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VIBRATION ANALYSIS OF CURVED BEAM USING FGM

MD RASHID AKHTAR and AAS MOHAMMAD

Department of Mechanical Engineering
Jamia Millia Islamia Delhi
New Delhi 110025, India

Abstract

FGM is an advanced composite material which is now extensively used in many engineering sector specially aviation because of high strength to weight ratio. If the combination of metal and ceramic is done then it shows high level of thermal properties which can be used at high temperature maintaining its strength. The FGM used for the present vibration analysis of the curved beam is a combination of Alumina (Al_2O_3) and Aluminium. It was analysed with different boundary conditions like simply supported, clamped-clamped and one end fixed other end free. The different values of volume fraction index were used for the analysis of the beam. The material properties vary in thickness direction and power law is used for the calculation of the effective property at any desired thickness form the mid plane.

The beam formulation is done using higher order shear deformation theory. The aspect ratio was kept fixed while changing the L/R ratio the variation in frequency was monitored. The main purpose for analysis the vibration characteristics of curved beam is to utilize the results for the better design of leaf spring using FGM.

The dependence of support conditions and radius of curvature on the frequency response is highlighted. Comparison and convergence study has been performed to the present formulation. The result and the analysis of the frequency of vibration can be used to optimize the frequency of leaf spring to have better ride quality.

Thus we observe that as R is increased keeping L constant the value of frequency of vibration decreases. It was also observed that as the boundary conditions are changed from clamped- clamped to simply-supported the frequency of vibration decreases for different values of volume fraction index.

1. Introduction

FGM materials are being tried in various fields of engineering applications because of its nice and adjustable properties according to the

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requirement. It is also being widely used in aviation because of good strength to weight ratio. In the present study the vibration analysis of curved beam is being done to have an idea of its vibration characteristics so that it can be used as a leaf spring in automobiles. As we all know that Leaf Spring is very important element of an automobile suspension system. It absorbs the vehicle vibration and gives the rider a comfortable ride. Leaf spring is one of those springs which are mainly used for heavy vehicles.

The advantage of using a leaf spring is that it can be guided along a definite path as it deflects to acts as structural member in addition to energy absorbing device.

Lots of efforts have been made by different people to reduce the vibration of leaf spring so that the ride could be made more comfortable. In this paper the analysis of master leaf was modeled as a curved beam and vibration was studied for FGM and for different L/R ratios with different boundary conditions. The aspect ratio (b/h) of the beam was kept fixed as 10.

V. Bheemreddyet.al [1] developed a mathematical model for glass/epoxy prepreg which simulated the resin flow, heat transfer, consolidation and curing of cavity molded flex beams. This would significantly improve the cost effectiveness.

Bhandari Manish et al. [2] did the static analysis of FGM plate using sigmoid law and validation was done with the published results. The variation in young's modulus, non dimensional deflection as well as non dimensional stress was also calculated and compared.

M. Mahmood et al. [3] did the analysis of composite leaf spring using ANSYS V 5.4 software and tried to optimize the weight of the spring. Abdul Rahim Abu Talib et al. [4] tried to analyse the elliptic spring made for both light and heavy trucks. They tried to optimize the spring parameters for different ellipticity ratios. Vinkel Arora et al. [5] did the fatigue life assessment of leaf spring of 65Si7 using analytical and graphical methods. They also did the life assessment using SAE design manual approach and it was compared with experimental results.

Mesut Simsek et al. [6] studied the vibration of straight beam within the frame work of third order shear deformation theory. They calculated the six dimensionless frequency parameters of different beams having different h/L ratios for different boundary conditions.

Ankit Gupta and Mohammad Talha [7] introduced the geometrically nonlinear vibrations response of FGM plates. They proposed displacement based new hyperbolic higher-order shear and normal deformation theory (HHSNDT). The performance of a curved beam with coupled polynomial distributions was investigated by P. Raveendranath et al. [8]. The result shows excellent convergence of natural frequencies even for very thin deep arches and higher vibrational modes.

