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2-VERTEX SWITCHING OF CONNECTED UNICYCLIC JOINTS IN GRAPHS

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Abstract

A graph $G'(V, E')$ is created from G by eliminating all edges between s and its complement $V - s$ and any non-edges between s and $V - s$ are added as edges for a simple graph $G(V, E)$ and a non empty subset $s \subset V$. We write G^v for $G\{v\}$ when $s = v$, and the associated switching is referred to as vertex switching. $|S|$-vertex switching is another name for it. 2-vertex switching occurs when $|S|$ equals 2. If B is connected and maximal, a joint at σ in G is a subgraph of G that includes $G[\sigma]$. If B is connected, we refer to it as a $\hat{c}$-joint, otherwise, we refer to it as a d-joint. An acyclic graph is one that has no cycles. The term "tree" refers to a linked acyclic network. In this article, for a graph G, we provide necessary and sufficient criteria for G^{σ}, the switching of G at $\sigma = \{m, n\}$ to be connected and unicyclic graph when $mn \in E(G)$ and $mn \notin E(G)$.

1. Introduction

For any graph $G(V, E)$ with $|V(G)| = p$, the graph $G'(V, E')$ is defined

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as the graph generated from G by deleting all edges between σ and its counterpart, $V - \sigma$, and any non-edges between σ and $V - \sigma$ are added as edges where $\sigma \subseteq V$. Seidel [1, 8] defined switching, which is also known as $|\sigma|$-vertex switching [9, 11]. When $|\sigma| = 2$, it is called as 2-vertex switching. Highly irregular graphs and its chromatic number are studied in [16]. Harge discussed in detail about switching of a vertices in a graph in [2, 4] A graph which contains exactly one cycle is called an unicylic graph. In [6, 13] the concept of self vertex switchings were studied. A survey in two graphs and reconstruction of graphs were studied in [12, 14]. Switching classes and Euler graphs were discussed in [10].

In 2008, the concept of branches and joints in graphs were introduced by Vilfred V. et al., [10]. A joint at σ in G is a subgraph B of G that includes $G[\sigma]$ if $B - \sigma$ is connected and maximum. If B is connected, we refer to it as a $\hat{c}$-joint, otherwise, we refer to it as a d-joint. B is a total joint if $B = \sigma + (B - \sigma)$. In [3] graphs were characterized for self vertex switching of trees. In [3, 15] C. Jayasekaran, et al., analysed the graphs for 2-vertex switching of joints and characterized trees for 2-vertex self switching in [7]. For standard symbols and definitions we refer F. Harary [3].

For the graph G in Figure 1.1, G^{σ}, $G[\sigma]$ and $G - \sigma$ is shown in Figures 1.2 to 1.4 respectively, where $\sigma = \{m, n\}$. Figures 1.5, 1.6 and 1.7 shows the $\hat{c}$-joint, d-joint and the total joint respectively.

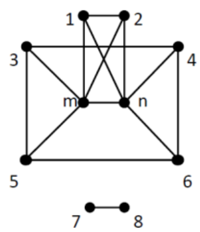

Figure. 1.1. G

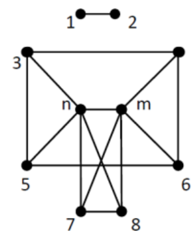

Figure. 1.2. G^{σ}

Figure. 1.3. $G[\sigma]$

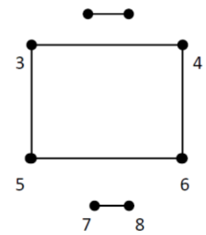

Figure. 1.4. $G - \sigma$

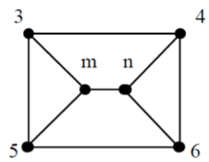

Figure 1.5. $\hat{c}$**-joint**

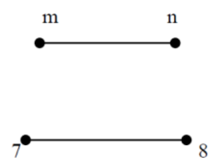

Figure 1.6. $\hat{d}$**-joint**

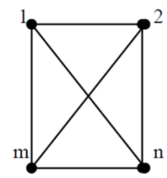

Figure 1.7. Total joint

When the transformer circuit faces an problem during the supply from one station to another the concept of two vertex switching is used to minimize the time for the power cut by switching the transformer circuit from one node to another which is one of the major application.

Consider the following outcomes, since they will be needed in the following sections.

