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SOLUTION OF THE ABEL INTEGRAL EQUATION OF SECOND KIND IN COUPLING OF LAPLACE TRANSFORM METHOD AND LEIBNITZ LINEAR DIFFERENTIAL EQUATION

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Abstract

In this paper, we introduce a numerical method for the solution of Abel integral equation of the second kind via Leibnitz linear differential equation by using Laplace transform method. The approach of the method is based on the assumption that the solution function is differentiable on $[0, \infty)$. The proposed method is very efficient and gives the exact solution of Abel integral equation of the second kind by passing through a simple integration process. Abel integral equation of the second kind occurs in several models in physical science, astrophysics, applied science etc. The numerical approach of the method is very simple and illustrates the accuracy, validity, and stability of the solution in form of the exact solution.

1. Introduction

Niels Henrik Abel was the first mathematician, who took the initiative of solving the integral equations of singular type in 1823. Abel integral equation

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[1] is one of the integral equations which is derived directly from a concrete problem of physics, without passing through a differential equation. It appears in several models in astrophysics, solid mechanics, physical sciences and applied sciences. N. Zeilon [2] in 1924, gave an idea of the solution of Abel integral equation on a finite segment. The various methods for solving Abel integral equation and fractional differential equations are given in [3-5]. He [6-7] developed the homotopy perturbation transform method. Numerical solutions of Abel integral equations are given in [8, 23, 28, 29] by using the different Wavelets methods. The homotopy perturbation transform method is used in [9-11, 19]. The different polynomial methods are given in [13, 14]. Various transform methods are used in [12, 18, 21-22]. Analytical solutions are given in [16, 17]. Fractional calculus is used in [20]. The variational iteration method is used in [27]. The Laplace decomposition method is used in [15]. Application of Chebyshev polynomials for solving the Abel integral equation of the first and second kind is given in [24]. The Taylor-Collocation method is discussed in [25]. The solution of Abel integral equation using the differential transform method is given in [26]. An approximate analytical solution of the generalized Abel integral equation of second kind in coupling of various transform methods is discussed in [30].

The main purpose of this paper is to introduce a numerical approach for solution for Abel integral equation of the second kind in coupling of Laplace transform with Leibnitz linear differential equation.

2. Some Basic Definitions and Terminologies

Definition 2.1. Laplace transform: The Laplace transform [22] of function a $f(t)$ defined for $t \geq 0$ is defined by the improper integral

$$L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) dt, \text{ provided this integral converges for some } s.$$

Definition 2.2. Inverse Laplace transform: If $L\{f(t); s\} = F(s)$ then the function $f(t) = L^{-1}\{F(s)\}$ is the inverse Laplace transform of $F(S)$.

Definition 2.3. Convolution property of Laplace transforms: If $L\{f(t); s\} = F(s)$ and $L\{g(t); s\} = G(s)$ then $L\{(f(t) * g(t)); s\} = F(s). G(s)$, where $f(t) * g(t) = \int_0^t f(u)g(t-u)du,$ is called the finite convolution of $f(t)$ and $g(t)$.

Definition 2.4. Leibnitz linear differential equation and its solution: An ordinary differential equation of the form $f'(t) + P(t)f(t) = Q(t),$ where P, Q are function t only is called Leibnitz linear differential equation, which is a special case of Bernoulli equation. One parameter solution $f(t)$ of this differential equation is given by

$$f(t) = e^{-\int P(t)dt}\left(\int Q(t)e^{\int P(t)dt}dt + c\right), \text{ where } c \text{ is integration constant.}$$

Definition 2.5. Laplace transform of the derivative of a function: The Laplace transform of derivative of function $f(t)$ is defined by the formula

$$L\{f'(t); s\} = sL\{f(t); s\} - f(0).$$

Remark 2.1. $\int_0^t \frac{u^\beta}{\sqrt{t-u}} du = \frac{\sqrt{\pi}\sqrt{(\beta+1)}}{\sqrt{\left(\beta+\frac{3}{2}\right)}} t^{\beta+\frac{1}{2}},$ where $\beta > -1.$

2.2. $L\left\{\int_0^t f(u)du; s\right\} = \frac{1}{s}. L\{f(t); s\}.$

3. Description of the Numerical Method for the Solution for Abel Integral Equation of the Second Kind

Consider the Abel integral equation of the second kind in general form as

$$f(t) = g(t) + \tau\int_0^t \frac{f(u)}{\sqrt{t-u}} du, \ \tau \text{ is non-zero constant and } t \geq 0, \tag{1}$$

where $g(t)$ is known and $f(t)$ is unknown function.

