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HYBRID TYPE FIXED POINT THEOREM ON GENERALIZED *b*-METRIC SPACE

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Abstract

The objective of the article is to introduce a FPT (fixed point theorem) for self-mapping on generalized b-metric space. To demonstrate this result, we have used F-contraction instead of simple contraction. This theorem is an extended version of existing results in b-metric space.

1. Introduction

Banach [4] in 1922 derived FPT for metric space, which is recognized as the well known theorem BCP (Banach Contraction Principle). Frechet [7] defined the notion of metric space in 1906. Many authors have generalized the idea of metric space like metric-like spaces [2], b-metric spaces [6], partial metric spaces [16], partial b-metric spaces [15]. In 1989, Bakhtin [3] and Bourbaki [13] gave the theory of b-metric space. Further, Czerwik [6] interpreted b-metric space in 1993 along with a perspective of generalization of BCP. Using generalisation of FPT in b-metric spaces, some authors proved their results on it ([8], [9]). Kamran [10] presented a new type of generalized metric space in 2017. Wardowski [17] gave a significant contraction abbreviated as 'F-contraction' in 2012 and derived a FPT regarding 'F-contraction'. A lot of work has been done by many authors on this theory using the F-contraction mapping and its extensions ([11], [12], [14]). Further, generalization of F-contraction was done by many authors and several FP

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theorems have been proved using F-contraction in various papers ([1], [14], [5]). Following are some of the definitions used in this paper.

2. Preliminaries (concepts and methods)

Definition 2.1 [7]. A mapping $h : G \times G \to R$ is known as a metric where G be any non-void set if

1. $h(i', j') > 0$ and $h(i', j') = 0$ iff $j' = i'$, for all j', i' belongs to G
2. $h(i', j') = h(j', i') \forall j', i'$ belongs to G
3. $h(i', j') + h(j', k') \geq h(i', k')$ for all k', j', i' belongs to G

where (G, h) be the metric space.

Definition 2.2 [6]. Suppose A be a non void set with $\lambda \geq 1$, then a mapping $h^* : A \times A \to [0, \infty)$ be known as b-metric, if $\forall k', j', i'$ belongs to A:

1. $h^*(i', j') = 0$ iff $j' = i'$
2. $h^*(i', j') = h^*(j', i')$
3. $h^*(i', k') \leq \lambda[h^*(i', j') + h^*(j', k')]$

where $(A, (i', j'), \lambda)$ is b-metric space.

Definition 2.3 [10] (2014). Consider $k_b : A \to [0, \infty)$ be function is known as an extended b-metric with $b : A \times A \to [1, \infty)$, if $\forall j', i', k'$ belongs to A, we have:

1. $k_b(i', j') = 0$ iff $j' = i'$
2. $k_b(i', j') = k_b(j', i')$
3. $k_b(i', j') \leq b(j', i')[k_b(i', j') + k_b(j', k')]$

where (A, k_b) is an extended b-metric space.

Definition 2.4 [17]. Consider a mapping $E : (0, \infty) \to R$ which satisfying, (F1) E is strictly increasing.

(F2) For each seq. (sequence) $\{p_m\}_{m \in N}$, $p_m = 0$ if and only if $\lim_{m \to \infty} F(p_m) = -\infty$.

(F3) $\exists l \in (0, 1)$ such that $\lim_{p \to 0^+} p^l F(p) = 0$.

A mapping $\mathcal{M} : A \to A$ is known as Wardowski F-contraction if $\exists\, \tau > 0$ such that $h^*(\mathcal{M}_{i'}, \mathcal{M}_{j'}) > 0 \Rightarrow \tau + F(h^*(\mathcal{M}_{i'}\mathcal{M}_{j'})) \leq F(h^*(i', j'))$ for all i', j' belongs to A.

3. Main Results

The main purpose of the current work is to derive a FPT for self-mapping on the generalized b-metric space using F-contraction. Also, in addition to the definition 1.4, we add one more condition on F i.e. F is continuous.

