Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 383-392

ON MERSENNE AND MERSENNE -LUCAS SEDENIONS

B. MALINI DEVI and S. DEVIBALA

Department of Mathematics
Sri Meenakshi Govt. Arts
College for Women (A), Madurai, India
E-mail: bmalini22.mdu@gmail.com
devibala27@yahoo.com

Abstract

In this study, we introduce new class of sedenion numbers associated with the Mersenne numbers. We define Mersenne sedenion and Mersenne-Lucas sedenion by using the Mersenne numbers. We obtain generating functions and Binet formulas for the Mersenne and Mersenne-Lucas sedenion and some of interesting identities of these numbers.

1. Introduction

Sedenions are obtained by applying the Cayley-Dickson construction to the octonions and form a 16-dimensional non-associative and non-commutative algebra over the set of real numbers. Many different classes of sedenion number sequences such as Fibonacci sedenion, Lucas sedenion, Pell sedenion have been obtained by a number of authors in many different ways [1-7].

In this paper we aim at developing new classes of sedenion numbers associated with the Mersenne and Mersenne-Lucas numbers. We introduce the Mersenne sedenion and Mersenne-Lucas sedenion and obtain some of their identities.

2. Method of Analysis

The Mersenne numbers $\{M_n\}$ is defined by the recurrence relation

$$M_n = 3M_{n-1} - 2M_{n-2}, \; n \geq 2 \text{ with } M_0 = 0, \; M_1 = 1.$$

2020 Mathematics Subject Classification: 11B83.
Keywords: Sedenion, Mersenne sedenion, Mersenne-Lucas sedenion.
Received May 17, 2021; Accepted June 7, 2021

The Mersenne-Lucas numbers $\{ML_n\}$ is defined recurrently by

$$ML_n = 3ML_{n-1} - 2ML_{n-2},\ n \geq 2 \text{ with } ML_0 = 2,\ ML_1 = 3.$$

The generating functions for these sequences are

$$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1 - 3x + 2x^2}$$

$$\sum_{n=0}^{\infty} ML_n x^n = \frac{2 - 3x}{1 - 3x + 2x^2}$$

The Binet formulas for these sequences are

$$M_n = \alpha^n - \beta^n$$

$$ML_n = \alpha^n + \beta^n,$$

where $\alpha = 2, \beta = 1$ are roots of the characteristic equation $x^2 - 3x + 2 = 0$

A sedenion $\mathbb{S}$ can be written as $S = \sum_{i=0}^{15} a_i e_i$, where $a_0, a_1, \ldots, a_{15}$ are reals.

We take the basis elements of $\mathbb{S}$ as $\{e_0, e_1, \ldots, e_{15}\}$, where e_0 is the unit element and $e_1, e_2, \ldots, e_{15}$ are imaginaries.

Multiplication table for the basis of $\mathbb{S}$.

.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	-0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
2	2	-3	-0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
3	3	2	-1	-0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
4	4	-5	-6	-7	-0	1	2	3	12	13	14	15	-8	-9	-10	-11
5	5	4	-7	6	-1	-0	-3	2	13	-12	15	-14	9	-8	11	-10
6	6	7	4	-5	-2	3	-0	-1	14	-15	-12	13	10	-11	-8	9

7	7	-6	5	4	-3	-2	1	-0	15	14	-13	-12	11	10	-9	-8
8	8	-9	-10	-11	-12	-13	-14	-15	-0	1	2	3	4	5	6	7
9	9	8	-11	10	-13	12	15	-14	-1	-0	-3	2	-5	4	7	-6
10	10	11	8	-9	-14	-15	12	13	-2	3	-0	-1	-6	-7	4	5
11	11	-10	9	8	-15	14	-13	12	-3	-2	1	-0	-7	6	-5	4
12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	-0	-1	-2	-3
13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	-0	3	-2
14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	-0	1
15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	-0