We can write the equation (1) in form

$$f(t) - g(t) = \tau\int_0^t \frac{f(u)}{\sqrt{t-u}} du = \tau(f(t) * t^{\frac{-1}{2}}). \tag{2}$$

Operating Laplace transform on the both sides of the equation (2), we have

$$L\{f(t) - g(t); s\} = \tau L\left\{\int_0^t \frac{f(u)}{\sqrt{t-u}}\, du\right\} = \tau L\left\{f(t) * t^{\frac{-1}{2}}; s\right\}.$$

By using the convolution property of Laplace transforms, we get

$$L\{f(t) - g(t); s\} = \tau L\{f(t); s\} L\{t^{\frac{-1}{2}}; s\} = \tau L\{f(t)\}\sqrt{\frac{\pi}{s}},$$

$$\Rightarrow \tau L\{f(t); s\} = L\{f(t) - g(t); s\}\sqrt{\frac{s}{\pi}},$$

$$\Rightarrow \tau L\{f(t); s\} = \frac{s}{\pi} L\{f(t) - g(t); s\}\sqrt{\frac{\pi}{s}}, \tag{3}$$

Again, using convolution property of Laplace transforms in equation (3), we have

$$\tau L\{f(t); s\} = \frac{s}{\pi} L\left\{\int_0^t \frac{(f(u) - g(u))}{\sqrt{t-u}}\, du; s\right\}. \tag{4}$$

Define $F(t) = \int_0^t \frac{(f(u) - g(u))}{\sqrt{t-u}}\, du,$ and so $F(0) = 0.$

Since the Laplace transform of the derivative of $F(t)$ is given by

$$L\{F'(t); s\} = sL\{F(t); s\} - F(0).$$

$$\Rightarrow L\{F'(t); s\} = sL\{F(t); s\} \text{ as } F(0) = 0. \tag{5}$$

Now from the equation (4) and (5), we have

$$\tau L\{f(t); s\} = \frac{1}{\pi} L\{F'(t); s\} = \frac{1}{\pi} L\left\{\frac{d}{dt}\int_0^t \frac{(f(u) - g(u))}{\sqrt{t-u}}\, du; s\right\}. \tag{6}$$

Taking inverse Laplace transform on both sides in the equation (6), we have

$$\tau\pi f(t) = \frac{d}{dt}\int_0^t \frac{f(u)}{\sqrt{t-u}}\, du - \frac{d}{dt}\int_0^t \frac{g(u)}{\sqrt{t-u}}\, du. \tag{7}$$

On making use of the equation (2) in equation (7), we have

$$\tau\pi f(t) = \frac{1}{\tau}\{f'(t) - g'(t)\} - \frac{d}{dt}\int_0^t \frac{g(u)}{\sqrt{t-u}}\,du. \tag{8}$$

$$= \frac{1}{\tau}f'(t) - \frac{1}{\tau}g'(t) - \frac{d}{dt}\int_0^t \frac{g(u)}{\sqrt{t-u}}\,du.$$

$$\Rightarrow f'(t) - \tau^2\pi f(t) = g'(t) + \tau\frac{d}{dt}\int_0^t \frac{g(u)}{\sqrt{t-u}}\,du \tag{9}$$

which is a Leibnitz type linear differential equation.

The one parameter solution of the eq. (9) is given by

$$f(t) = e^{\pi\tau^2 t}\int [\frac{d}{dt}\{g(t) + \tau\int_0^t \frac{g(u)}{\sqrt{t-u}}\,du\}]e^{-\pi\tau^2 t}dt + ce^{\pi\tau^2 t}. \tag{10}$$

where c is integration constant given by taking $t = 0$ in eq. (1) and (10).

