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COMMON FIXED POINT UNDER GENERALIZED CONDITION ON COMPLEX QUASI-PARTIAL METRIC SPACE AND APPLICATIONS

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Abstract

In this paper, we prove some common fixed points for weakly compatible mappings satisfying generalized condition (B) on complex quasi-partial metric spaces. Some examples are given to illustrate the main results.

1. Introduction

The Banach contraction principle is one of the fundamental results of nonlinear functional analysis to prove the existence and uniqueness of fixed points of certain self-maps of metric spaces. There are many generalizations

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of metric spaces such as partial metric spaces, generalized metric spaces, cone metric spaces, and quasi metric spaces. Recently, Azam et al. [4] obtained the generalization of Banach's contraction principal by introducing the concept of complex valued metric space and established some common fixed point theorems for mappings involving rational expressions which are not meaningful in cone metric spaces.

The partial metric was introduced by Matthews [12, 13], it differs from a metric in that points are allowed to have nonzero self-distances (i.e., $d(x, x) \geq 0)$ and the triangle inequality is modified to account for positive self-distance but the property of symmetric and modified version of triangle inequality is satisfied. Matthews [12, 13] obtained, among other results, a partial metric version of the Banach fixed point theorem. After the appearance of partial metric spaces, some authors started to generalize Banach contraction mapping theorem to partial metric spaces and focus on. Valero [18], Oltra and Valero [17], and Altun et al. [9] gave some generalizations of the result of Matthews. Partial quasi metric space was introduced by Künzi et al. [10] by dropping the symmetry condition in the definition of a partial metric. Karapinar et al. [11] called it a quasi-partial metric space and gave the first fixed point result in a quasi-partial metric space. Later, some more results on fixed point theory on partial metric spaces were published in [23].

The study of fixed point theorems concerning rational inequalities in complex valued metric spaces have been increasing vigorously [1, 2, 8, 9, 19 and 20]. Along this direction, P. Dhivya et al. [8] introduced the notion of fixed point results on ordered complex partial metric spaces, which is broader than complex valued metric spaces. In this article, we prove some common fixed points of weakly compatible mappings satisfying generalized condition (B) on complex quasi-partial metric spaces. Our results generalize, extend and improve many results existing in the literature ([5-9, 16] and so on) illustrating the importance of the generalized condition (B) for quadruple of mappings in a complex quasi partial metric space. Two examples are given to illustrate this work.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows. $z_1 \precsim z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$, $\text{Im}(z_1) \leq \text{Im}(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied.

(C1) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$

(C2) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$

(C3) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$

(C4) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C3) is satisfied.

Definition 2.1 [4]**.** Let X be a nonempty set whereas be the set of complex numbers. Suppose that the mapping $d : X \times X \to C$, satisfies the following conditions.

1. $0 \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$

2. $d(x, y) = d(y, x)$ for all $x, y \in X$

3. $(x, y) \preccurlyeq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 2.2 [12, 13]**.** Let $X \neq \phi$. A partial metric is a function $p : X \times X \to R^+$ satisfying

1. $p(x, y) = p(y, x)$ (symmetry);

2. if $0 \leq p(x, x) = p(x, y) = p(y, y)$, then $x = y$ (non-negativity and indistancy implies equality);

3. $p(x, x) \leq p(x, y)$ (Small self-distances);

4. $(x, z) + (y, y) \leq p(x, y) + p(y, z)$ (Triangularity); for all $x, y, z \in X$. The pair (X, p) is called a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on with a base of the family of open-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 2.3 [10]. A quasi-partial metric is a function $q : X \times X \to R^+$ satisfying

1. $q(x, x) \leq q(y, x)$ (Small self-distances);

2. $q(x, x) \leq q(x, y)$ (Small self-distances);

3. $x = y$ iff $q(x, x) = q(x, y)$ and $q(y, y) = q(y, x)$ (Indistancy implies equality and vice versa);

4. $q(x, z) + q(y, y) \leq q(x, y) + q(y, z)$ (Triangularity);

for all $x, y, z \in X$. The pair (X, q) is called a quasi-partial metric space. Note that, if $(x, y) = q(y, x)$ for all $x, y \in X$, then (X, q) becomes a partial metric space. It is easy to see that for a partial metric p on X, the function $d_p : X \times X \to R^+$ defined by

$$(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a (usual) metric on X. Analogously for a quasi-partial metric p on X, the function $d_p : X \times X \to R^+$ defined by

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y)$$

is a (usual) metric on X.

Definition 2.4 [6]. A complex partial metric on a non-empty set X is a function $p_c : X \times X \to C^+$ such that for all $x, y, z \in X$.

1. $0 \leq p_c(x, x) = p_c(x, y)$ (small self-distances)

2. $p_c(x, y) = p_c(y, y)$ (symmetry)

3. $p_c(x, x) = p_c(x, y) = p_c(y, y)$ if and only if $x = y$ (equality)

4. $p_c(x, y) \leq p_c(x, z) + p_c(z, y) - p_c(z, z)$ (Triangularity)

For the complex partial metric p_c on X, the function $d_{pc} : X \times X \to C^+$ given by

$$d_{pc} = 2p(x, y) - p_c(x, x) - p_c(y, y)$$

is a (usual) metric on X. Each complex partial metric p_c on X generates a topology τ_{pc} on X with the base family of open p_c-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$ where $\{B_{pc}(x, \epsilon) = y \in X : (x, y) < p_c(x, x) + \epsilon\}$ for all $x \in X$ and $0 < \epsilon \in C^+$.