Theorem 1.1 [5]. *If* G *is of order* $p \geq 3$ *and let* $\sigma = \{m, n\} \subseteq V(G)$ *and* $mn \notin E(G)$. *If* B *is a* $\hat{c}$*-joint at* σ *in* G, *then* B^{σ} *is a* $\hat{c}$*-joint at* σ *in* G^{σ} *iff* $B - \sigma$ *is connected,* $0 < d_B(m) \leq |V(B)| - 3$ *and* $0 < d_B(n) \leq |V(B)| - 3$.

Theorem 1.2 [5]. *If* G *is of order* $p \geq 3$ *and let* $\sigma = \{m, n\} \subseteq V(G)$ *and* $mn \notin E(G)$. *If* B *is a d-joint at* σ *in* G, *then* B^{σ} *is a* $\hat{c}$*-joint at* σ *in* G^{σ} *iff* $B - \sigma$ *is connected and either* $d_B(m) = 0$ *and* $0 \leq d_B(m) \leq |V(B)| - 3$ *or* $d_B(n) = 0$ *and* $0 \leq d_B(n) \leq |V(B)| - 3$.

Theorem 1.3 [5]. *If* G *is of order* $p \geq 3$ *and let* $\sigma = \{m, n\} \subseteq V(G)$ *and* $mn \in E(G)$. *If* B *is a* $\hat{c}$*-joint at* σ *in* G, *then* B^{σ} *is a* $\hat{c}$*-joint if and only* $B - \sigma$ *is connected and either* $0 < d_B(m) \leq |V(B)| - 2$ *or* $0 < d_B(n) \leq |V(B)| - 2$.

Theorem 1.4 [5]. *If* G *is of order* $p \geq 3$ *and let* $\sigma = \{m, n\} \subseteq V(G)$ *and*

$mn \in E(G)$. If B is a $\hat{c}$-joint at σ in G, then B^{σ} is a d-joint at σ in G_{σ} iff $B-\sigma$ is connected and $d_B(m) = d_B(n) = |V(B)|-1$.

Theorem 1.5 [5]. *If G is of order $p \geq 3$ and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \in E(G)$. If B is a d-joint at σ is G, then iff $B-\sigma$ is connected and $d_B(m) = d_B(n) = 1$.*

2. Main Results

2. 2-Vertex Switching of Connected Unicyclic Graphs

We present necessary and sufficient requirements for a graph G in this study, for which G^{σ} at $\sigma = \{m, n\}$ to be connected and unicyclic graph when $mn \in E(G)$ and $mn \notin E(G)$.

We use this to describe two vertex switching of unicyclic graphs that are connected.

Theorem 2.1. *For a graph G of order $p \geq 5$ and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \notin E(G)$. If B is a $\hat{c}$-joint at σ in G, then B^{σ} is a $\hat{c}$-joint and unicyclic iff $|V(B)| \geq 5$ and one of the following holds*:

(i) *$B-\sigma$ is connected, acyclic and $\{d_B(m), d_B(n)\} = \{|V(B)|-4, |V(B)|-3\}$.*

(ii) *$B-\sigma$ is connected, unicyclic and $d_B(m) = d_B(n) = |V(B)|-3$.*

Proof. Let B be a $\hat{c}$-joint at σ in G such that B^{σ} is a $\hat{c}$-joint and unicyclic. By Theorem 1.1 we have, $B-\sigma$ is connected, $0 < d_B(n) \leq |V(B)|-3$ and $0 < d_B(m) \leq |V(B)|-3$. Since B^{σ} is unicyclic and $B-\sigma$ is either acyclic or unicyclic.

Case 1. $B-\sigma$ is acyclic.

If $d_B(m) < |V(B)|-4$, then there exist at least three vertices a, b, c in $V(B)-\sigma$ which are not-adjoint to m in B which implies m is adjoint to a, b and C in B^{σ}. Since $B-\sigma$ is connected, there exist $a-b$, $b-c$ and $a-c$

paths in B and hence in B^{σ}. Now the edges am, bm and cm and the paths $a-b$, $b-c$ and $a-c$ form at least three cycles in B^{σ}, which is a contradiction to B^{σ} is unicyclic. Hence either $d_B(m) = |V(B)| - 4$ or $d_B(m) = |V(B)| - 3$.

Subcase 1.a. $d_B(m) = |V(B)| - 4$.