M. Kawakami [9] presented an approximate method to study the analysis for both the in-plane and out-of-plane free vibration of horizontally curved beams with arbitrary shapes and variable cross-sections. S. M. Ibrahim et al. [10] investigated large amplitude periodic forced vibration of curved beams under periodic excitation using a three-noded beam element.

Mohammad Amir and Mohammad Talha [11] analyzed the thermoelastic vibration of shear deformable functionally graded material (FGM) curved beams with micro structural defects by the finite element method. The material properties of FGM beams are allowed to vary continuously in the thickness direction by a simple power-law distribution in terms of the volume fractions of the constituents. Mohammad Amir and Mohammad Talha [12] studied the imperfection sensitivity in the vibration behavior of functionally graded arches with micro structural defects (porosity).

2. Mathematical Formulation

2.1. Displacement Field:

The displacement field is expressed in terms of axial and transverse displacements of the mid-plane, along with the curvilinear coordinates x and y direction. The displacement field is based on traction-free conditions at the inner and outer surface of the curved beam14 is written as,

$$u(x, z, t) = u_0(x, t) + z\varnothing_x(x, t) - \frac{4z^2}{h^2}\left[\varnothing_x(x, t) + \frac{\partial w_0}{\partial x}(x, t)\right]$$

$$w(x, z, t) = w_0(x, t)$$

$$let\ \frac{\partial w_0}{\partial x} = \theta_x$$

$$u(x, z, t) = u_0(x, t) + z\varnothing_x(x, t) - \frac{4z^2}{3 * h^2}[\varnothing_x(x, t) + \theta_x(x, t)]$$

$$u(x, z, t) = u_0(x, t) + \varnothing_x\left(z - \frac{4z^2}{3h^2}\right) - \frac{4z^2}{3h^2}\theta_x(x, t)$$

$$w(x, z, t) = w_0(x, t).$$

Where, u and w represents the displacements of a point along the (x, z) coordinates. $u_0, w_0, \varnothing_x, \theta_x$ are four unknown displacement functions of mid-plane.

$$\begin{bmatrix} u(x, z, t) \\ w(x, z, t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \left(z - \frac{4z^3}{3h^2}\right) & -\frac{4z^3}{3h^2} \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \\ \varnothing_x \\ \theta_x \end{bmatrix}.$$

The basic field variables can be symbolized mathematically as, $\{\lambda_0\} = \{u_0, w_0, \varnothing_x, \theta_x\}^T$

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & \left(z - \frac{4z^3}{3h^2}\right) & -\frac{4z^3}{3h^2} \\ 0 & 1 & 0 & 0 \end{bmatrix}[\lambda_0].$$

The displacement field in the compressed form can be written as

$$[U] = [\breve{N}][\lambda_0].$$

Where, $[U] = \begin{bmatrix} u \\ w \end{bmatrix}$ and $[\breve{N}] = \begin{bmatrix} 1 & 0 & \left(z - \frac{4z^3}{3h^2}\right) & -\frac{4z^3}{3h^2} \\ 0 & 1 & 0 & 0 \end{bmatrix}.$

2.2. Strain-displacement relations:

$$\varepsilon_{xx} = 1/\left(1+\frac{z}{R}\right)\left(\frac{\partial u}{\partial x}+\frac{w}{R}\right).$$

Using linear strain displacement relation the strain vectors may be defined as

$$\varepsilon_{xx} = 1/\left(1+\frac{z}{R}\right)\left\{\frac{\partial u_0}{\partial x}+z\,\frac{\partial \varnothing_x}{\partial x}-\frac{4z^2}{h^2}\left[\frac{\partial \varnothing_x}{\partial x}+\frac{\partial \theta_x}{\partial x}\right]+\frac{w_0}{R}\right\}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z}+1/\left(1+\frac{z}{R}\right)\left(\frac{\partial w}{\partial x}+\frac{u}{R}\right)$$

$$=\left(\frac{\partial u_0}{\partial x}+\varnothing_x-\frac{4z^2}{h^2}[\varnothing_x+\theta_x]\right)+1/\left(1+\frac{z}{R}\right)\left\{\frac{\partial w_0}{\partial x}-\frac{u_0}{R}-\frac{z}{R}(\varnothing_x)-\frac{4z^2}{3h^2}[\varnothing_x+\theta_x]\right\}.$$