Definition 2.5. A complex quasi partial metric on a nonempty set X is a function $qp_c : X \times X \to C^+$ which satisfies

(CQPM1) If $qp(x, x) = qp_c(x, y) = qp_c(y, y)$, that $x = y$ (equality)

(CQPM2) $qp_c(x, x) \leq qp_c(x, y)$ (small self-distances)

(CQPM3) $qp_c(x, x) \leq qp_c(y, x)$ (small self-distances)

(CQPM4) $qp_c(x, y) \leq qp_c(x, z) + qp_c(z, y) - qp_c(z, z)$, for all $x, y, z \in X$. (Triangularity)

A complex quasi partial metric space (CQPMS) is equal to (X, qp_c) such that X is a nonempty set and qp_c is a complex quasi partial metric on X. Note that, if $q(x, y) = qp_c(x, y)$ for all $x, y \in X$, then (X, qp_c) becomes a complex partial metric space.

It is easy to see that for a complex partial metric p_c on X, the function $d_{pc} : X \times X \to R^+$ defined by

$$d_p(x, y) = 2p_c(x, y) - p_c(x, x) - p_c(y, y)$$

is a (usual) complex metric on X. Analogously for a complex quasi-partial metric qp_c on X, the function $d_{qpc} : X \times X \to C^+$ defined by

$$d_{qpc}(x, y) = qp_c(x, y) + qp_c(y, x) - qp_c(x, x) - qp_c(y, y)$$

is a (usual) metric on X. Each complex quasi partial metric qp_c on X generates a topology $\tau_{qPc,}$ on with the base family of open qp_c-balls $\{B_{qp}(x, \epsilon) : x \in X, \epsilon > 0\}$, where $\{B_{qpc}(x, \epsilon) = y \in X : q(x, y) < qp_c(x, x) + \epsilon\}$ for all $x \in X$ and $0 < \epsilon \in C^+$.

Example 2.6. Let $X = [0, \infty)$ endowed with complex quasi partial metric q_c is defined by $q_c : X \times X \to C^+$ with $q(x, y) = \max\{x, y\} + i \max\{x, y\} + x$, for all $x, y \in X$.

A complex valued metric space is a complex partial metric space. But a complex partial metric pace need not be a complex valued metric space. The following example illustrates such a complex partial metric space.

Example 2.7 [6]**.** Let $X = [0, \infty)$ endowed with complex partial metric p_c is defined by $p_c : X \times X \to C^+$ with $(x, y) = \max\{x, y\} + i \max\{x, y\}$, for all $x, y \in X$.

It is easy to verify that (X, p_c) is a complex partial metric space and note that self-distance need not be zero, for example $(1, 1) = 1 + i \neq 0$. Now the metric induced by p_c is follows,

$$d(x, y) = 2p_c(x, y) - p_c(x, x) - p_c(y, y)$$

without loss of generality suppose $x \geq y$ then

$$d(x, y) = 2\{\max\{x, y\} + i \max\{x, y\}\} - \{x + ix\} - \{y + iy\}, \text{ for all } x, y \in X.$$

therefore, $d_{pc}(x, y) = |x - y| + i|x - y|$.

It is also true for complex quasi partial metric spaces.

Example 2.8. The following example illustrates such a complex quasi partial metric space. It is easy to verify that (X, qp_c) is a complex quasi partial metric space and note that self-distance need not be zero, for example $p(2, 2) = 2 + i \neq 0$. Now the metric induced by qp_c is follows, $p_c(x, y) = qp_c(x, y) + qp_c(x, y) - qp_c(x, x) - qp_c(y, y)$ without loss of generality suppose $x \geq y$ then

$$d_{qpc}(x, y) = \{ \max\{x, y\} + i \max\{x, y\} + x + \max\{y, x\} + i \max\{y, x\} + y\}$$

$$- [\{x + ix\} + x] - [\{y + iy\} + y], \text{ for all } x, y \in X.$$

therefore,

$$d_{qpc}(x, y) = | x - y | + i| x - y |.$$

Theorem 2.9. *Let (X, qp_c) be a complex quasi partial metric space, then (X, qp_c) is T_0 topology.*

Proof. Suppose $x, y \in X$ and $x \neq y$, from condition (CQPM1) and (CQPM3) in Definition (3.1), we get $qp(x, x) < qp_c(x, y)$ or $qp_c(y, y) < qp_c(x, y)$. We suppose that $qp(x, x) < qp(x, y)$, which implies that $0 < qp_c(x, y) - qp_c(x, x)$. Now let $c_x^+ \in C^+$ such that $0 < c_x^+ < qp_c(x, y) - qp_c(x, x)$. So we find that. Then $x \in B_{qp_c}(x, c^+x)$ and $y \notin B_{qp_c}(x, c^+x)$. Then we conclude that (X, qp_c) is T_0 topology. ∎

Definition 2.10. Let (X, qp_c) be a complex quasi partial metric space (CQPMS). A sequence (x_n) in a CQPMS (X, qp_c) is converges to $x \in X$, if for every $0 < \epsilon \in \mathbb{C}^+$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we get $(x_n) \in B_q(x, \epsilon)$. Then said to be a limit of (x_n), which is denoted by $\lim_{n \to \infty} x_n = x$ or $(x_n) \to x$.

Lemma 2.11. *Let (X, qp_c) be a complex quasi partial metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is converges to $x \in X$ if and only if $| qp(x, x_n) - qp_c(x, x) | < \epsilon$.*

Proof. Suppose that (x_n) converges to x, for a given real number $\epsilon > 0$, let

$$c_\epsilon = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}$$

Then $0 < c_\epsilon \in \mathbb{C}$ and there is a natural number N, such that $x_n \in B_q(x, c_\epsilon)$ for all $n \geq N$ i.e.

$$qp_c(x_n, x) < c_\epsilon + pc(x, x).$$

So that when

$$n \geq N, |\, qp_c(x_n, x) - qp_c(x, x) \,| < \epsilon.$$

It follows that

$$qp_c(x_n, x) \to qp_c(x, x) \ (as\ n \to \infty).$$

Conversely, suppose that $qp_c(x_n, x) \to qp_c(x, x)$ $(as\ n \to \infty)$. For each $0 < c_\epsilon \in \mathbb{C}$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|\, z \,| < \delta \Rightarrow z < c_\epsilon.$$