The n^{th} Mersenne sedenion is

$$\widehat{M_n} = \sum_{s=0}^{15} M_{n+s} e_s$$

The n^{th} Mersenne-Lucas sedenion is

$$\widehat{ML_n} = \sum_{s=0}^{15} ML_{n+s} e_s$$

3. Generating Functions and Binet formulas for the Mersenne and Mersenne-Lucas Sedenions

Theorem 1. *The generating functions for the Mersenne and Mersenne-Lucas sedenions are*

$$f(x) = \frac{\widehat{M_0} + (\widehat{M_1} - 3\widehat{M_0})x}{1 - 3x + 2x^2}$$

$$g(x) = \frac{\widehat{ML_0} + (\widehat{ML_1} - 3\widehat{ML_0})x}{1 - 3x + 2x^2}$$

Proof. Define $f(x) = \sum_{i=0}^{\infty} \widehat{M_\iota} x^i$

$$f(x) = \widehat{M_0} + \widehat{M_1}x + \sum_{i=2}^{\infty} \widehat{M_\iota}x^i$$

$$-3xf(x) = -3x\widehat{M_0} - 3\sum_{i=2}^{\infty} \widehat{M_{\iota-1}}x^i$$

$$2x^2 f(x) = 2\sum_{i=2}^{\infty} \widehat{M_{\iota-2}}x^i$$

Adding these equations, we get

$$(1-3x+2x^2)f(x) = \widehat{M_0} + \widehat{M_1}x - 3x\widehat{M_0} + \sum_{i=2}^{\infty}(\widehat{M_\iota} - 3\widehat{M_{\iota-1}} + 2\widehat{M_{\iota-2}})x^i$$

$$\therefore f(x) = \frac{\widehat{M_0} + (\widehat{M_1} - 3\widehat{M_0})x}{1-3x+2x^2}$$

And define $g(x) = \sum_{i=0}^{\infty} \widehat{ML_\iota}x^i$

$$g(x) = \widehat{ML_0} + \widehat{ML_1}x + \sum_{i=2}^{\infty} \widehat{ML_\iota}x^i$$

$$-3xg(x) = -3x\widehat{ML_0} - 3\sum_{i=2}^{\infty} \widehat{ML_{\iota-1}}x^i$$

$$2x^2 g(x) = 2\sum_{i=2}^{\infty} \widehat{ML_{\iota-2}}x^i$$

Adding these equations, we get

$$(1-3x+2x^2)g(x) = \widehat{ML_0} - 3x\widehat{ML_0} + \widehat{ML_1}x$$

$$+\sum_{i=2}^{\infty}(\widehat{ML_\iota} - 3\widehat{ML_{\iota-1}} + 2\widehat{ML_{\iota-2}})x^i$$

$$\therefore g(x) = \frac{\widehat{ML_0} + (\widehat{ML_1} - 3\widehat{ML_0})x}{1-3x+2x^2}$$

Theorem 2. *For any integer* n, *the* n^{th} *Mersenne sedenion is*

$$\widehat{M_n} = \alpha^n A - \beta^n B$$

and the n^{th} *Mersenne-Lucas sedenion is*

$$\widehat{ML_n} = \alpha^n A - \beta^n B,$$

where $A = \sum_{s=0}^{15} \alpha^s e_s$ *and* $B = \sum_{s=0}^{15} \beta^s e_s$.

Proof. From the definition, we have

$$\widehat{M_{n+1}} - \widehat{M_n} = \sum_{s=0}^{\infty} (M_{n+s+1} - M_{n+s}) e_s$$

From the identity $M_{n+1} - M_n = \alpha^n$, we obtain

$$\widehat{M_{n+1}} - \widehat{M_n} = \alpha^n A \tag{1}$$

Similarly, we have

$$\widehat{M_n} - 2\widehat{M_{n-1}} = \beta^n B \tag{2}$$

Adding equations (1) and (2) and using the identity $M_{n+1} - 2M_{n-1}$, $= ML_n$ we get

$$\widehat{ML_n} - \alpha^n A + \beta^n B$$

And subtracting these equations, we get

$$\widehat{M_n} - \alpha^n A - \beta^n B.$$

When using the binet formulas to obtain identities for the Mersenne and Mersenne-Lucas sedenions, we require AB, BA, A^2, B^2, these products are given in the next lemma.