Lemma 3.1. *Consider (A, k_b) as an extended b-metric space with a convergence seq. $\{i'_n\}$ in A where $\lim_{n \to \infty} i'_n = i'$ then for $j' \in A$ we have*

$$k_b(i', j') \leq \lim_{n \to \infty} \inf k_b(i'_n, j') \leq \lim_{n \to \infty} \sup k_b(i'_n, a) \leq k_b(i', j').$$

Theorem 3.2. *Consider (A, k_b) be complete extended b-metric space and $\mathcal{M} : A \to A$ is F-contraction mapping, then $\mathcal{M}$ has Picard operator with unique fixed point.*

Proof. Assume $i'_0 \in A$. Consider the sequence $\{i'_n\}$ where $i'_n = \mathcal{M}i'_n$, $n = 1, 2, 3 \ldots.$ Denote $k_b(i'_n, i'_{n+1})$ by μ_n and let $\mu_n > 0 \forall n = 1, 2, 3 \ldots.$ Then, as $\mathcal{M}$ is a F-contraction mapping $\alpha < 1$, $\mathcal{M}i'_n \neq i'_n \forall_n \in N$, Then

$$\tau + F(k_b(\mathcal{M}i', \mathcal{M}j')) \leq \alpha[F(k_b(i', \mathcal{M}i')) + F(k_b(i', \mathcal{M}j')) + F(k_b(j', \mathcal{M}j))]. \quad (1)$$

Letting $i' = i'_n$ and $j' = i'_n$

$\mathcal{M}i' = \mathcal{M}i'_{n-1} = i'_n$,

$\mathcal{M}j' = \mathcal{M}i'_n = i'_{n+1}$.

Then Using (1), we have

$$\tau + F(k_b(\mathcal{M}i'_{n-1}, \mathcal{M}i'_n)) \leq \alpha[F(k_b(i'_{n-1}, \mathcal{M}i'_{n-1}))$$

$$+F(k_b(i'_{n-1}\mathcal{M}i'_n)) + F(k_b(i'_n, \mathcal{M}i'_n))]$$

$$\tau + F(k_b(i'_n, i'_{n+1})) \leq \alpha[F(k_b(i'_{n-1}, i'_n)$$

$$+F(k_b(i'_{n-1}, i'_{n+1})) + F(k_b(i'_n, i'_{n+1}))] \tag{2}$$

using Definition 2.3 in (2)

$$F(k_b(i'_{n-1}, i'_{n+1})) \leq b(i'_{n-1}, i'_{n+1})[F(k_b(i'_{n-1}, i'_n)) + F(k_b(i'_n, i'_{n+1}))]$$

$$\tau + F(k_b(i'_n, i'_{n+1})) \leq \alpha[F(k_b(i'_{n-1}, i'_n)) + b(i'_{n-1}, i'_{n+1})\{F(k_b(i'_{n-1}, i'_n))$$

$$+F(k_b(k_b(i'_n, i'_{n+1}))\} + F(k_b(i'_n, i'_{n+1}))].$$

Using $k_b(i'_n, i'_{n+1}) = \mu_n$

$$\tau + F(\mu_n) \leq \alpha[F(\mu_{n-1}) + b(i'_{n-1}, i'_{n+1})\{F(\mu_{n-1}) + F(\mu_n)\} + F(\mu_n)]$$

$$\tau + F(\mu_n) \leq \alpha F(\mu_{n-1}) + \alpha b(i'_{n-1}, i'_{n+1})F(\mu_{n-1})$$

$$+\alpha b(i'_{n-1}, i'_{n+1})F(\mu_n) + \alpha F(\mu_n)$$

$$F(\mu_n)(1 - \alpha[b(i'_{n-1}, i'_{n+1}) + 1]) \leq \alpha[1 + b(i'_{n-1}, i'_{n+1})]F(\mu_{n-1}) - \tau$$

$$F(\mu_n) \leq \frac{\alpha}{[1 - \alpha(b(i'_{n-1}, i'_{n+1}) + 1)]}\{F(\mu_{n-1})[1 + b(i'_{n-1}, i'_{n+1})]\}$$