For this $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ we have

$$|\, qp_c(x_n, x) - qp_c(x, x) \,| < \delta$$

which implies that

$$qp_c(x_n, x) < c_\epsilon + qp_c(x, x).$$

for all $n \geq N$. Hence x_n converges to x. ■

Note that let (X, qp_c) be a complex quasi partial metric space. If

$$qp_c(x_n, x) \to qp_c(x, x) \ (as\ n \to \infty).$$

Then

$$qp_c(x_n, x) \to qp_c(x, x) \ (as\ n \to \infty).$$

Lemma 2.12. *Let (X, qp_c) be a complex quasi partial metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\, q(x_n, x_m) - qp_c(x, x) \,| < \epsilon$.*

Proof. Suppose that (x_n) converges to x, for a given real number $\epsilon > 0$, let

$$c_\epsilon = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 < c_\epsilon \in \mathbb{C}$ and there is a natural number N, such that $x_n \in B_{qp}(x, c_\epsilon)$ for all $n, m \geq N$ i.e.

$$qp_c(x_n, x_m) < c_\epsilon + pc(x, x).$$

So that when

$$n, m \geq N, |qp_c(x_n, x_m) - qp_c(x, x)| < \epsilon.$$

It follows that

$$qp_c(x_n, x) \to qp_c(x, x) \ (as \ n \to \infty).$$

Conversely, suppose that $qp_c(x_n, x) \to qp_c(x, x) \ (as \ n \to \infty)$. For each $0 < c_\epsilon \in \mathbb{C}$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z < c_\epsilon.$$

For this $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$ we have

$$|qp_c(x_n, x_m) - qp_c(x, x)| < \delta$$

which implies that

$$qp_c(x_n, x_m) < c_\epsilon + pc(x, x).$$

for all $n, m \geq N$. Hence $\{x_n\}$ is a Cauchy sequence.

Definition 2.13. Let (X, qp_c) be a complex quasi partial metric space CQPMS. A sequence $\{x_n\}$ in a CQPMS (X, qp_c) is called Cauchy if there is $c \in \mathbb{C}^+$, such that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$|qp_c(x_n, x_m) - c| < \epsilon.$$

Lemma 2.14. *Let (X, qp_c) be a complex quasi partial metric space. A sequence $\{x_n\}$ is Cauchy sequence in the CQPMS (X, qp_c) then $\{x_n\}$ is Cauchy in a metric space (X, qp_c).*

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, qp_c). There is $c \in \mathbb{C}^+$ such that for every real $\epsilon > 0$, there is $N \in \mathbb{N}$, for all $n, m \geq N$ $| qp_c(x_n, x_m) - c | < \frac{\epsilon}{4}$.

Hence

$$d_{qpc}(x_n, x_m) = (qp_c(x_n, x_m) - c) + (qp_c(x_n, x_m) - c) - (qp_c(x_n, x_n) - c)$$
$$- (qp_c(x_n, x_m) - c).$$

for all $n, m \geq N$, we have

$$| qp_c(x_n, x_m) - c | < \epsilon.$$

That is

$$d_q(x_n, x_m) \to 0 \, (as\ n, m \to \infty). \qquad \blacksquare$$

Let X be a complex quasi partial metric space and $A \subseteq X$. A point $x \in X$ is called an interior point of set A, if there exists $0 < r \in \mathbb{C}$ such that $B_q(x, r) = \{y \in X : qp_c(x, y) < qp_c(x, x) + r\} \subseteq A$. A subset A is called open, if each point of A is an interior point of A. A point $x \in X$ is said to be a limit point of A, for every $0 < r \in \mathbb{C}$, $B_q(x, r) \cap A - \{x\} \neq \phi$. A subset $B \subseteq X$ is called closed, contains all its limit points.

Definition 2.15. Let (X, qp_c) be a complex partial metric space (CQPMS).

(1) A CQPMS (X, qp_c) is said to be complete if a Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau_{qPc,}$ to a point $x \in X$ such that $\lim\limits_{n, m \to \infty} qp_c(x_n, x_m) = qp_c(x, x)$.

(2) A mapping $T : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that $T(B_{qpc}(x_0, \delta)) \subset B_{qpc}(T(x_0), \epsilon)$.

Definition 2.16. A subset of a complex quasi-partial metric space CQPMS (X, qp_c) is closed if whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that

$\{x_n\}$ converges to some $x \in X$, then $x \in \mathcal{M}$.

Lemma 2.17. *Let* (X, qp_c) *be a CQPMS. Then the following statement hold true.*

(1) *If* $qp_c(x, y) = 0$, *then* $x = y$

(2) *If* $x \neq y$, *then* $qp_c(x, y) > 0$ *and* $q_c(y, x) > 0$

Definition 2.18. Let A and S be self-mappings on a set X. A point $x \in X$ is called a coincidence point of A and S if $Ax = Sx = w$, where w is called a point of coincidence of A and S.