As u_0 is a function of x and t only i.e. $\frac{\partial u_0}{\partial z} = 0.$

$$\gamma_{xz} = 1/\left(1+\frac{z}{R}\right)\left\{\left(1+\frac{z}{R}\right)\left(\varnothing_x-\frac{4z^2}{h^2}[\varnothing_x+\theta_x]\right)+\frac{\partial w_0}{\partial x}-\frac{u_0}{R}-\frac{z}{R}(\varnothing_x)\right.$$

$$\left.-\frac{4z^3}{3*R*h^2}[\varnothing_x+\theta_x]\right\}$$

$$\gamma_{xz} = 1/\left(1+\frac{z}{R}\right)\left\{\left(\varnothing_x+\frac{\partial w_0}{\partial x}-\frac{u_0}{R}\right)-\frac{4z^2}{h^2}[\varnothing_x+\theta_x]-\frac{8z^3}{3*R*h^2}[\varnothing_x+\theta_x]\right\}$$

$$\begin{bmatrix}\varepsilon_{xx}\\ \gamma_{xz}\end{bmatrix} = 1/\left(1+\frac{Z}{R}\right)$$

$$\left(\begin{bmatrix}\frac{\partial u_0}{\partial x}+\frac{w_0}{R}\\ \varnothing_x+\frac{\partial w_0}{\partial x}-\frac{u_0}{R}\end{bmatrix}+z\begin{bmatrix}\frac{\partial \varnothing_x}{\partial x}\\ 0\end{bmatrix}+z^2\begin{bmatrix}0\\ -\frac{4}{h^2}[\varnothing_x+\theta_x]\end{bmatrix}+z^3\begin{bmatrix}-\frac{4}{3h^2}\left(\frac{\partial \varnothing_x}{\partial x}+\frac{\partial \theta_x}{\partial x}\right)\\ -\frac{8}{3R*h^2}[\varnothing_x+\theta_x]\end{bmatrix}\right)$$

$$\begin{bmatrix}\varepsilon_{xx}\\ \gamma_{xz}\end{bmatrix}=\left(1/\left(1+\frac{z}{R}\right)\right)\left(\begin{bmatrix}\epsilon_1^0\\ \epsilon_2^0\end{bmatrix}+z\begin{bmatrix}k_1^1\\ 0\end{bmatrix}+z^2\begin{bmatrix}0\\ k_2^2\end{bmatrix}+z^3\begin{bmatrix}k_1^3\\ k_2^3\end{bmatrix}\right)$$

$$\begin{bmatrix}\varepsilon_{xx}\\ \gamma_{xz}\end{bmatrix} = 1/\left(1+\frac{z}{R}\right)\begin{bmatrix}1 & 0 & z & 0 & 0 & z^3 & 0\\ 0 & 1 & 0 & z & z^2 & 0 & z^3\end{bmatrix}\begin{bmatrix}\epsilon_1^0\\ \epsilon_2^0\\ k_1^1\\ k_2^2\\ k_1^3\\ k_2^3\end{bmatrix}$$

$$[T] = 1/\left(1+\frac{z}{R}\right)\begin{bmatrix}1 & 0 & z & 0 & 0 & z^3 & 0\\ 0 & 1 & 0 & z & z^2 & 0 & z^3\end{bmatrix}$$

where $\varepsilon_1^0 = \frac{\partial u_0}{\partial x} + w_0/R$

$$\varepsilon_2^0 = \varnothing_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R}$$