Since $mn \notin E(G)$, there exist two vertices a, b in $V(B) - \sigma$ such that a and b are not-adjoint to m in B. Implying that m is adjoint to a and b in B^{σ}. Since $B - \sigma$ is connected, there exists an $a-b$ path in B^{σ}. Now, the edges am, bm and the path $a-b$ form a cycle C_1 in B^{σ}.

If $d_B(n) < |V(B)| - 3$, there exist at least two vertices, x, y in $V(B) - \sigma$, which are not-adjoint to n in B. Implying that n is adjoint to both x and y in B^{σ}.

If $\{a, b\} = \{x, y\}$, then the edges $nx = na$, $ny = bn$ and the path $a-b$ form a cycle C_2 in B^{σ} different from C_1.

If $\{a, b\} \neq \{x, y\}$, then the $x-y$ path in B^{σ} and the edges nx and ny form a cycle C_3 in B^{σ} different from C_1.

If $a = x$ and $b \neq y$, then the $x-y$ path in B^{σ}, and the edges $na = nx$ and ny form a cycle C_4 in B^{σ} different from C_1.

Hence in all three cases, we get a cycle in addition to C_1 in B^{σ} which is a contradiction to B^{σ} is unicyclic. Hence $d_B(n) = |V(B)| - 3$.

Subcase 1.b. $d_B(m) = |V(B)| - 3$.

Since $mn \notin E(G)$, there is only one vertex in $V(B) - \sigma$ say a, which is not adjoint to m in B. As a result, a is adjoint to m in B^{σ} and hence ma is an edge in B^{σ}.

Now, $0 < d_B(n) \leq |V(B)| - 3.$ We can show that either $d_B(n) = |V(B)| - 3$ or $d_B(n) = |V(B)| - 4$ by using a similar argument as in

Case 1. If $d_B(n) = |V(B)| - 3,$ there exists only one vertex, b, in $V(B) - \sigma$ and b is not-adjoint to n in B. Implying bn is an edge in B^{σ}. Since $B - \sigma$ is acyclic and $mn \notin E(G)$, nb and am do not form a cycle in B^{σ} and hence we have B^{σ} is acyclic which is a contradiction to B^{σ} is unicyclic. Hence $d_B(n) = |V(B)| - 4$.

Case 2. $B - \sigma$ is unicyclic.

Let C_1 be the only cycle in $B - \sigma$ in G. Then C_1 is also a cycle of $B - \sigma$ in G^{σ}. We have $0 < d_B(m) \leq |V(B)| - 3$ and $0 < d_B(n) \leq |V(B)| - 3$ in G. If $d_B(m) < |V(B)| - 3,$ then there is at least two vertices, say a and b in $V(B) - \sigma$ that are not-adjoint to m in B. Now $mn \notin E(G)$ implies that m is adjoint to a and b in B^{σ}. Since $B - \sigma$ is connected, there is $a - b$ path in B and in B^{σ}. Now the edges am, bm and path $a - b$, form a cycle C_2 in B^{σ} different from C_1, which is a contradiction to B^{σ} is unicyclic. Hence $d_B(m) = |V(B)| - 3.$ Similarly $d_B(n) = |V(B)| - 3.$ From case 1, we have $d_B(n) = |V(B)| - 4$ and B is connected. Hence $d_B(n) \geq 1$ implies that $|V(B)| \geq 5.$ Also from case 2, $d_B(n) = |V(B)| - 3$ and $B - \sigma$ is unicyclic. This implies $|V(B) - \sigma| \geq 3$ and hence $|V(B)| \geq 5.$

Conversely, assume the conditions in the statement.

Case A. $B - \sigma$ is connected, acyclic and $\{d_B(m), d_B(n)\} = \{|V(B)| - 4, |V(B)| - 3\}.$

Without loss of generality, let $d_B(m) = |V(B)| - 3$ and $d_B(n) = |V(B)| - 4.$ By Theorem 1.1, B^{σ} is connected. Now $mn \notin E(G)$ and $d_B(m) = |V(B)| - 3,$ implying there exist only a vertex in $V(B) - \sigma,$ say a, which is not adjoint to m in B and adjoint to m in B^{σ} and hence ma is an edge in B^{σ}. Also $d_B(n) = |V(B)| - 4$ implies that there exists exactly two vertices in $V(B) - \sigma$, say u and v, such that n is not adjoint to both u and v in B and n is adjoint to both u and v in B^{σ}. Thus v_m and v_n are edges in B^{σ}. Since $B - \sigma$ is connected, $m - n$ is a path in $B - \sigma$ and in B^{σ}. Clearly the edge v_m, path $m - n$ and edge nv forms a unique cycle in B^{σ}. Hence B^{σ} is unicyclic.

