 4.18. *Let* $(X, \tau, \mathcal{S})$ *be an sfts. Then* $P = \widetilde{P} \vee (P \wedge \Phi_{\mathcal{S}}(P))$ *for* $P \in I^X$.

Proof. Clearly $\widetilde{P} \leq P$, and hence $\widetilde{P} \vee (P \wedge \Phi_{\mathcal{S}}(P)) \leq P, \ldots, (i)$

Again, for any fuzzy point $y_\alpha \leq P$, either $y_\alpha \leq \Phi_{\mathcal{S}}(P)$ or else $y_\alpha \leq \widetilde{P}$.

Thus in either case, $y_\alpha \leq \widetilde{P} \vee (P \wedge \Phi_{\mathcal{S}}(P))$ which gives

$P \leq \widetilde{P} \vee (P \wedge \Phi_{\mathcal{S}}(P)), \ldots, (ii)$

Thus the proof is complete in view of (i) and (ii).

Theorem 4.19. *For any fuzzy set* P *in a sfts* $(X, \tau, \mathcal{S})$, $\widetilde{P} \wedge P \wedge \Phi_{\mathcal{S}}(\widetilde{P}) = 0_X$.

Proof. It suffices to show that $y_\alpha \leq \Phi_{\mathcal{S}}(\widetilde{P}) \Rightarrow y_\alpha \not\leq \widetilde{P}$.

Now, $y_\alpha \leq \Phi_{\mathcal{S}}(\widetilde{P}) \Rightarrow$ for each $U \in Q(y_\alpha)$, $\widetilde{P} * U \in \mathcal{S}$, hence $P * U \in \mathcal{S}$ (as $\widetilde{P} \leq P$) for each $U \leq Q(y_\alpha)$. Thus $y_\alpha \leq \Phi_{\mathcal{S}}(P)$ so that $y_\alpha \not\leq \widetilde{P}$. □

Theorem 4.20. *Let* $(X, \tau, \mathcal{S})$ *be an sfts. Then the following conditions are equivalent*:

(a) *For any* $P \in I^X$, $P \wedge \Phi_{\mathcal{S}}(P) = 0_X \Rightarrow P \notin \mathcal{S}$.

(b) *For any* $P \in I^X$, $\widetilde{P} \notin \mathcal{S}$.

(c) *For any* $\tau_{\mathcal{S}}$*-closed set P in* X, $\widetilde{P} \notin \mathcal{S}$.

(d) *For any* $P \in I^X$, *if* $B = P \vee \Phi_{\mathcal{S}}(P)$, *then* $\widetilde{B} \notin \mathcal{S}$.

Proof. $(a) \Rightarrow (b)$: Let $x \in \text{supp}\,(\widetilde{P})$ and $x_\alpha \leq \widetilde{P}(0 < \alpha \leq 1)$. Then $x_\alpha \notin \Phi_{\mathcal{S}}(P)$ and hence $x_\alpha \notin \Phi_{\mathcal{S}}(\widetilde{P})$ (as $\widetilde{P} \leq P$). We claim that $x \in \text{supp}\,(\Phi_{\mathcal{S}}(\widetilde{P}))$. If not, then $\Phi_{\mathcal{S}}(\widetilde{P})(x) = \alpha_0(\text{say}) < \alpha$, where $\alpha_0 > 0$. Then $x_{\alpha_0} \leq \Phi_{\mathcal{S}}(\widetilde{P}) \leq \Phi_{\mathcal{S}}(P)$. On the other hand, $\alpha_0 < \alpha$ and $x_\alpha \in \widetilde{P} \Rightarrow x_{\alpha_0} \leq \widetilde{P} \Rightarrow x_{\alpha_0} \not\leq \Phi_{\mathcal{S}}(P)$. Hence we arrive at a contradiction. We thus find that $\text{supp}\,(\widetilde{P}) \cap \text{supp}\,(\Phi_{\mathcal{S}}(\widetilde{P})) = \phi$, i.e., $\widetilde{P} \wedge \Phi_{\mathcal{S}}(\widetilde{P}) = 0_X$ and then by (a), it follows that $\widetilde{P} \notin \mathcal{S}$.

$(b) \Rightarrow (c)$: Obvious.

(c) $(c) \Rightarrow (d)$: $B = \Psi_{\mathcal{S}}(P)$ is $\tau_{\mathcal{S}}$-closed and hence $\widetilde{B} \notin \mathcal{S}$ by (c).

$(d) \Rightarrow (a)$: Let $B = P \vee \Phi_{\mathcal{S}}(P)$. In view of (d) it is only to be shown that $\widetilde{B} = P$. Indeed, $x_\alpha \leq \widetilde{B} \Rightarrow x_\alpha \leq B, x_\alpha \not\leq \Phi_{\mathcal{S}}(P)$ (by Theorem 3.10) $\Rightarrow x_\alpha \leq P$ (as $B = P \vee \Phi_{\mathcal{S}}(P)$); and again, $x_\alpha \leq P \Rightarrow x_\alpha \not\leq \Phi_{\mathcal{S}}(P)$ (as $P \wedge \Phi_{\mathcal{S}}(P) = 0_X) = \Phi_{\mathcal{S}}(B)$, but $x_\alpha \in B$ (as $P \leq B) \Rightarrow x_\alpha \in \widetilde{B}$. □

Conflicts of interest:

No conflict of interest was declared by the author.

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