Definition 2.19 [9]**.** Let X be a non-empty set. Two mappings $A, S : X \to X$ are said to be weakly compatible if they commute at their coincidence point, i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

Definition 2.20 [3]**.** Let A, B, S and T be four self-mappings of a quasi-partial metric space (x, q). The pair of mappings (A, S) satisfies generalized condition (B) associated with (B, T) ((A, S) is a generalized almost (B, T)-contraction) if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that for all $x, y, z \in X$ we have

$$\begin{aligned} q(Sx, Ty) \leq \delta \max\{&qp_c(Ax, By), qp_c(Ax, Sx), qp_c(By, Ty), \\ &\frac{(qp_c(Sx, By) + qp_c(Ax, Ty))}{2}\} \\ &+ L \min\{qp_c(Ax, Sx), qp_c(By, Ty), qp_c(Ax, Ty), qp_c(By, Sx)\}. \end{aligned} \tag{3.1}$$

3. Main Result

Theorem 3.1. *Let* $\mathcal{A}, \mathcal{B},$ *and* $\mathcal{T}$ *be self-mappings of a complex quasi-partial metric space* (X, qp_c). *If the pair of mappings* $(\mathcal{A}, \mathcal{S})$ *satisfies generalized condition* (*B*) *associated with* $(\mathcal{B}, \mathcal{T})$ *for all* $x, y, z \in X$ *and we have*

1. $\mathcal{T}X \subset \mathcal{A}X$ *and* $\mathcal{S}X \subset \mathcal{B}X$,

2. $\mathcal{A}X$ *or* $\mathcal{B}X$ *is closed*,

3. $(\delta + 2L) < 1$, *then the pairs* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *have a coincidence point. Further,* $\mathcal{A}, \mathcal{B}, \mathcal{S}$ *and have a unique common fixed point, provided that the pairs* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *are weakly compatible.*

Proof. Let $x_0 \in X$. Since $\mathcal{S}X \subset \mathcal{B}X$, there exists a point $x_1 \in X$, such that $y_1 = \mathcal{B}x_1 = \mathcal{S}x_0$. Suppose there exists a point $y_2 = Tx_1$ corresponding to this point y_1. Also since $\mathcal{T}X \subset \mathcal{A}X$, there exists $x_1 \in X$, such that $y_2 = \mathcal{A}x_2 \subset \mathcal{T}x_1$. Continuing in this manner, we can define a sequence $\{y_n\}$ in X as follows

$$y_{2n+1} = \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n},$$

$$y_{2n+2} = \mathcal{A}x_{2n+2} = \mathcal{T}x_{2n+1}.$$

Now

$$qp_c(y_{2n+1}, y_{2n+2}) = qp_c(\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1})$$

$$\leq \delta \max\{qp_c(\mathcal{A}x_{2n}, \mathcal{B}\mathcal{B}x_{2n+1}), qp_c(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}), qp_c(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}),$$

$$\frac{(qp_c(\mathcal{S}x_{2n}, \mathcal{B}x_{2n+1}) + qp_c(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}))}{2}\}.$$

$$+ L \min\{qp_c(\mathcal{A}x2n, \mathcal{S}x2n), qp_c(\mathcal{B}x2n+1, \mathcal{T}x2n+1), qp_c(\mathcal{A}x2n, \mathcal{T}x2n+1),$$

$$qp_c(\mathcal{B}x2n+1, \mathcal{S}x2n)\} \leq \delta \max\{qp_c(y_{2n}, y_{2n+1})$$
$$\frac{qp_c(y_{2n}, y_{2n+1}), qp_c(y_{2n+1}, y_{2n+2})}{2}\}$$

$$+ L \min\{qp_c(y_{2n}, y_{2n+1}), qp_c(y_{2n+1}, y_{2n+1}), qp_c(y_{2n}, y_{2n+1}), qp_c(y_{2n+1}, y_{2n+1})\}$$

$$= \delta \max\{qp_c(y_{2n}, y_{2n+1}), qp_c(y_{2n+1}, y_{2n+2}), + L \min\{qp_c(y_{2n}, y_{2n+2}),$$

$$qp_c(y_{2n+1}, y_{2n+1})\}, (qp(y_{2n+1}, y_{2n+1}) + qp_c(y_{2n}, y_{2n+2}))$$

Now the following four cases arise:

Case I. When

$$\max\{qp_c(y_{2n}, y_{2n+1}), qp_c(y_{2n+1}, y_{2n+2})\} = qp_c(y_{2n}, y_{2n+1})$$

and

$$\min\{qp_c(y_{2n},\, y_{2n+2}),\, qp_c(y_{2n+1},\, y_{2n+1})\} = qp_c(y_{2n},\, y_{2n+1})$$

then

$$qp(y_{2n+1},\, y_{2n+2}) \leq \delta\ qp_c(y_{2n},\, y_{2n+1}) + Lqp_c(y_{2n},\, y_{2n+2})$$

$$\leq \delta\ qp_c(y_{2n},\, y_{2n+1}) + L\{qp_c(y_{2n+1}, qp\ \ y_{2cn}(+y_{22})n, -y_2 qp_{n+c1}()y + 2n + 1,$$

$$y_{2n+1})\} \leq (\delta + L)p_c(y_{2n},\, y_{2n+1}) + L\ qp_c(y_{2n+1},\, y_{2n+2}) \text{ i.e.,}$$

$$(1 - L)qp(y_{2n+1},\, y_{2n+2}) \leq (\delta + L)qp_c(y_{2n},\, y_{2n+1})$$

i.e.,

$$qp_c(y_{2n+1},\, y_{2n+2}) \leq \frac{(\delta + L)}{(1 - L)} qp_c(y_{2n},\, y_{2n+1}).$$

Now let

$\mu_1 = \dfrac{(\delta + L)}{(1 - L)}$, since $(\delta + 2L) < 1$ and $L \geq 0$, then $\mu_1 < 1$. Therefore

$$q(y_{2n+1},\, y_{2n+2}) \leq \mu_1 qp_c(y_{2n},\, y_{2n+1}).$$

Case II. When

$$\max\{qp_c(y_{2n},\, y_{2n+1}),\, qp_c(y_{2n+1},\, y_{2n+2})\} = qp_c(y_{2n},\, y_{2n+1})$$

and

$$\min\{qp_c(y_{2n},\, y_{2n+2}),\, qp_c(y_{2n+1},\, y_{2n+1})\} = qp_c(y_{2n},\, y_{2n+1}).$$