Case B. $B - \sigma$ is connected, unicyclic and $d_B(m) = d_B(n) = |V(B)| - 3$.

By Theorem 1.1, B^{σ} is connected. Since $d_B(m) = |V(B)| - 3$, m is not-adjoint to exactly one vertex, say x, of $V(B) - \sigma$ in B implies mx is an edge in B^{σ}. Similarly, $d_B(n) = |V(B)| - 3$ implies n is not-adjoint to exactly one vertex of $V(B) - \sigma$ in B^{σ}, say y, and ny is an edge in B^{σ}. Now $B - \sigma$ is unicyclic and $mn \notin E(G)$, the addition of the edges mx and ny (for $x \neq y$) and the edge mx and nx (for $x = y$) do not form another cycle in B^{σ}. Hence B^{σ} is unicyclic.

Thus in both cases we have B^{σ} is connected and unicyclic.

Corollary 2.1. *Let G be a connected graph and let $\sigma = \{m, n\}$ be a subset of $V(G)$ such that $mn \notin E(G)$. Let G be connected. Then G^{σ} is unicyclic and connected iff $p \geq 5$ and either*:

(i) $G - \sigma$ *is connected, acyclic and* $\{d_G(m), d_G(n)\} = \{|V(G)| - 3, |V(G)| - 4\}$ *or*

(ii) $G - \sigma$ *is connected, unicyclic and* $d_G(m) = d_G(n) = |V(G)| - 3$.

Theorem 2.2. *If G is of order $p \geq 3$ and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \in E(G)$. If B is a $\hat{c}$-joint at σ in G, then B^{σ} is a $\hat{c}$-joint and unicyclic iff $|V(B)| \geq 5$ and one of the following holds*

(i) $B - \sigma$ *is connected, acyclic and either* $d_B(n) = d_B(m) = |V(B)| - 2$ *or* $\{d_B(m), d_B(n)\} = \{|V(B)| - 3, |V(B)| - 1\}$.

(ii) *$B - \sigma$ is connected, unicyclic and* $\{d_B(n), d_B(m)\} = \{|V(B)| - 3, |V(B)| - 1\}$.

Proof. If B is a $\hat{c}$-joint so that B^{σ} is a $\hat{c}$-joint and unicyclic. By Theorem 1.3 we have $B - \sigma$ is connected and either $0 < d_B(m) \leq |V(B)| - 2$ or $0 < d_B(n) \leq |V(B)| - 2$. Without sacrificing generality, let $0 < d_B(m) \leq |V(B)| - 2$. Since $mn \in E(G)$, we have $1 \leq d_B(n) \leq |V(B)| - 1$. We have $B - \sigma$ is either acyclic or unicyclic since B^{σ} is unicyclic.

Case 1. $B - \sigma$ is acyclic.

If $d_B(m) < |V(B)| - 3$, then there exist at least three vertices a, b and c in $V(B) - \sigma$ which are not-adjoint to m in B. Implying m is adjoint to a, b and c in B^{σ}. Since $B - \sigma$ is connected, there exist $a - b$, $b - c$ and $a - c$ paths in B and hence in B^{σ}. Now the edges am, bm, cm and the paths $a - b$, $b - c$ and $a - c$, form at least three different cycles in B^{σ}, which is a contradiction to B^{σ} is unicyclic. Hence either $d_B(m) = |V(B)| - 2$ or $d_B(m) = |V(B)| - 3$. Similarly if $d_B(m) < |V(B)| - 3$, then B^{σ} is not unicyclic. Hence either $d_B(n) = |V(B)| - 1$ or $d_B(n) = |V(B)| - 2$ or $d_B(n) = |V(B)| - 3$.

Subcase 1.a. $d_B(m) = |V(B)| - 3$.

$mn \in E(G)$ shows that m is not-adjoint to only two vertices, say a and b of $V(B) - \sigma$ in B. This shows that m is adjoint to a and b in B^{σ}. As $B - \sigma$ is connected, there is an $a - b$ path in B^{σ}. Now, the edge am, the path $a - b$ and the edge bm form a cycle C_1 in B^{σ} without the edge mn. If $d_B(n) \leq |V(B)| - 2$, there is at least one vertex, x, in $V(B) - \sigma$, which is not-adjoint to n in B. Hence xn is an edge in B^{σ}. Now the edges xn, nm and ma and the path $a - x$ form cycle C_2 in B^{σ} with the edge mn, which is a contradiction to B^{σ} is unicyclic. This implies that $d_B(n) = |V(B)| - 1$.