Lemma 1.

$AB = -65, 533e_0 - 11, 473e_1 + 34, 427e_2 + 6, 583e_3 - 57, 373e_4 - 21, 673e_5 +13, 007e_6 - 39, 013e_7 + 48, 659e_8 - 11, 049e_9 + 36, 239e_{10} + 22, 875e_{11} - 50, 113e_{12} - 5, 909e_{13} + 45, 491e_{14} + 25, 977e_{15}$

$BA = -65, 533e_0 + 11, 351e_1 - 34, 417e_2 - 19, 109e_3 - 22, 465e_4 + 21, 739e_5 - 12, 877e_6 + 39, 217e_7 - 64, 513e_8 + 12, 075e_9 - 34, 717e_{10} - 18, 777e_{11} + 58, 307e_{12} + 22, 295e_{13} - 13745_{14} + 35, 355e_{15}$

$$A^2 = \left(2 - \sum_{s=0}^{15} 2^{2s}\right)e_0 + 2\sum_{s=1}^{15} 2^s e_s$$

$$B^2 = -16e_0 + 2\sum_{s=1}^{15} e_s$$

Proof. From the definition of A and B, and using the multiplication table for the basis of sedenions, we computed these results.

Theorem 3. *For* $n \geq 1,\ r \geq 1,$ *we have the following identities*:

i. $\widehat{M_{n+1}} - \widehat{M_n} = 2^n \sum_{s=0}^{15} 2^s e_s$

ii. $\widehat{M_{n+1}} + \widehat{M_n} = 3(2^n)\sum_{s=0}^{15} 2^s e_s - 2\sum_{s=0}^{15} e_s$

iii. $\widehat{ML_{n+1}} - \widehat{ML_n} = 2^n \sum_{s=0}^{15} 2^s e_s$

iv. $\widehat{ML_{n+1}} + \widehat{ML_n} = 3(2^n)\sum_{s=0}^{15} 2^s e_s + 2\sum_{s=0}^{15} e_s$

Proof. From the definition

$$\widehat{M_{n+1}} - \widehat{M_n} \sum_{s=0}^{15} M_{n+s+1}e_s - \sum_{s=0}^{15} M_{n+s}e_s$$

$$= \sum_{s=0}^{15} (M_{n+s+1} - M_{n+s})e_s$$

With $M_{n+1} - M_n = 2^n$, we get

$$\widehat{M_{n+1}} - \widehat{M_n} = 2^n \sum_{s=0}^{15} 2^s e_s$$

Again, by using the identity $M_{n+1} + M_n = 3(2^n) - 2$,

$$\widehat{M_{n+1}} + \widehat{M_n} = 3(2^n)\sum_{s=0}^{15} 2^s e_s - 2\sum_{s=0}^{15} e_s$$

Similarly, by using the identities $ML_{n+1} - ML_n = 2^n$ and $ML_{n+1} - ML_n = 3(2^n) + 2$ we obtain the results (iii) and (iv).

Theorem 4. (*Catalan's Identities*). *For any positive integers n and r such that* $r \leq n$, *we get*

i. $\widehat{M}_{n+r}\widehat{M}_{n-r} - \widehat{M}_n^{\,2} = -2^n\left[\frac{BA}{2^r} + 2AB(2^{r-1} - 1)\right]$

ii. $\widehat{M}_{n-r}\widehat{M}_{n+r} - \widehat{M}_n^{\,2} = -2^n\left[2^r BA + \frac{BA}{2^r} + (1 - 2^{r-1})\right]$

iii. $\widehat{ML}_{n+r}\widehat{ML}_{n-r} - \widehat{ML}_n^{\,2} = 2^n\left[\frac{BA}{2^r} + 2AB(2^{r-1} - 1)\right]$