$$k_1^1 = \frac{\partial \varnothing_x}{\partial x}$$

$$k_2^2 = -\frac{4}{h^2}[\varnothing_x + \theta_x]$$

$$k_1^3 = -\frac{4}{3h^2}\left(\frac{\partial \varnothing_x}{\partial x} + \frac{\partial \theta_x}{\partial x}\right)$$

$$k_2^3 = -\frac{8}{3R * h^2}[\varnothing_x + \theta_x]$$

$$\begin{bmatrix}\varepsilon_1^0\\ \varepsilon_2^0\\ k_1^1\\ k_2^2\\ k_1^3\\ k_2^3\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x} & \frac{1}{R} & 0 & 0\\ -\frac{1}{R} & \frac{\partial}{\partial x} & 1 & 0\\ 0 & 0 & \frac{\partial}{\partial x} & 0\\ 0 & 0 & -\frac{4}{h^2} & -\frac{4}{h^2}\\ 0 & 0 & \frac{-4}{3h^2} & \frac{-4}{3h^2}\\ 0 & 0 & \frac{-8}{3 * R * h^2} & \frac{-8}{3 * R * h^2}\end{bmatrix}\begin{bmatrix}u_0\\ w_0\\ \varnothing_x\\ \theta_x\end{bmatrix}$$

$$\{\varepsilon^0\} = [B][\lambda_0]$$

$$\text{Where, } B = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{1}{R} & 0 & 0 \\ -\frac{1}{R} & \frac{\partial}{\partial x} & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & -\frac{4}{h^2} & -\frac{4}{h^2} \\ 0 & 0 & \frac{-4}{3h^2} & \frac{-4}{3h^2} \\ 0 & 0 & \frac{-8}{3*R*h^2} & \frac{-8}{3*R*h^2} \end{bmatrix} \text{ and } \lambda_0 = \begin{bmatrix} u_0 \\ w_0 \\ \varnothing_x \\ \theta_x \end{bmatrix}$$

2.3. The linear constitutive relations of an isotropic beam is given as:

$$\begin{Bmatrix} \sigma_{xx} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix}$$

Where $Q_{11} = E/(mu^2)$ and $Q_{44} = E/(2^*(1+mu))$.

$$\begin{Bmatrix} \sigma_{xx} \\ \tau_{xz} \end{Bmatrix} = QT\varepsilon^0.$$

3. Finite Element Formulation

3.1. Formulation

A two noded element with four degrees of freedom per node is being utilized to discretize the beam geometry. The generalized displacement vector of the model is expressed by

$$\begin{bmatrix} u_0 \\ w_0 \\ \varnothing_x \\ \theta_x \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_1 & N_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_0^1 \\ u_0^2 \\ w_0^1 \\ w_0^2 \\ \phi_x^1 \\ \phi_x^2 \\ \theta_x^1 \\ \theta_x^2 \end{bmatrix}.$$

The shape functions N_1 and N_2 are as follows

$$N_1 = 1 - x/Le$$

$$N_2 = x/Le.$$

3.2. Strain Energy of the curved beam

$$\Pi^e = \frac{1}{2}\int \varepsilon_i^T \sigma_i dv$$

$$= 1/2\int \varepsilon^{0T} T^T QT\varepsilon^0 dv.$$

3.3. Kinetic energy of beam

$$\Delta = 1/2\int \rho U'^T U'^T dV$$

$$\Delta = 1/2\int \rho\lambda_0'^T \breve{N}^T \breve{N}\lambda_0' dV.$$

3.4. Governing equation

Free vibration analysis of curved beam has been obtained using the variational principle, which is the abstraction of the principle of virtual displacement.