4. Uniqueness of the Solution obtained with Exact Solution for Abel Integral Equation of the Second Kind

From equation (9) whose solution is given by equation (10), by taking Laplace transform, we have

$$L\{f'(t); s\} - \tau^2\pi L\{f(t); s\} = L\{g'(t); s\} + \tau L\left\{\frac{d}{dt}\int_0^t \frac{g(u)}{\sqrt{t-u}}\,du; s\right\}.$$

$$sL\{f(t); s\} - f(0) - \tau^2\pi L\{f(t); s\} = sL\{g(t); s\} - g(0) + \tau sL\left\{\int_0^t \frac{g(u)}{\sqrt{t-u}}\,du; s\right\}.$$

Since from equation (1), $f(0) = g(0)$ and so

$$sL\{f(t); s\} - \tau^2\pi L\{f(t); s\} = sL\{g(t); s\} - g(0) + \tau sL\{g(t); s\}L\{\frac{1}{\sqrt{t}}; s\}$$

$$\Rightarrow (s - \tau^2\pi)L\{f(t); s\} = (s + \tau\sqrt{\pi s})L\{g(t); s\}$$

$$\Rightarrow L\{f(t); s\} = \frac{(s + \tau\sqrt{\pi s})}{(s - \tau^2\pi)}L\{g(t); s\}$$

$$\Rightarrow f(t) = L^{-1}\left[\frac{\sqrt{s}}{(\sqrt{s} - \tau\sqrt{\pi})} L\{g(t); s\}\right] \tag{11}$$

which is the exact solution of the equation (1) and so solution of the equation (1) given by (10) and (11) is unique.

5. Numerical Implementation of the Method

In this section, we shall test the applicability of the newly proposed method by several examples.

Example 5.1. Consider the Abel integral equation of second kind as follows [9, 13, 17, 19, 24, 30]

$$f(t) = t + \frac{4}{3} t^{\frac{3}{2}} - \int_0^t \frac{f(u)}{\sqrt{t-u}} du, \text{ with exact solution } t. \tag{12}$$

The solution of the equation (12) is given by

$$f(t) = e^{\pi\tau^2 t} \int [\frac{d}{dt}\{g(t) + \tau \int_0^t \frac{g(u)}{\sqrt{t-u}} du\}] e^{-\pi\tau^2 t} dt + ce^{\pi\tau^2 t}, \text{ where } \tau = -1 \text{ and}$$

$g(t) = t + \frac{4}{3} t^{\frac{3}{2}}$ and so

$$f(t) = e^{\pi t} \int [\frac{d}{dt}\{(t + \frac{4}{3} t^{\frac{3}{2}}) - \int_0^t \frac{(u + \frac{4}{3} u^{\frac{3}{2}})}{\sqrt{t-u}} du\}] e^{-\pi t} dt + ce^{\pi t}$$

$$= e^{\pi t} \int [\frac{d}{dt}\{(t + \frac{4}{3} t^{\frac{3}{2}}) - (\frac{4}{3} t^{\frac{3}{2}} + \frac{\pi}{2} t^2)\}] e^{-\pi t} dt + ce^{\pi t}$$

$$= e^{\pi t} \int (1 - \pi t) e^{-\pi t} dt + ce^{\pi t}$$

$$= e^{\pi t}(e^{-\pi t} t) + ce^{\pi t}$$

$$= t + ce^{\pi t}. \tag{13}$$

Putting $t = 0$ in equations (12) and (13), we have $c = 0$.

Hence $f(t) = t$, which is the exact solution.

Example 5.2. Consider the Abel integral equation of second kind as follows [9, 13, 14, 17, 19, 23, 24, 25, 29]

$$f(t) = t^2 + \frac{16}{15} t^{\frac{5}{2}} - \int_0^t \frac{f(u)}{\sqrt{t-u}}\, du, \text{ with exact solution } t^2. \tag{14}$$

The solution of the equation (14) is given by

$$f(t) = e^{\pi\tau^2 t} \int [\frac{d}{dt}\{g(t) + \tau\int_0^t \frac{g(u)}{\sqrt{t-u}}\, du\}] e^{-\pi\tau^2 t} dt + ce^{\pi\tau^2 t}, \text{ where } \tau = -1 \text{ and}$$

$g(t) = t^2 + \frac{16}{15} t^{\frac{5}{2}}$, and so

$$f(t) = e^{\pi t} \int [\frac{d}{dt}\{(t^2 + \frac{16}{15} t^{\frac{5}{2}}) - \int_0^t \frac{(u^2 + \frac{16}{15} u^{\frac{5}{2}})}{\sqrt{t-u}} du\}] e^{-\pi t} dt + ce^{\pi t}$$

$$= e^{\pi t} \int [\frac{d}{dt}\{(t^2 + \frac{16}{15} t^{\frac{5}{2}}) - (\frac{16}{15} t^{\frac{5}{2}} + \frac{\pi}{3} t^3)\}] e^{-\pi t} dt + ce^{\pi t}$$

$$= e^{\pi t} \int (2t - \pi t^2) e^{-\pi t} dt + ce^{\pi t}$$

$$= t^2 + ce^{\pi t}. \tag{15}$$

Putting $t = 0$ in equation (14) and (15), we have $c = 0$.