iv. $\widehat{ML}_{n-r}\widehat{ML}_{n+r} - \widehat{ML}_n^{\,2} = 2^n\left[2^r BA + \frac{AB}{2^r}(1 - 2^{r-1})\right]$,

where $A = \sum_{s=0}^{15} 2^s e_s$ and $B = \sum_{s=0}^{15} e_s$

Proof. Using binet formula for Mersenne sedenions and applying lemma 1, we have

$$\widehat{M}_{n+r}\widehat{M}_{n-r} - \widehat{M}_n^{\,2} = (2^{n+r}A - B)(2^{n-r}A - B) - (2^n A - B)^2$$

$$= -2^n\left[\frac{BA}{2^r} + 2AB(2^{r-1} - 1)\right]$$

Similarly, $\widehat{M}_{n-r}\widehat{M}_{n+r} - \widehat{M}_n^{\,2} = -2^n\left[2^r BA + \frac{BA}{2^r}(1 - 2^{r+1})\right]$.

Also, by using the binet formula for Mersenne-Lucas sedenions and applying Lemma 1, we obtain

$$\widehat{ML}_{n+r}\widehat{ML}_{n-r} - \widehat{ML}_n^{\,2} = (2^{n+r}A + B)(2^{n-r}A + B) - (2^n A + B)^2$$

$$= 2^n\left[\frac{BA}{2^r} + 2AB(2^{r-1} - 1)\right]$$

Similarly, $\widehat{ML}_{n-r}\widehat{ML}_{n+r} - \widehat{ML}_n^{\,2} = 2^n\left[2^r BA + \frac{AB}{2^r}(1 - 2^{r+1})\right]$.

Theorem 5. (*Cassini's Identity*). *For any positive integer* n, *we have*

i. $\widehat{M_{n+1}}\widehat{M_{n-1}} - \widehat{M_n}^2 = -2^n BA$

ii. $\widehat{M_{n-1}}\widehat{M_{n+1}} - \widehat{M_n}^2 = -2^n [4BA - 3AB]$

iii. $\widehat{ML_{n+1}}\widehat{ML_{n-1}} - \widehat{ML_n}^2 = 2^{n-1} BA$

iv. $\widehat{ML_{n-1}}\widehat{ML_{n+1}} - \widehat{ML_n}^2 = 2^n [4BA - 3AB]$,

where $A = \sum_{s=0}^{15} 2^s e_s$ *and* $B = \sum_{s=0}^{15} e_s$.

Proof. By substituting $r = 1$ in Catalan's identity, we obtain these identities.

Theorem 6 (d'Ocagne's Identity)**.** *For any positive integers m and n, if* $m > n$ *then*

i. $\widehat{M_m}\widehat{M_{n+1}} - \widehat{M_{m+1}}\widehat{M_n} = 2^n (2^{m-n} AB - BA)$

ii. $\widehat{ML_m}\widehat{ML_{n+1}} - \widehat{ML_{m+1}}\widehat{ML_n} = 3(2^n)(2^{m-n} AB + BA)$

Proof. Using Binet formulas for Mersenne and Mersenne-Lucas sedenions and Lemma 1, we obtain

$$\widehat{M_m}\widehat{M_{n+1}} - \widehat{M_{m+1}}\widehat{M_n} = (2^m A - B)(2^{n+1} A - B) - (2^{m+1} A - B)(2^n A - B)$$

$$= -2^{n+1} BA - 2^m AB + 2^n BA + 2^{m+1} AB$$

$$= -2^n BA + 2^m AB$$

$$= 2^n (2^{m-n} AB - BA).$$

Similarly,

$$\widehat{ML_m}\widehat{ML_{n+1}} - \widehat{ML_{m+1}}\widehat{ML_n} = (2^m A - B)(2^{n+1} A + B)$$

$$- (2^{m+1} A + B)(2^n A + B)$$

$$= 2^{n+1} BA + 2^m AB + 2^n BA + 2^{m+1} AB$$

$$= 3(2^{n})(2^{m-n}AB + BA)$$

Theorem 7. *For any positive integer n, we have*

$$\widehat{ML_n}^2 - \widehat{M_n}^2 = 2^{n+2}AB \text{ and } \widehat{ML_n}^2 + \widehat{M_n}^2 = 2(2^{2n}A^2 + B^2)$$