Then

$$q(y_{2n+1},\, y_{2n+2}) \leq \delta\ qp_c(y_{2n},\, y_{2n+1}) + Lqp_c(y_{2n+1},\, y_{2n+1}) \text{ i.e.,}$$

$$q(y_{2n+1},\, y_{2n+2}) \leq \delta\ qp_c(y_{2n},\, y_{2n+1}) + Lqp_c(y_{2n},\, y_{2n+1}) \text{ i.e.,}$$

$$q(y_{2n+1},\, y_{2n+2}) \leq (\delta + L)qp_c(y_{2n},\, y_{2n+1}).$$

Now let $\mu_2 = (\delta + L)$. Since $(\delta + 2L) < 1$, then $\mu_2 < 1$. Therefore

$$qp(y_{2n+1},\, y_{2n+2}) \leq \mu_2 qp_c(y_{2n},\, y_{2n+1}).$$

Case III. When

$$\max\{qp_c(y_{2n},\, y_{2n+1}),\, qp_c(y_{2n+1},\, y_{2n+2})\} = qp_c(y_{2n+1},\, y_{2n+2})$$

and

$$\min\{qp_c(y_{2n},\, y_{2n+2}),\, qp_c(y_{2n+1},\, y_{2n+1})\} = qp_c(y_{2n},\, y_{2n+2}),$$

then

$$qp(y_{2n+1},\, y_{2n+1}) \leq \delta\; qp_c(y_{2n+1},\, y_{2n+2}) + Lqp_c(y_{2n},\, y_{2n+2})$$

or

$$(1-\delta)qpc(y_{2n+1},\, y_{2n+2}) \leq L\{qp_c(y_{2n},\, y_{2n+1}) + qp_c(y_{2n+1},\, y_{2n+2})$$
$$- qp_c(y_{2n+1},\, y_{2n+1})\}$$

i.e.,

$$(1-\delta-L)q(y_{2n+1},\, y_{2n+2}) \leq Lqp_c(y_{2n},\, y_{2n+1}) \text{ i.e.,}$$

$$qp_c(y_{2n+1},\, y_{2n+2}) \leq \frac{1}{(1-(\delta+L))} qp_c(y_{2n},\, y_{2n+1})$$
$$- qp_c(y_{2n+1},\, y_{2n+1})\}$$

Let $\mu_3 = \frac{1}{(1-(\delta+L))}$, since $(\delta + 2L) < 1$, then $\mu_3 < 1$. Therefore

$$qp(y_{2n+1},\, y_{2n+2}) \leq \mu_3 qp_c(y_{2n},\, y_{2n+1}).$$

Case IV. When

$$\max\{qp_c(y_{2n},\, y_{2n+1}),\, qp_c(y_{2n+1},\, y_{2n+2})\} = qp_c(y_{2n+1},\, y_{2n+2})$$

and

$$\min\{qp_c(y_{2n},\, y_{2n+2}),\, qp_c(y_{2n+1},\, y_{2n+1})\} = qp_c(y_{2n+1},\, y_{2n+1}),$$

then

$$q(y_{2n+1},\, y_{2n+2}) \leq \delta\; qp_c(y_{2n+1},\, y_{2n+2}) + Lqp_c(y_{2n+1},\, y_{2n+1}) \text{i.e.,}$$

$$(1-\delta)q(y_{2n+1},\, y_{2n+2}) \leq Lqp_c(y_{2n},\, y_{2n+1}) \text{i.e.,}$$

$$qp_c(y_{2n+1},\, y_{2n+2}) \leq \frac{L}{(1-\delta)} qp_c(y_{2n},\, y_{2n+1})$$

Let $\mu_4 = \frac{L}{(1-\delta)}$, Since $(\delta + 2L) < 1$, then $\mu_4 < 1$. Therefore

$$qp(y_{2n+1}, y_{2n+1}) \leq \mu_4 qp_c(y_{2n+1}, y_{2n+2}).$$

Choose $\mu = \max\{\mu_1, \mu_2, \mu_3, \mu_4\}$. Therefore $0 < \mu < 1$ and we get

$$q(y_{2n+1}, y_{2n+2}) \leq \mu_2 qp_c(y_{2n}, y_{2n+1}) \leq \mu_2 qp_c(y_{2n-1}, y_{2n})$$

$$\leq \mu_3 qp_c(y_{2n-2}, y_{2n-1}) \leq \ldots \leq \mu^{2n+1} qp_c(y_0, y_1).$$

So by induction we get

$$| qp(yn, y_{n+1}) | \leq \mu^n | qp_c(y_0, y_1) |$$

Which tends to 0 as n tends to ∞.

So $\{y_n\}$ is convergent and hence its subsequence $\{y_{2n+2}\} = \{\mathcal{A}x_{2n+2}\}$ is also convergent to z. Let $\mathcal{A}X$ be closed.

So $z \in \mathcal{A}X$, i.e., there exists $u \in X$ such that $z = \mathcal{A}X$. We claim $z = \mathcal{S}u$. If not, by using (3.1), we get

$$qp_c(\mathcal{S}u, \mathcal{T}x_{2n+1})$$

$$\leq \delta \max\{qp_c(\mathcal{A}u, \mathcal{B}x_{2n+1}), qp_c(\mathcal{A}u, \mathcal{S}u), qp_c(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}),$$

$$\frac{1}{2}(qp_c(\mathcal{S}u, \mathcal{B}x_{2n+1}) + qp_c(\mathcal{A}u, \mathcal{T}x_{2n+1})) + L \min\{qp_c(\mathcal{A}u, \mathcal{S}u),$$

$$qp_c(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), qp_c(\mathcal{A}u, \mathcal{T}x_{2n+1}) qp_c(\mathcal{B}x_{2n+1}, \mathcal{S}u)\}.$$

Letting $n \to \infty$, then

$$qp_c(\mathcal{S}u, z) \leq \delta \max\{qp_c(\mathcal{A}u, z), qp_c(\mathcal{A}u, \mathcal{S}u), qp_c(z, z), \frac{1}{2}(qp_c(\mathcal{S}u, z)$$

$$+ qp_c(\mathcal{A}u, z))\} + L \min\{qp_c(\mathcal{A}u, \mathcal{S}u), qp_c(z, z), qp_c(\mathcal{A}u, z), qp_c(z, \mathcal{S}u)\}$$

i.e.,

$$qp_c(\mathcal{S}u, z) \leq (\delta + L) qp_c(\mathcal{S}u, z),$$

a contradiction to (3). Hence, $qp(\mathcal{S}u, z) = 0$, i.e., $\mathcal{S}u = z$.