$$- \frac{\tau}{[1 - \alpha(b(i'_{n-1}, i'_{n+1}) + 1)]}. \tag{3}$$

Similarly

$$F(\mu_{n-1}) \leq \frac{\alpha[1 + b(i'_{n-2}, i'_n)]}{[1 - \alpha(b(i'_{n-2}, i'_n) + 1)]}\{F(\mu_{n-2})[1 + b(i'_{n-2}, i'_n)]\}$$

$$- \frac{\tau}{[1 - \alpha(b(i'_{n-2}, i'_n) + 1)]}.$$

Putting value of $F(\mu_{n-1})$ in equation (3)

$$F(\mu_n) \leq \frac{\alpha}{[1-\alpha(b(i'_{n-1}, i'_{n+1})+1)]}\left\{\frac{\alpha^2[1+b(i'_{n-1}, i'_{n+1})][1+b(i'_{n-2}, i'_n)][1+b(i'_{n-3}, i'_{n-1})]}{[1-\alpha(b(i'_{n-2}, i'_n)+1)][1-\alpha(b(i'_{n-3}, i'_{n-1})+1)]}\right.$$

$$F(\mu_{n-3}) - \frac{\tau\alpha[1+b(i'_{n-1}, i'_{n+1})][1+b(i'_{n-2}, i'_n]}{[1-\alpha(b(i'_{n-2}, i'_n)+1)][1-\alpha(i'_{n-3}, i'_{n-1})+1)]}\Bigg\} - \frac{\tau}{[1-\alpha(b(i'_{n-2}, i'_n)+1]}$$

$$F(\mu_n) \leq \frac{\alpha^3[1+b(i'_{n-1}, i'_{n+1})][1+b(i'_{n-2}, i'_n)][1+b(i'_{n-3}, i'_{n-1}]}{[1-\alpha(b(i'_{n-1}, i'_{n+1})+1)][1-\alpha(b(i'_{n-2}, i'_n)+1)][1-\alpha(b(i'_{n-3}, i'_{n-1})+1]}$$

$$F(\mu_{n-3}) - \frac{\tau\alpha^2[1+b(i'_{n-1}, i'_{n+1})][1+b(i'_{n-2}, i'_n)]}{[1-\alpha(b(i'_{n-1}, i'_{n+1})+1)][1-\alpha(b(i'_{n-2}, i'_n)+1)][1-\alpha(b(i'_{n-3}, i'_{n-1})+1]}$$

$$- \frac{\tau}{[1-\alpha(b(i'_{n-2}, i'_n)+1]}$$

hence by induction

$$F(\mu_n) \leq \frac{\alpha^n[1+b(i'_{n-1}, i'_{n+1})][1+b(i'_{n-2}, i'_n)]\ldots[1+b(i'_0, i'_2)]}{[1-\alpha(b(i'_{n-1}, i'_{n+1})][1-\alpha(b(i'_{n-2}, i'_n)+1)]\ldots[1-\alpha(b(i'_0, i'_2)+1)]} F(\mu_0)$$

$$- \frac{\tau\alpha^{n-1}[1+b(i'_{n-1}, i'_{n+1})][1+b(i'_{n-2}, i'_n)]\ldots[1+b(i'_0, i'_2)]}{[1+b(i'_{n-1}i'_{n+1})][1-\alpha(b(i'_{n-2}, i'_n)+1)]\ldots[1-\alpha(b(i'_0, i'_2)+1)]}$$

$$- \frac{\tau n}{[1-\alpha(b(i'_{n-2}, i'_n)+1]}$$

$$F(\mu_n) \leq \alpha^n F(\mu_0) - \tau\alpha^{n-1} - \frac{\tau n}{[1-\alpha(b(i'_{n-2}, i'_n)+1)]}$$

$$F(\mu_n) \leq \alpha^n F(\mu_0) - \tau\alpha^{n-1} - \frac{\tau n}{[1-\alpha(b(i'_{n-2}, i'_n)+1)]}$$

