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PROPERTIES OF OSCILLATORY MEAN INVOLVING POWER EXPONENTIAL MEAN AND POWER MEAN

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Abstract

The oscillatory mean involving power exponential mean and power mean and its dual are introduced. Further, different kinds of Schur convexities are discussed.

1. Introduction

For h, k are positive real numbers, then

$$A(h, k) = \frac{h+k}{2},\ G(h, k) = \sqrt{hk},\ H(h, k) = \frac{2hk}{h+k} \text{ and } F_1 = (h^h k^k)^{\frac{1}{h+k}}$$

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are respectively called arithmetic, geometric, harmonic and power exponential mean. Lokesha et al. introduced the oscillatory mean and r^{th} oscillatory mean in [3, 4]. Further studied the various remarkable inequalities involving power mean, logarithmic mean and identric mean. Also obtained the best possible values. Results on convexities and Schur conditions are found in ([1, 2, 17, 19, 20, 21], [6]-[9]) and studies on power type means in ([10]-[17], [22, 23]).

For $h > k > 0, r$ be a real number, $\beta \in (0, 1)$ then, oscillatory mean involving power exponential mean and power mean is given by;

$$O_{pemp} = O_{pemp}(h, k, \beta, r) = \beta(h^h k^k)^{\frac{1}{h+k}} + (1-\beta)\left(\frac{h^r + k^r}{2}\right)^{\frac{1}{r}} \tag{1.1}$$

and its dual is given by

$$O_{pemp}^{(d)} = O_{pemp}^{(d)}(h, k, \beta, r) = (h^h k^k)^{\frac{\beta}{h+k}} * \left(\frac{h^r + k^r}{2}\right)^{\frac{(1-\beta)}{r}}. \tag{1.2}$$

When $r = 1$, the oscillatory mean involving power exponential mean and arithmetic mean is given by;

$$O_{pemp} = O_{pemp}(h, k, \beta, 1) = \beta(h^h k^k)^{\frac{1}{h+k}} + (1-\beta)\left(\frac{h+k}{2}\right) \tag{1.3}$$

and its dual is given by

$$O_{pemp}^{(d)} = O_{pemp}^{(d)}(h, k, \beta, r) = (h^h k^k)^{\frac{\beta}{h+k}} * \left(\frac{h+k}{2}\right)^{1-\beta}. \tag{1.4}$$

Various researchers have studied several homogeneous functions and obtained some identities involving means and established remarkable mean inequalities.

Lemma 1.1[22]**.** *With usual notations*, *recall*

$$(h-k)\left(\frac{\partial\varphi}{\partial h} - \frac{\partial\varphi}{\partial k}\right) \geq 0\,(\leq 0) \tag{1.5}$$

$$(\ln h - \ln k)\left(h\frac{\partial\varphi}{\partial h} - k\frac{\partial\varphi}{\partial k}\right) \geq 0\ (\leq 0) \tag{1.6}$$

$$(h - k)\left(h^2\frac{\partial\varphi}{\partial h} - k^2\frac{\partial\varphi}{\partial k}\right) \geq 0\ (\leq 0) \tag{1.7}$$

are respectively called the Schur, Schur geometric and harmonic convex(concave) conditions.

2. Results

In this section, Schur convexities of the r^{th} oscillatory mean involving power exponential mean and power mean and its dual are discussed. Further, declared results when $r = 1$ for oscillatory mean involving power exponential mean and arithmetic mean.

Theorem 2.1. *For* $h > k > 0$, r *be a real number*, $\beta \in (0, 1)$ *then*, r^{th} *oscillatory mean* $O_{pemp}(h, k, \beta, r)$ *and its dual* $O_{pemp}^{(d)}(h, k, \beta, r)$ *are Schur convex*, *if* $r \geq 1$.

Proof. From the definition, the r^{th} oscillatory mean of power exponential mean and power mean is given by;

$$O_{pemp} = O_{pemp}(h, k, \beta, r) = \beta(h^h k^k)^{\frac{1}{h+k}} + (1 - \beta)\left(\frac{h^r + k^k}{2}\right)^{\frac{1}{r}} \tag{2.1}$$