$$\frac{\partial E}{\partial x} = 0 = \int \varepsilon^T \sigma dv + \int \rho U'^T U' dV$$

$$= b\int_0^L \int_{-h/2}^{h/2} \varepsilon^{0T} T^T QT\varepsilon^0 dzdx + \int_0^L \int_{-h/2}^{h/2} \rho\lambda_0'^T \breve{N}^T \breve{N}\lambda_0' dzdx$$

$$= b\int_0^L \varepsilon^{0T}[D]\varepsilon^\circ dx + b\int_0^L \lambda_0'^T [m]\lambda_0' dx.$$

Where, $D = \int_{-h/2}^{h/2} T^T QTdz$ and $m = \int_{-h/2}^{h/2} \breve{N}^T \breve{N}dz$

$$= b\int_0^L \lambda_0^T B^T DB\lambda_0 dx + b\int_0^L \lambda_0'^T [m]\lambda_0' dxX$$

$$= KU + M\ddot{U}$$

$$KU = \lambda MU, \text{ where } \lambda = \omega^2.$$

Where λ is the eigenvalue and U is the global displacement vector.

4. Result and Discussion

The frequency of vibration of curved FGM beam was found for two different support conditions using MATLAB program. The support conditions are clamped-clamped simply supported and Fixed-free. The convergence table is also included in the paper. The specification of the beam used was as follows:

Material Properties of FGM constituents:

Material	Young's Modulus(N/m^2)	Density (kg/m3)	Poisson's ratio
Alumina (Al2O3)	380*(10^9)	3960	0.30
Aluminium (Al)	70*(10^9)	2702	0.30

Case 1. Depth $h = 0.002m$,

Case 2. Depth $h = 0.002m$, Width $b = 0.02m$; $R = 8, 12m$. Length $a = 0.5m$, Support condition: Simply-Supported Support condition: Clamped-Clamped

Case 1:

Table 1. Convergence Table for Clamped-Clamped Condition, $R = 8$.

No. of elements	Natural frequency of vibration with different values of volume fraction index, n			
	n=1	n=2	n=4	n=10
50	1786	1.57E+03	1.38E+03	1.19E+03
100	1292	1.13E+03	999.1496	873
200	1131	993	877.4593	770
400	1083	952	842	741
600	1074	943	835	735.5
800	1070	940	833	733.5
900	1070	939.96	832	732.9

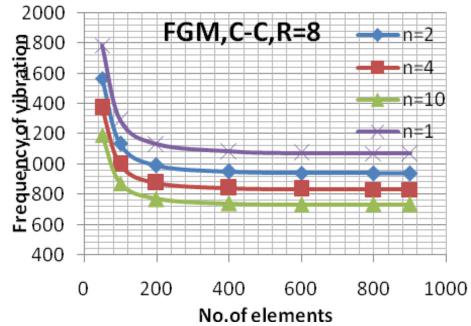

Figure 1. Vibration for Clamped-Camped condition.

Table 2. Convergence Table for Clamped-Clamped Condition, $R = 12$.

No. of elements	Natural frequency of vibration with diff. values of n			
	n=1	n=2	n=4	n=10
50	1.63E+03	1.43E+03	1.25E+03	1.09E+03
100	1.07E+03	9.37E+02	828	727
200	8.70E+02	764	680	603
400	812	714	637	567
600	801	704	629	560
800	797.41	701	626	558
900	796.34	700	625	557.6

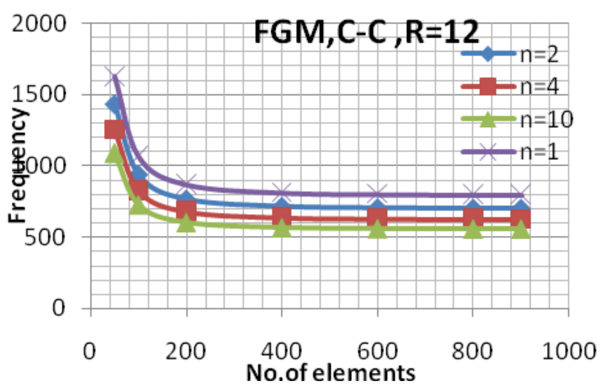

Figure 2. Vibration for Clamped-Camped.