$$\leq F(\mu_0) - \frac{\tau n}{[1-\alpha(b(i'_{n-2}, i'_n)+1)]}. \tag{4}$$

For the case $n \to \infty$, we get $\lim_{n\to\infty} F(\mu_n) = -\infty$ so that $\lim_{n\to\infty} \mu_n = 0$. From condition (F3), there exist $l \in (0, 1)$ such that $\lim_{n\to\infty} (\mu_n)^l F(\mu_n) = 0$. Multiplication of (4) with $(\mu_n)^l$ yields

$$0 \leq (\mu_n)^l F(\mu_n) + \frac{\tau n}{[1-\alpha(b(i'_{n-2}i'_n)+1)]}(\mu_n)^l \leq (\mu_n)^l F(\mu_0).$$

For the case n tends to ∞, we get $\lim_{n\to\infty} n(\mu_n)^l = 0$. Using the Theorem 3.2 in [11], it is easy to prove $\{i'_n\}$ as a Cauchy sequence $\exists w \in \mathcal{M}$ such that

$\lim_{n\to\infty} i'_n = \hat{e}$. Applying lemma (3.1) we get,

$$\lim_{n\to\infty} k_b(\hat{e}, i'_n) \le \lim_{n\to\infty} \sup k_b(\hat{e}, i'_n) \le k_b(\hat{e}, \hat{e}) = 0.$$

Also using equation (1), we have $\forall n \in N$

$$\tau + F(k_b(\mathcal{M}\hat{e}, \mathcal{M}i'_n)) \le \alpha[F(k_b(\hat{e}, \mathcal{M}\hat{e})) + F(k_b(\hat{e}, \mathcal{M}i'_n)) + F(k_b(i'_n, i'_{n+1}\hat{e}))].$$

Hence in the limit as $n \to \infty$ and using (2) and (3) we get

$$\tau + \lim_{n\to\infty} F(k_b(\mathcal{M}\hat{e}, \mathcal{M}i'_n)) \le -\infty$$

$$\Rightarrow \lim_{n\to\infty} F(g_b(\mathcal{M}\hat{e}, \mathcal{M}i'_{n+1}) \le -\infty.$$

This implies

$$\lim_{n\to\infty} k_b(\mathcal{M}\hat{e}, i'_{n+1}) = \lim_{n\to\infty} k_b(\mathcal{M}\hat{e}, \mathcal{M}i'_n) = 0.$$

Since the convergent sequence $\{i'_n\}$ converges to both $\hat{e}$ and $\mathcal{M}\hat{e}$, it must be the case that $\mathcal{M}\hat{e} = \hat{e}$.

$'\hat{e}'$ is a fixed point of $\mathcal{M}$.

Later, we have to prove fixed point is unique, let if possible, $\hat{e}'$ be another fixed point of $\mathcal{M}$ with $\hat{e} \ne \hat{e}'$. Then

$$\tau + F\, k_b(\mathcal{M}\hat{e}, \mathcal{M}\hat{e}') \le \alpha[F\, k_b(\hat{e}, \mathcal{M}\hat{e}) + F\, k_b(\hat{e}', \mathcal{M}\hat{e}')],$$

or

$$\tau + F(k_b(\hat{e}, \hat{e}')) \le \alpha F(k_b(\hat{e}, \hat{e}')) < F(k_b(\hat{e}, \hat{e}'))$$

or

$$F(k_b(\hat{e}, \hat{e}')) < F(k_b(\hat{e}, \hat{e}')) - \tau < F(k_b(\hat{e}, \hat{e}'))$$

$$F(k_b(\hat{e}, \hat{e}')) < \alpha F(k_b(\hat{e}, \hat{e}'))$$

which is not possible.

Hence $\mathcal{M}$ has a unique fixed point.

4. Conclusions

We have proved a FPT on generalized b-metric space for self mapping by using F-contraction. In this theorem, this condition is used on extended b-metric space to find Picard operator having a unique fixed point.

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