Proof.

$$\widehat{ML_n}^2 - \widehat{M_n}^2 = (2^n A + B)^2 - (2^n A - B)^2$$

$$= 2^{2n}A^2 + B^2 + 2^{n+1}AB - 2^{2n}A^2 - B^2 + 2^{n+1}AB$$

$$= 2^{n+2}AB$$

$$\widehat{ML_n}^2 + \widehat{M_n}^2 = (2^n A + B)^2 + (2^n A - B)^2$$

$$= 2^{2n}A^2 + B^2 + 2^{n+1}AB + 2^{2n}A^2 + B^2 - 2^{n+1}AB$$

$$= 2(2^{2n}A^2 + B^2)$$

Theorem 8. *For any positive integers n, r and s, we have*

$$\widehat{ML_{n+r}}\widehat{M_{n+s}} - \widehat{ML_{n+s}}\widehat{M_{n+r}} = 2^n(BA - AB)(2^s + 2^r)$$

Proof.

$$\widehat{ML_{n+r}}\widehat{M_{n+s}} - \widehat{ML_{n+s}}\widehat{M_{n+r}} = (2^{n+r}A + B)(2^{n+s}A - B)$$

$$- (2^{n+s}A + B)(2^{n+r}A - B)$$

$$= 2^{n+s}(BA - AB) + 2^{n+r}(BA - AB)$$

$$= 2^n(BA - AB)(2^s + 2^r)$$

Theorem 9. *For any positive integer's n and r, we have*

$$\widehat{M_{n+r}}M_{n+r} - \widehat{ML_{n-r}}M_{n-r} = 2^{2(n-r)}(2^{4r} - 1)A - 2^{n-r}(2^{2r} - 1)(A + B)$$

$$\widehat{M_{n+r}}ML_{n+r} - \widehat{M_{n-r}}ML_{n-r} = 2^{2(n-r)}(2^{4r} - 1)A - 2^{n+r}(2^{2r} + 1)(A - B) - 2B$$

$$\widehat{ML_{n+r}}ML_{n+r} - \widehat{ML_{n-r}}ML_{n-r} = 2^{2(n-r)}(2^{4r} + 1)A + 2^{n-r}(2^{2r} + 1)(A + B) + 2B$$

Proof. Using Binet formulas for Mersenne and Mersenne-Lucas sequences, and Binet formulas for Mersenne and Mersenne-Lucas Sedenions we obtain these relations.

References

[1] G. Bilgici, Fibonacci and Lucas sedenions, Journal of Integer Sequences 20(2) (2017), 3.

[2] A. Cariow and G. Cariowa, Rationalized algorithm for computing the product of two sedenions, Cent. Eur. J. Comp. Sci. 2(4) (2012), 389-397.

[3] R. E. Cawagas, On the structure and zero divisors of the Cayley-Dickson sedenion algebra, Discuss. Math. Gen. Algebra Appl. 24(2004), 251-265.

[4] A. Dasdemir and G. Bilgici, Gaussian Mersenne numbers and generalized Mersenne quaternions, Notes on Number Theory and Discrete Mathematics 25(3) (2019), 87-96.

[5] A. F. Horadem, Quaternion recurrence relations, Ulam Quarterly 2(2) (1993), 23-33.

[6] Koshy T. Fibonacci and Lucas, Numbers with Application, Wiley-Interscience Publication, New York, 2001.

[7] Y. Tasyurdu and A. Akpinar, Perrin octonions and Perrin sedenions, Konuralp Journal of Mathematics 8(2) (2020), 384-390.