Subcase 1.b. $d_B(m) = |V(B)| - 2.$

$mn \in E(G)$ implies that there is only one vertex, a, in $V(B) - \sigma$ which is not-adjoint to m in B. This implies that m is adjoint to a in B^{σ}.

If $d_B(n) = |V(B)| - 3,$ then there is two vertices, say b and c, in $V(B) - \sigma$ so that b and c are not-adjoint to n in B. This implies n is adjoint to b and c in B^{σ}. If $b = a,$ then the edges $mn, am, cn, nb = na$ and the path $a - c$ form three different cycles $(amna;$ the edges an, nc and the $c - a$ path; the edges am, mn, vc and the path $c - a)$ in B^{σ}. If $b \neq a,$ then the edges am, bn, nc, mn and the paths $a - b, b - c$ and $a - c$ form three different cycles in B^{σ}. In both cases we get a contradiction to B^{σ} is unicyclic.

If $d_B(n) = |V(B)| - 2,$ then there is only one vertex, say b, in $V(B) - \sigma$ not-adjoint to n in B. This shows that bn is an edge in B^{σ}. Hence the edges am, bn, mn and the path $a - b$ form a cycle in B^{σ}.

If $d_B(n) = |V(B)| - 1,$ then n is adjoin to every vertices in $V(B) - \sigma$ in B. This shows that n is not-adjoint to the vertices of $V(B) - \sigma$ in B^{σ}. Since $B - \sigma$ is acyclic and am, mn are edges, we have B^{σ} is acyclic, which is a contradiction to B^{σ} is unicyclic.

Hence we have $d_B(n) = |V(B)| - 2.$

Case 2. $B - \sigma$ is unicyclic.

Let C_1 be the only cycle in $B - \sigma$ in G. Clearly C_1 is also a cycle in B^{σ}. If $d_B(m) < |V(B)| - 2,$ then there exist at least two vertices, a and b, in $V(B) - \sigma$ not-adjoint to m in B. This shows that m is adjoin to a and b in B^{σ}. Since $B - \sigma$ is connected, there exists an $a - b$ path in B and in B^{σ}. Clearly, the edge ma, path $a - b$ and edge bm form a cycle C_2 in B^{σ} different from C_1, which is a contradiction to B^{σ} is unicyclic. Hence $d_B(m) = |V(B)| - 2.$ This shows there is a vertex x in $B - \sigma$ and is not-adjoint to m in B. Hence xm is an edge in B^{σ}.

If $0 < d_B(n) < |V(B)| - 1$, then $V(B)$ has at least one vertex, say y, that is not neighbouring to n in B. This illustrates that in B^{σ}, y is adjoint to n. There is a xy path in B and in B^{σ} since $B - \sigma$ is linked. The edges yn, vm, mx, and the path xy create a cycle C_3 in B^{σ}, which is different from C_1, which is unicyclic and we get a contradiction. As a result $d_B(n) = |V(B)| - 1$.

Similarly if $0 < d_B(v) < |V(B)| - 2$, then we have either $B - \sigma$ is connected, acyclic and either $d_B(m) = d_B(n) = |V(B)| - 2$ or $d_B(n) = |V(B)| - 3$ and $d_B(m) = |V(B)| - 1$ or $B - \sigma$ is connected, unicyclic, $d_B(n) = |V(B)| - 2$ and $d_B(m) = |V(B)| - 1$. Thus we conclude that either $B - \sigma$ is connected, acyclic and either $d_B(m) = d_B(n) = |V(B)| - 2$ or $\{d_B(m), d_B(n)\} = \{|V(B)| - 3, |V(B)| - 1\}$ or $B - \sigma$ is connected, unicyclic and $\{d_B(m), d_B(n)\} = \{|V(B)| - 2, |V(B)| - 1\}$.

Conversely, assume the conditions given in the statement. We have three cases.

Case A. $B - \sigma$ is connected, acyclic and $\{d_B(m), d_B(n)\} = \{|V(B)| - 3, |V(B)| - 1\}$.