By finding the partial derivatives w.r.t h and k, gives

$$\frac{\partial O_{pemp}}{\partial h} = \frac{\beta F_1}{(h + k)^2}\left[h + k + h\ln\frac{h}{k}\right] + \frac{(1 - \beta)h^{r-1}}{2}\left(\frac{h^h + k^k}{2}\right)^{\frac{1}{r}-1} \tag{2.2}$$

$$\frac{\partial O_{pemp}}{\partial k} = \frac{\beta F_1}{(h + k)^2}\left[h + k + h\ln\frac{k}{h}\right] + \frac{(1 - \beta)h^{r-1}}{2}\left(\frac{h^h + k^k}{2}\right)^{\frac{1}{r}-1} \tag{2.3}$$

Then for $r > 1$, $(h - k)\left(\frac{\partial O_{pemp}}{\partial h} - \frac{\partial O_{pemp}}{\partial k}\right)$

$$= \frac{\beta F_1}{(h+k)^2}[h(\ln h - \ln k)(h-k)] + \left(\frac{1-\beta}{2}\right)\left(\frac{h^r + k^r}{2}\right)^{\frac{1}{r}-1}$$

$$(h^{r-1} - k^{r-1})(h-k) > 0$$

If $r = 1$, $(h-k)\left(\frac{\partial O_{pemp}}{\partial h} - \frac{\partial O_{pemp}}{\partial k}\right) = \frac{\beta F_1}{(h+k)^2}[h(\ln h - \ln k)(h-k)] > 0$

Thus Schur convexity condition holds for $r \geq 1$. From the definition, the dual r^{th} oscillatory mean of power exponential mean and power mean is given by;

$$O_{pemp}^{(d)} = O_{pemp}^{(d)}(h, k, \beta, r) = (h^h k^k)^{\frac{\beta}{h+k}} * \left(\frac{h+k}{2}\right)^{1-\beta}. \tag{2.4}$$

Take log on both sides gives

$$\ln(O_{pemp}^{(d)}) = \left(\frac{\beta}{h+k}\right)(h \ln h + k \ln k) + \left(\frac{1-\beta}{r}\right)(\ln(h^r + k^k) - \ln 2) \tag{2.5}$$

By finding the partial derivatives w.r.t h and k, gives

$$\frac{\partial O_{pemp}^{(d)}}{\partial h} = O_{pemp}^{(d)}\left[\frac{\beta}{(h+k)^2}\{h + k + k(\ln h - \ln k)\} + (1-\beta)h^{r-1}\right] \tag{2.6}$$

$$\frac{\partial O_{pemp}^{(d)}}{\partial k} = O_{pemp}^{(d)}\left[\frac{\beta}{(h+k)^2}\{h + k + h(\ln h - \ln k)\} + (1-\beta)k^{r-1}\right] \tag{2.7}$$

Then for $r > 1$

$$(h-k)\left(\frac{\partial O_{pemp}^{(d)}}{\partial h} - \frac{\partial O_{pemp}^{(d)}}{\partial k}\right)$$

$$= O_{pemp}^{(d)}\left[\frac{\beta(h-k)(\ln h - \ln k)}{(h+k)} + (1-\beta)(h-k)(h^{r-1} - k^{r-1})\right] > 0$$

If $r = 1$, $(h-k)\left(\frac{\partial O_{pema}}{\partial h} - \frac{\partial O_{pema}}{\partial k}\right) = \left(\frac{h+k}{2}\right)\left[\frac{\beta(h-k)(\ln h - \ln k)}{(h+k)}\right] > 0$

Thus Schur convexity condition holds for $r \geq 1$. Hence the proof of theorem 2.1.

Theorem 2.2. *For* $h > k > 0, r$ *be a real number,* $\beta \in (0, 1)$ *then,* r^{th} *oscillatory mean* $O_{pemp}(h, k, \beta, r)$ *and its dual* $O^{(d)}_{pemp}(h, k, \beta, r)$ *are geometric-Schur convex, if* $r \geq 1$.

Proof. From equations (2.1), (2.2) and (2.3), for $r > 1$, leads to

$$(\ln h - \ln k)\left(h\frac{\partial O_{pemp}}{\partial h} - k\frac{\partial O_{pemp}}{\partial k}\right) = \frac{\beta(h^h k^k)^{\frac{1}{h+k}}}{(h+k)^2}[(\ln h - \ln k)(h^2 - k^2)$$

$$+ hk(\ln h - \ln k)^2] + \left(\frac{1-\beta}{2}\right)\left(\frac{h^r + k^r}{2}\right)^{\frac{1}{r}-1}(h^r - k^r)(\ln h - \ln k) > 0.$$

If $r = 1$,

$$(\ln h - \ln k)\left(h\frac{\partial O_{pemp}}{\partial h} - k\frac{\partial O_{pemp}}{\partial k}\right)$$

$$= \frac{\beta(h^h k^k)^{\frac{1}{h+k}}}{(h+k)^2}[(\ln h - \ln k)(h^2 - k^2) + hk(\ln h - \ln k)^2].$$

Thus Schur geometric convexity condition holds for $r \geq 1$.