Table 3. Convergence Table for Simply Supported Condition, $R = 8$.

No. of elements	Natural frequency of vibration with different values of volume fraction index ,n			
	n=1	n=2	n=4	n=10
50	1.31E+03	1.17E+03	1.03E+03	874.13
100	1.18E+03	1.05E+03	921.2	787.4
200	1.01E+03	891.12	811.2	741.67
300	897.08	791.31	728.17	674.50
400	853.50	753.27	696.7	649
500	832.56	734.9	681.7	637
600	820.96	724.87	673.4	630
700	813.8	718.70	668.38	626.77
800	809.26	714.67	665.08	624.15
900	806	711.89	662.81	622.35

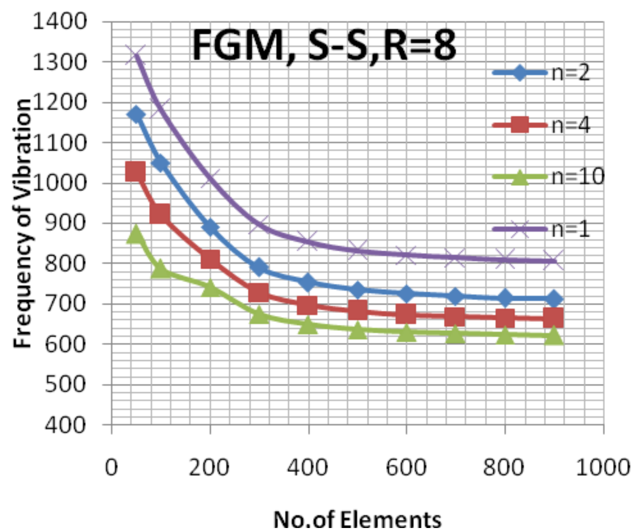

Figure 3. Vibration for Simply- Supported.

Table 4. Convergence Table for Simply Supported Condition, $R = 12$.

No. of elements	Natural frequency of vibration with different values of n			
	n=1	n=2	n=4	n=10
50	1.03E+03	9.14E+02	8.03E+02	683.5005
100	8.62E+02	7.72E+02	679	577.7378
200	8.06E+02	719	634	542.9083
400	784	697	616	531
600	779	692	612	528
800	777	690	611	527.9459
900	777.08	689	611.0052	527.69

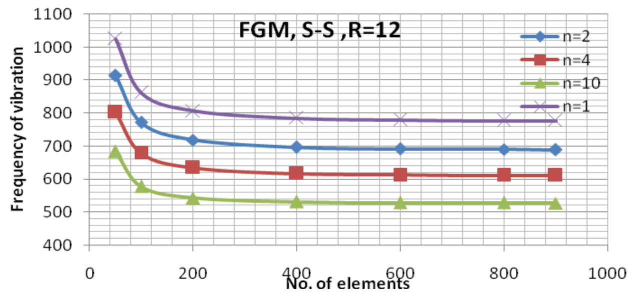

Figure 4. (Convergence graph) Vibration for Simply Supported condition, $R = 12$.

Conclusion

The vibration analysis of the FGM curved beam was done for different values of radius of curvature as it is very important aspect of spring design. MATLAB was used for calculating the frequency of vibration. The effect of end condition on the vibration of leaf spring was studied for $R = 8, 12$m. During the analysis it was found that as the value of volume fraction index (n) is increased the value of frequency of vibration in both the clamped-clamped condition and simply supported condition decreases. Also it was observed that if the Radius of curvature is increased the value of frequency of vibration reduces for same boundary condition but for clamped-clamped condition the frequency decreases much rapidly compared to simply supported condition. Also the weight of the beam is reduced with higher values of volume fraction index. Optimizing these parameters can give a better design options for leaf springs as well.

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