Without sacrificing generality, let $d_B(m) = |V(B)| - 1$, $d_B(n) = |V(B)| - 3$. By Theorem 1.3, B^{σ} is connected since $d_B(v) = |V(B)| - 3 < d_B(u) - 2$. Now $mn \in E(G)$ and $d_B(m) = |V(B)| - 1$ shows that m is adjoin to vertices of $V(B) - \sigma$ in B and not adjoin to the vertices of $V(B) - \sigma$ in B^{σ}. Also $d_B(n) = |V(B)| - 3$ implies that there is exactly two vertices in $V(B) - \sigma$, say u and v, so that n is not adjoint to both u and v in B and n is adjoint to u and v in B^{σ}. Thus vm and vn are edges in B^{σ}. Since $B - \sigma$ is connected, there exist a $u - v$ path in $B - \sigma$ and in B^{σ}. Clearly the edge mv, path $v - u$ and edge vn make a unique cycle in B^{σ}. Hence B^{σ} is unicyclic.

Case B. $B - \sigma$ is connected, acyclic and $d_B(m) = d_B(n) = |V(B)| - 2.$

By Theorem 1.3, B^σ is connected. Since $mn \in E(G)$ and $d_B(m) = |V(B)| - 2,$ m is not adjoint to exactly one vertex, say x, of $V(B) - \sigma$ in B which shows that m is adjoin to x in B^σ and mx is an edge in B^σ. Similarly, $d_B(n) = |V(B)| - 2$ shows that n is adjoin to exactly one vertex of $V(B) - \sigma$ in B^σ, say y so ny is an edge in B^σ. If $x = y$, then $mxynm$ is a cycle in B^σ. If $x \neq y$, then there is an $x - y$ path in B^σ since $B - \sigma$ is connected. Now, $B - \sigma$ is acyclic and $mn \in E(G)$, the edge mx, the path $x - y$ and the edge ny (for $x \neq y$) form a cycle in B^σ. Hence in both possible ways, B^σ is unicyclic.

Case C. $B - \sigma$ is connected, unicyclic and $\{d_B(m), d_B(n)\} = \{|V(B)| - 2, |V(B)| - 1\}$.

Without sacrificing generality, $d_B(m) = |V(B)| - 1$ and $d_B(n) = |V(B)| - 2.$ By Theorem 1.3, B^σ is connected since $d_B(v) = |V(B)| - 2.$ Now $mn \in E(G)$ and $d_B(m) = |V(B)| - 1,$ implies m is adjoint to all the vertices of $V(B) - \sigma$ in B and not adjoint to all vertices of $V(B) - \sigma$ in B^σ. Also $d_B(n) = |V(B)| - 2$ implies that m is not-adjoint to exactly one vertex, say x, of $V(B) - \sigma$ in B which implies that m is adjoint to x in B^σ and ux is an edge in B^σ. Since $B - \sigma$ is unicyclic and the vertex n is adjoint to only one vertex of $B - \sigma$ in B^σ, B^σ is unicyclic.

Thus from the above three cases we have B^σ is connected and unicyclic. Hence the Theorem.

Corollary 2.2. *If G is of order p and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \in E(G)$. If G is a $\hat{c}$-joint at σ in G, then G^σ is a $\hat{c}$-joint and unicyclic iff one of the following is true*:

(i) *$G - \sigma$ is connected, acyclic and either $d_G(m) = d_G(n) = |V(G)| - 2$ or $\{d_G(m), d_G(n)\} = \{|V(G)| - 3, |V(G)| - 1\}$.*

(ii) $G - \sigma$ *is connected, unicyclic and* $\{d_G(m), d_G(n)\} = \{| V(G)| - 1, | V(G)| - 2\}$.

Theorem 2.3. *For a graph G of order* $p \geq 4$ *and let* $\sigma = \{m, n\} \subseteq V(G)$ *such that* $mn \notin E(G)$. *If B is a d-joint at* σ *in G, then* B^{σ} *is a* $\hat{c}$*-joint and unicyclic iff* $B = K_1 \cup P_3$ *and* $\{d_B(m), d_B(n)\} = \{0, 1\}$.