From equations (2.4), (2.5), (2.6) and (2.7) for $r > 1$, leads to

$$(\ln h - \ln k)\left(h\frac{\partial O^{(d)}_{pemp}}{\partial h} - k\frac{\partial O^{(d)}_{pemp}}{\partial k}\right)$$

$$= O^{(d)}_{pemp}[\frac{\beta(h^2 - k^2)(\ln h - \ln k) + 2hk(\ln h - \ln k)^2}{(h+k)}$$

$$+ (1-\beta)(\ln h - \ln k)(h^r - k^r)] > 0.$$

If $r = 1$,

$$(\ln h - \ln k)\left(h\frac{\partial O^{(d)}_{pemp}}{\partial h} - k\frac{\partial O^{(d)}_{pemp}}{\partial k}\right)$$

$$= O^{(d)}_{pemp}\left[\frac{\beta(h^2 - k^2)(\ln h - \ln k) + 2hk(\ln h - \ln k)^2}{(h+k)^2}\right] > 0.$$

Thus geometric-Schur convexity condition holds for $r \geq 1$. Hence the proof of theorem 2.2.

Theorem 2.3. *For* $h > k > 0$, r *be a real number*, $\beta \in (0, 1)$ *then*, r^{th} *oscillatory mean* $O_{pemp}(h, k, \beta, r)$ *and its dual* $O_{pemp}^{(d)}(h, k, \beta, r)$ *are harmonic-Schur convex*, *if* $r \geq 1$.

Proof. From equations (2.1), (2.2) and (2.3), for $r > 1$, leads to

$$(h-k)\left(h^2 \frac{\partial O_{pemp}}{\partial h} - k^2 \frac{\partial O_{pemp}}{\partial k}\right)$$

$$= \frac{\beta(h^h k^k)^{\frac{1}{h+k}}}{(h+k)^2}[(h-k)^2(h+k)^2 + hk^2(h-k)(h-k)(\ln h - \ln k)$$

$$+ h^2 2k(h-k)(\ln h - \ln k) + 2k^3(k-h)\ln k]$$

$$+ \left(\frac{1-\beta}{2}\right)\left(\frac{h^r + k^r}{2}\right)^{\frac{1}{r}-1}(h-k)(h^{r+1} - k^{r+1}) > 0.$$

If $r = 1$, $(h-k)\left(h^2 \frac{\partial O_{pemp}}{\partial h} - k^2 \frac{\partial O_{pemp}}{\partial k}\right)$

$$= \frac{\beta(h^h k^k)^{\frac{1}{h+k}}}{(h+k)^2}[(h-k)^2(h+k)^2 + hk^2(h-k)(h-k)(\ln h - \ln k)$$

$$+ h^2 2k(h-k)(\ln h - \ln k) + 2k^3(k-h)\ln k]$$

$$+ \left(\frac{1-\beta}{2}\right)(h-k)(h^2 - k^2) > 0.$$

Thus Schur harmonic convexity condition holds for $r \geq 1$.

From equations (2.4), (2.5), (2.6) and (2.7) for $r > 1$, leads to

$$(h-k)\left(h^2 \frac{\partial O_{pemp}^{(d)}}{\partial h} - k^2 \frac{\partial O_{pemp}^{(d)}}{\partial k}\right)$$

$$= O_{pemp}^{(d)}\Big[\frac{\beta\{(h-k)^2(h+k)^2 + hk(h^2-k^2)(\ln h - \ln k)\}}{(h+k)^2}$$

$$+ (1-\beta)(h-k)(h^{r+1} - k^{r+1})\Big] > 0.$$

If $r = 1$, $(h-k)\left(h^2 \frac{\partial O_{pemp}^{(d)}}{\partial h} - k^2 \frac{\partial O_{pemp}^{(d)}}{\partial k}\right)$

$$= O_{pemp}^{(d)}\Big[\frac{\beta\{(h-k)^2(h+k)^2 + hk(h^2-k^2)(\ln h - \ln k)\}}{(h+k)^2}$$

$$+ (1-\beta)(h-k)(h^2 - k^2)\Big] > 0.$$

Thus harmonic-Schur convexity condition holds for $r \geq 1$. Hence the proof of theorem 2.3.

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