So $\mathcal{A}u = \mathcal{S}u$, i.e., $\mathcal{A}$ and $\mathcal{S}$ have a coincidence point. Since $\mathcal{S}X \subset \mathcal{B}X$, there exists $v \in X$ such that $z = \mathcal{S}u = \mathcal{B}v$.

We claim that $v = z$. If not, by using (3.1) we get

$$qp_c(\mathcal{S}u, \mathcal{T}v) \leq \delta \max\{qp_c(\mathcal{A}u, \mathcal{B}v), qp_c(\mathcal{A}u, \mathcal{S}u), qp_c(\mathcal{B}v, \mathcal{T}v), \frac{1}{2}(qp_c(\mathcal{S}u, \mathcal{B}v) + qp_cq(\mathcal{A}u, \mathcal{T}v))\} + L\min\{qp_c(\mathcal{A}u, \mathcal{S}u), qp_c(\mathcal{B}v, \mathcal{T}v), qp_c(\mathcal{A}u, \mathcal{T}v), qp_c(\mathcal{B}v, \mathcal{S}u)\}$$

i.e.,

$$qp_c(z, \mathcal{T}v) \leq \delta \max\{qp_c(z, z), qp_c(z, z), qp_c(z, \mathcal{T}v), \frac{1}{2}(qp_c(z, z) + qp_c(z, \mathcal{T}v))\} + L\min\{qp_c(z, z), qp_c(z, \mathcal{T}v), qp_c(z, \mathcal{T}v), qp_c(z, z)\}$$

i.e.,

$$qp_c(z, \mathcal{T}v) \leq \delta qp_c(z, \mathcal{T}v) + Lqp_c(z, \mathcal{T}v)$$

i.e.,

$$qp_c(z, \mathcal{T}v) \leq (\delta + L)\, qp_c(z, \mathcal{T}v),$$

a contradiction to (3). Hence, $q(z, \mathcal{T}v) = 0$, i.e., $\mathcal{T}v = z$. So $\mathcal{B}v = \mathcal{T}v$, i.e., $\mathcal{B}$ and $\mathcal{T}$ have a coincidence point.

If we assume that $\mathcal{B}X$ is closed, then an argument analogous to the previous argument establishes that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ have a coincidence point. Hence, $\mathcal{A}u = \mathcal{S}u = \mathcal{B}v = \mathcal{T}v = z$.

Since $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible,

$$\mathcal{A}z = \mathcal{A}\mathcal{S}u = \mathcal{S}\mathcal{A}u = \mathcal{S}z, \text{ and}$$

$$\mathcal{B}z = \mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}v = \mathcal{T}z.$$

Now we will show that $z = \mathcal{A}z$. If not, by using (3.1) we get

$$qp_c(\mathcal{S}z, \mathcal{T}v) \leq \delta \max\{qp_c(\mathcal{A}z, \mathcal{B}v), qp_c(\mathcal{A}z, \mathcal{S}z), qp_c(\mathcal{B}v, \mathcal{T}v), \frac{1}{2}(qp_c(\mathcal{S}z, \mathcal{B}v)$$

$$+ qp_c(\mathcal{A}z, \mathcal{T}v))\}$$

$$+ L\min\{qp_c(\mathcal{A}z, \mathcal{S}z), qp_c(\mathcal{B}v, \mathcal{T}v), qp_c(\mathcal{A}z, \mathcal{T}z), qp_c(\mathcal{B}v, \mathcal{S}z)\},$$

$$qp_c(\mathcal{A}z, z) \leq \delta\max\{qp_c(\mathcal{A}z, z), d(z, z), \frac{1}{2}(qp_c(\mathcal{A}z, z) + qp_c(\mathcal{A}z, z))\}$$

$$+ L\min\{qp_c(\mathcal{S}z, \mathcal{S}z), qp_c(z, z), qp_c(\mathcal{A}z, z), qp_c(z, \mathcal{A}z)\}$$

i.e.,

$$qp_c(\mathcal{A}z, z) \leq \delta qp_c(\mathcal{A}z, z) + Lqp_c(\mathcal{A}z, z)$$

i.e.,

$$qp_c(\mathcal{A}z, z) \leq (\delta + L)qp_c(\mathcal{A}z, z),$$

a contradiction to (3). So $qp_c(\mathcal{A}z, z) = 0$, then $z = \mathcal{A}z$. Similarly we can prove that $z = \mathcal{B}z$. Hence, $z = \mathcal{A}z = \mathcal{B}z = \mathcal{S}z = \mathcal{T}z$, i.e., z is a common fixed point for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$.