Proof. Let B be a d-joint at $\sigma = \{m, n\}$ in G such that B^{σ} is a $\hat{c}$-joint and unicyclic. By Theorem 1.2 we have $B - \sigma$ is connected and either $d_B(m) = 0$ and $0 \leq d_B(n) \leq | V(B)| - 3$ or $d_B(n) = 0$ and $0 \leq d_B(m) \leq | V(B)| - 3$. Without loss of generality we take $d_B(m) = 0$ and $0 \leq d_B(n) \leq | V(B)| - 3$. Since $d_B(m) = 0$, m is adjoin to every vertices of $B - \sigma$ in B^{σ}. If $| V(B)| \geq 5$, then $| V(B) - \sigma| \geq 5 - 2 = 3$ and therefore there is at least three vertices in $B - \sigma$, say a, b and c, which are not-adjoint to m in B. Then a, b and c are adjoin to m in B^{σ} and hence ma, mb and mc are all edges in B^{σ}. As $B - \sigma$ is connected, there exist $a - b$, $b - c$ and $a - c$ paths in $B - \sigma$ and hence in B^{σ}. Clearly the edges am, bm and cm and the paths $a - b$, $b - c$ and $a - c$, form at least three different cycles in B^{σ}, which is a contradiction to B^{σ} is unicyclic. Hence $| V(B)| < 5$, which implies that either $| V(B)| = 3$ or $| V(B)| = 4$.

If $| V(B)| = 3$, then $| V(B) - \sigma| = 3 - 2 = 1$. Let the vertex be a. Therefore $d_B(m) = 0$ and $d_B(n) \leq 3 - 3 = 0$ and hence B is $3K_1$. This implies that B^{σ} is P_3 which is a contradiction to B^{σ} is unicyclic.

If $| V(B)| = 4$, then $| V(B) - \sigma| = 4 - 2 = 2$. Let the two vertices be a and b. Since $d_B(m) = 0$, am and bm are edges in B^{σ}. Since $B - \sigma$ is connected, ab is an edge in $B - \sigma$ and hence in B^{σ}. Clearly the edges am, mb and ba in B^{σ} make a cycle C_1.

Now $d_B(n) \leq | V(B)| - 3 = 4 - 3 = 1$ implies that $d_B(n)$ is either 0 or 1. If $d_B(n) = 0$, then $B = 2K_1 \cup P_2$ where the K_1's are vertices m and n. Clearly

B^{σ} is $K_4 - mn$ which contains more than one cycle which is contradiction to B^{σ} is unicyclic.

If $d_B(n) = 1$, then $B = K_1 \cup P_3$, where K_1 is the vertex m and n is an end vertex of P_3. Clearly $B^{\sigma} = C_3(m)(0, 0, P_2)$ is unicyclic where n does not lie on the cycle. Hence $B = K_1 \cup P_3$, $d_B(m) = 0$ and $d_B(n) = 1$.

Similarly, if $d_B(n) = 0$ and $0 \leq d_B(m) \leq |V(B)| - 3$, then we can show that $B = K_1 \cup P_3$, $d_B(n) = 0$ and $d_B(m) = 1$. Thus $B = K_1 \cup P_3$ and $\{d_B(m), d_B(n)\} = \{0, 1\}$.

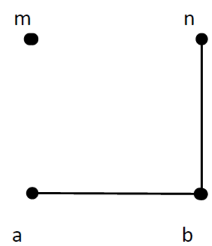

Figure 8. $B = K_1 \cup P_3$

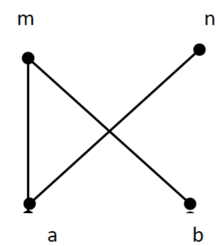

Figure 9. $B^{\sigma} = C_3(u)(0, 0, P_2)$

Conversely, let B be a d-joint, $B = K_1 \cup P_3$ and $\{d_B(m), d_B(n)\} = \{0, 1\}$. If $d_B(m) = 0$ and $d_B(v) = 1$, then $B^{\sigma} = C_3(u)(0, 0, P_2)$ and if $d_B(m) = 1$ and $d_B(n) = 0$, then $B^{\sigma} = C_3(m)(0, 0, P_2)$. In both the cases we get B^{σ} is a connected and unicyclic.