Uniqueness of the fixed point is an easy consequence of (3.1). ■

Example 3.2. Let $X = [0, 2]$ be a set endowed with complex quasi-partial metric $d_{qpc}(x, y) = |x - y| + i|x - y|$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{T}$ be self-mappings defined by

$$\mathcal{A}x = \begin{cases} \frac{x}{4}, & 0 \leq x \leq 1 \\ \frac{5}{8}, & 0 \leq x \leq 2 \end{cases} \qquad \mathcal{B}x = \begin{cases} \frac{3x}{4}, & 0 \leq x \leq 1 \\ \frac{3}{4}, & 0 \leq x \leq 2, \end{cases}$$

$$\mathcal{S}x = \begin{cases} \frac{x}{12}, & 0 \leq x \leq 1 \\ \frac{1}{4}, & 0 \leq x \leq 2 \end{cases} \qquad \mathcal{T}x = \begin{cases} \frac{x}{8}, & 0 \leq x \leq 1 \\ \frac{1}{8}, & 0 \leq x \leq 2 \end{cases}$$

Here

$\mathcal{A}X = \left[0, \frac{1}{4}\right] \cup \left\{\frac{5}{8}\right\}$ and $\mathcal{A}X = \left[0, \frac{3}{4}\right]$. So $\mathcal{T}X = \left[0, \frac{1}{8}\right] \subset \mathcal{A}X$ and $\mathcal{S}X = \left[0, \frac{1}{12}\right] \cup \left\{\frac{1}{4}\right\} \subset \mathcal{B}X$.

The point 0 is a coincidence point of the four mappings. Further $\mathcal{AS}0 - \mathcal{SA}0 = 0$ and $\mathcal{TB}0 = \mathcal{BT}0 = 0$, i.e., the two pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible.

Case I. For $x, y \in [0, 1]$, we have

$$d_{qp_c}(\mathcal{S}x, \mathcal{T}y) = \left|\frac{x}{12} - \frac{y}{8}\right| + i\left|\frac{x}{12} - \frac{y}{8}\right|$$

$$\leq \frac{4}{5}\left\{\left|\frac{x}{4} - \frac{3y}{4}\right| + i\left|\frac{x}{4} - \frac{3y}{4}\right|\right\}.$$

Case II. For $x \in [0, 1]$, and $y \in (1, 2]$, we have

$$d_{qp_c}(\mathcal{S}x, \mathcal{T}y) = \left|\frac{x}{12} - \frac{1}{8}\right| + i\left|\frac{x}{12} - \frac{1}{8}\right|.$$

$$\leq \frac{4}{5}\left\{\left|\frac{3x}{4} - \frac{1}{8}\right| + i\left|\frac{3x}{4} - \frac{1}{8}\right|\right\}.$$

$$\leq \frac{4}{5}\left\{\left|\frac{1}{4} - \frac{1}{8}\right| + i\left|\frac{1}{4} - \frac{1}{8}\right|\right\}.$$

Case III. For $\in (1, 2]$, and $y \in [0, 1]$, we have

$$d_{qp_c}(\mathcal{S}x, \mathcal{T}y) = \left|\frac{1}{4} - \frac{y}{8}\right| + i\left|\frac{1}{4} - \frac{y}{8}\right|.$$

$$\leq \frac{4}{5}\left\{\left|\frac{5}{8} - \frac{3y}{4}\right| + i\left|\frac{x}{4} - \frac{3y}{4}\right|\right\}.$$

Case IV. For $x, y \in (1, 2]$, we have

$$d_{qp_c}(\mathcal{S}x, \mathcal{T}y) = \left|\frac{1}{4} - \frac{1}{8}\right| + i\left|\frac{1}{4} - \frac{1}{8}\right|.$$

$$\leq \frac{4}{5}\left\{\left|\frac{3}{4} - \frac{1}{8}\right| + i\left|\frac{3}{4} - \frac{1}{8}\right|\right\}.$$

Consequently, all hypotheses of Theorem 3.1 are satisfied (for $\delta = \frac{4}{5}$ and $L = 0$) and 0 is the unique common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$.

If $\mathcal{A} = \mathcal{B}$ and $\mathcal{S} = \mathcal{T}$, we get the following corollary

Corollary 3.3. *Let $\mathcal{A}$ and $\mathcal{T}$ be self-mappings of a complex quasi-partial metric space (X, qp_c). If $\mathcal{A}$ satisfies generalized condition (B) associated with $\mathcal{T}$ for all $x, y \in X$ and we have*

1. $\mathcal{T}X \subset \mathcal{A}X$,

2. $\mathcal{A}X$ *is closed*,

3. $(\delta + 2L) < 1$, *then $\mathcal{A}$ and $\mathcal{T}$ have a coincidence point. Further, $\mathcal{A}$ and $\mathcal{T}$ have a unique common fixed point, provided that the pair $(\mathcal{A}, \mathcal{T})$ is weakly compatible.*

Corollary 3.4. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{T}$ be self-mappings of a complex quasi-partial metric space (X, qp_c). If the pairs of mappings $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ satisfy*

$$q(\mathcal{S}x, \mathcal{T}y)$$

$$\leq \delta max\{qp_c(\mathcal{A}x, \mathcal{B}y), qp_c(\mathcal{A}x, \mathcal{S}x), qp_c(\mathcal{B}y, \mathcal{T}y), \frac{(qp_c(\mathcal{S}x, \mathcal{B}y) + qp_c(\mathcal{A}x, \mathcal{T}y))}{2}\}$$

for all $x, y \in X$ and we have

1. $\mathcal{T}X \subset \mathcal{A}X$, *and* $\mathcal{S}X \subset \mathcal{B}X$,

2. $\mathcal{A}X$ *or* $\mathcal{B}X$ *is closed, then the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ have a coincidence point. Further, $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point, provided that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible.*

Proof. The proof follows similar lines to the proof of Theorem 3.1, using $L = 0$. ■

Corollary 3.5. *Let A and T be self-mappings of a complex quasi-partial metric space (X, qp_c). If the pair of mappings (A, T) satisfies*

$$q(Tx, Ty) \leq \delta max\left\{\begin{matrix} qp_c(Ax, Ay), pq_c(Ax, Tx), qp_c(Ay, Ty), \\ \frac{(qp_c(Tx, Ay) + qp_c(Ax, Ty))}{2} \end{matrix}\right\}$$

for all $x, y \in X$ *and we have*

1. $TX \subset AX$

2. AX *is closed, then the pair* (A, T) *has a coincidence point. Further,* A *and* T have a *unique common fixed point, provided that the pair* (A, T) *is weakly compatible.*

Proof. The proof follows similar lines to the proof of Theorem 3.1, using $= L = 0$, $a = B$ and $S = T$.