Corollary 2.3. *If G is of order $p \geq 4$ and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \notin E(G)$. If G is a d-joint at σ in G, then G^{σ} is a $\hat{c}$-joint and unicyclic iff $G = K_1 \cup P_3$ and $\{d_G(m), d_G(n)\} = \{0, 1\}$.*

Theorem 2.4. *For a graph G of order $p \geq 3$ and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \in E(G)$. If B is a d-joint at σ in G, then B^{σ} is a $\hat{c}$-joint and unicyclic iff $G = K_1 \cup P_3$ where K_2 is the edge uv.*

Proof. If B is a d-joint at $\sigma = \{m, n\}$ in G such that B^{σ} is a $\hat{c}$-joint and unicyclic. By Theorem 1.5 we have $B - \sigma$ is connected and

$d_B(m) = d_B(n) = 1$. Since mn is an edge, mn is a component of B. If $|V(B)| \geq 4$, then $|V(B) - \sigma| \geq 4 - 2 = 2$. Therefore there is at least two vertices in $B - \sigma$, say a and b such that they are adjoint to both u and v in B^{σ}. Now the edges mn, ma, mb, nb, na and nb form at least five different cycles in B^{σ} which is a contradiction to B^{σ} is unicyclic. Hence $|V(B)| = 3$. This implies that $G = K_1 \cup K_2$, where K_2 is the edge mn.

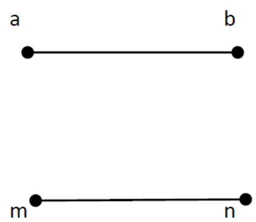

Figure 10. $B = 2K_2$

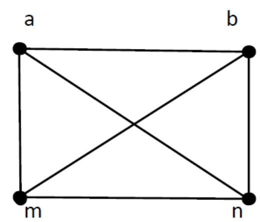

Figure 11. $B^{\sigma} = K_4$

Conversely, let $B = K_1 \cup K_2$, where K_2 is the edge mn. Clearly $B^{\sigma} = C_3$ is the unique cycle. Thus the theorem.

Corollary 2.4. *For a graph G of order* $p \geq 3$ *and let* $\sigma = \{m, n\} \subseteq V(G)$ *and* $mn \in E(G)$. *If G is a d-joint at* σ *in G, then* G^{σ} *is a* $\hat{c}$*-joint and unicyclic iff* $G = K_1 \cup K_2$ *where* K_2 *is the edge* mn.

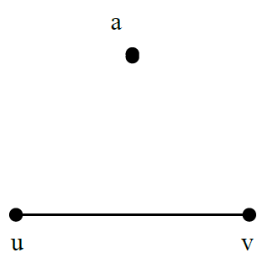

Figure 12. $G = K_1 \cup K_2$.

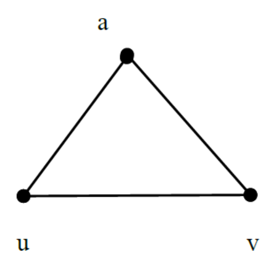

Figure 13. $G^{\sigma} = C_3$.

Theorem 2.5. *If G is of order* $p \geq 3$ *and* $\sigma = \{m, n\} \subseteq V(G)$ *and*

$mn \in E(G)$. If B is a $\hat{c}$-joint at σ in G, then B^{σ} is a d-joint and unicyclic iff $B - \sigma$ is connected, unicyclic and $d_B(m) = d_B(n) = |V(B)| - 1$.

Proof. If B is a $\hat{c}$-joint at σ in G and B^{σ} is a d-joint and unicyclic. By Theorem 1.4 we have $B - \sigma$ is connected and $d_B(m) = d_B(n) = |V(B)| - 1$. This implies that both u and v are adjoint to every vertices of $B - \sigma$. Hence $B - \sigma$ is a component of B^{σ}. Since $mn \in E(G)$, $B^{\sigma} = K_2 \cup (B - \sigma)$ where K_2 is the edge mn. Since B^{σ} is unicyclic, $B - \sigma$ is unicyclic.

Conversely, assume that $B - \sigma$ is connected, unicyclic and $d_B(m) = d_B(n) = |V(B)| - 1$. By Theorem 1.4, B^{σ} is a d-joint. Clearly $B^{\sigma} = K_2 \cup (B - \sigma)$. Since $B - \sigma$ is unicyclic, B^{σ} is also unicyclic.

Corollary 2.5. *If G is of order $p \geq 3$ and let $\sigma = \{m, n\} \subseteq V(G)$ and $mn \in E(G)$. If G is a $\hat{c}$-joint, then G^{σ} is a d-joint and unicyclic iff $G - \sigma$ is unicyclic, connected and $d_G(m) = d_G(n) = |V(G)| - 1$.*

Conclusion

Thus, in this article, we gave the necessary and sufficient conditions for G, for which G^{σ}, the switching of G at $\sigma = \{m, n\}$ to be connected and unicyclic graph when $mn \in E(G)$ and $mn \notin E(G)$.

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