Corollary 3.6. *Let* A *and* T *be self-mappings of a complex quasi-partial metric space* (X, qp_c). *If the pair of mappings* (A, T) *satisfies*

$$(Tx, Ty) \leq \delta max\{qp_c(Ax, Ay)\}$$

for all $x, y \in X$ *and we have*

1. $TX \subset AX$

2. AX *is closed, then the pair* (A, T) *has a coincidence point. Further,* A *and* T *have a unique common fixed point, provided that the pair* (A, T) *is weakly compatible.*

Proof. The proof follows similar lines to the proof of Theorem 3.1.

Theorem 3.7. *Let* A, B, S *and* T *be self-mappings of a complex quasi-partial metric space* (X, qp_c). *If there exist* $\delta \in (0, 1)$ *and* $L \geq 0$, *such that for all* $x, y \in X$, *the pairs of mappings* (A, S) *and* (B, T) *satisfy*

$$q(sx, Ty) \leq \delta max\{qp_c(Ax, By), qp_c(Ax, Sx), qp_c(By, Ty), qp_c(Ax, Ty),$$
$$qp_c(Sx, By)\} + Lmin\{qp_c(Ax, Sx), qp_c(By, Ty), qp_c(Ax, Ty), qp_c(By, Sx)\}. \quad (3.2)$$

1. $TX \subset AX$ *and* $SX \subset BX$

2. $(\delta + 2L) < 1$, *then the pairs* (A, S) *and* (B, T) *have a coincidence point. Further,* A, B, S *and* T *have a unique common fixed point, provided that the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. It can be proved following similar arguments to those given in the proof of Theorem 3.1.

Example 3.8. Let $X = [0, \infty)$ be endowed with the complex quasi-partial metric:

$d_{qpc}(x, y) = |x - y| + i|x - y|$. and let A, B, S and T be mappings defined by

$$Ax = \begin{cases} x, & 0 \le x \le 1 \\ 2, & x > 1 \end{cases} \qquad Bx = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ 1, & x > 1, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{10}, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases} \qquad Tx = \begin{cases} \frac{x}{5}, & 0 \le x \le 1 \\ \frac{1}{2}, & x > 1, \end{cases}$$

Here we have

$$\overline{TX} = \left[0, \frac{1}{5}\right] \cup \left\{\frac{1}{2}\right\} \subset [0, 1] \cup \{2\} = AX$$

$$\overline{SX} = \left[0, \frac{1}{10}\right] \cup \{1\} \subset \left[0, \frac{1}{2}\right] \cup \{1\} = BX.$$

The point is a coincidence point of the four mappings. Further $AS0 = SA0 = 0$ and $TB0 = BT0 = 0$, i.e., the two pairs (A, S) and (B, T) are weakly compatible.

Case I. For $x, y \in [0, 1]$ we have

$$d_{qp_c}(Sx, Ty) = \left|\frac{x}{10} - \frac{y}{5}\right| + i\left|\frac{x}{10} - \frac{y}{5}\right|$$

$$= \frac{1}{10}\{|2x - y| + i|2x - y|\}$$

$$\le \frac{5}{9}\left\{\left|x - \frac{y}{2}\right| + i\left|x - \frac{y}{2}\right|\right\}.$$

Case II. For $x \in [0, 1]$ and $y > 1$, we have

$$d_{qp_c}(Sx, Ty) = \left|\frac{x}{10} - \frac{y}{2}\right| + i\left|\frac{x}{10} - \frac{y}{2}\right|$$

$$\leq \frac{5}{9}\left\{\left|1 - \frac{y}{2}\right| + i\left|1 - \frac{y}{5}\right|\right\}.$$

Case III. For $\in [0, 1]$ and $x > 1$, we have

$$d_{qp_c}(Sx, Ty) = \left|1 - \frac{y}{5}\right| + i\left|1 - \frac{1}{2}\right|$$

$$\leq \frac{5}{9}\left\{\left|2 - \frac{y}{2}\right| + i\left|2 - \frac{y}{5}\right|\right\}.$$

Case IV. For $x, y > 1$, we have

$$d_{qp_c}(Sx, Ty) = \left|1 - \frac{1}{2}\right| + i\left|1 - \frac{1}{2}\right|$$

$$\leq \frac{5}{9}\{1 + i\}.$$

Consequently, all hypotheses of Theorem 3.2 are satisfied (for $\delta = \frac{5}{9}$ and $L = 0$) and 0 is the unique common fixed point of A, B, S and T.

For $A = B$ and $S = T$ Theorem 3.2 reduces to following corollary

Corollary 3.9. *Let A and T be self-mappings of a complex quasi partial metric space* (X, qp_c). *If there exist* $\delta \in (0, 1)$ *and* $L \geq 0$, *such that for all* $x, y \in X$, *the pair of mappings* (A, T) *satisfies*

$q(Tx, Ty)$

$$\{qp_c(Ax, Ay), qp_c(Ax, Tx), qp_c(Ay, Ty), qp_c(Ax, Ty), qp_c(Tx, Ay)\}$$

$$\{qp_c(Ax, Tx), qp_c(Ay, Ty), qp_c(Ax, Ty), qp_c(Ay, Tx)\}. \tag{3.3}$$

$\leq \delta max$

$+ L\ min$

1. $TX \subset AX$,

2. $(\delta + 2L) < 1$, *then the pair* (A, T) *has a coincidence point. Further, A and T have a unique common fixed point, provided that the pair* (A, T) *is weakly compatible.*

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