$f(t) = t^2$, which is the exact solution.

Example 5.3. Consider Abel integral equation of second kind as follows [9]

$$f(t) = \frac{\pi}{2} t + \sqrt{t} - \int_0^t \frac{f(u)}{\sqrt{t-u}}\, du, \text{ with exact solution } \sqrt{t}. \tag{16}$$

The solution of the equation (16) is given by

$$f(t) = e^{\pi\tau^2 t} \int [\frac{d}{dt}\{g(t) + \tau\int_0^t \frac{g(u)}{\sqrt{t-u}}\, du\}] e^{-\pi\tau^2 t} dt + ce^{\pi\tau^2 t}, \text{ where } \tau = -1 \text{ and}$$

$g(t) = \frac{\pi}{2} t + \sqrt{t}$ and so

$$f(t) = e^{\pi t}\int[\frac{d}{dt}\{(\frac{\pi t}{2} + \sqrt{t}) - \int_0^t \frac{(\frac{\pi}{2}u + \sqrt{u}}{\sqrt{t-u}}du\}]e^{-\pi t}dt + ce^{\pi t}$$

$$= e^{\pi t}\int[\frac{d}{dt}\{(\frac{\pi t}{2} + \sqrt{t}) - (\frac{2}{3}\pi t^{\frac{3}{2}} + \frac{\pi t}{2}\ )\}]e^{\pi t}dt + ce^{\pi t}$$

$$= e^{\pi t}\int(\frac{1}{2\sqrt{t}} + \pi\sqrt{t})e^{-\pi t}dt + ce^{\pi t}$$

$$= e^{\pi t}(e^{-\pi t}\sqrt{t}) + ce^{\pi t}$$

$$= \sqrt{t} + ce^{\pi t}. \tag{17}$$

Putting $t = 0$ in equation (16) and (17), we have $c = 0$.

Hence $f(t) = \sqrt{t}$, which is the exact solution.

Example 5.4. Consider the Abel integral equation of second kind as follows [9, 13, 17, 19, 23, 26, 30]

$$f(t) = 2\sqrt{t} - \int_0^t \frac{f(u)}{\sqrt{t-u}}du, \text{ with exact solution } 1 - e^{\pi t}erfc(\sqrt{t}) \tag{18}$$

and the complementary error function $erfc(t)$ is defined as $erfc(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-u^2}du.$

The solution of the equation (18) is given by

$$f(t) = e^{\pi\tau^2 t}\int[\frac{d}{dt}\{g(t) + \tau\int_0^t \frac{g(u)}{\sqrt{t-u}}du\}e^{-\pi\tau^2 t}dt + ce^{\pi\tau^2 t},$$ where $\tau = -1$ and

$g(t) = 2\sqrt{t}$ and so

$$f(t) = e^{\pi t}\int[\frac{d}{dt}\{2\sqrt{t} - \int_0^t \frac{(2\sqrt{u})}{\sqrt{t-u}}du\}]e^{-\pi t}dt + ce^{\pi t}$$

$$= e^{\pi t}\int[\frac{d}{dt}\{2\sqrt{t} - \pi t\}]e^{-\pi t}dt + ce^{\pi t}$$

$$= e^{\pi t}\int\left(\frac{1}{\sqrt{t}} - \pi\right)e^{-\pi t}dt + ce^{\pi t}$$

$$\Rightarrow f(t) = \left(1 + 2\sqrt{t} + \frac{4}{3}\pi t^{\frac{3}{2}} + \frac{8}{15}\pi^2 t^{\frac{5}{2}} + \cdots\right) + ce^{\pi t}. \tag{19}$$

Putting $t = 0$ in equations (18) and (19), we have $c = -1$.

$$f(t) = \left(1 + 2\sqrt{t} + \frac{4}{3}\pi t^{\frac{3}{2}} + \frac{8}{15}\pi^2 t^{\frac{5}{2}} + \cdots\right) - e^{\pi t}$$

$$= 1 - (1 - 2\sqrt{t} + \pi t - \frac{4}{3}\pi t^{\frac{3}{2}} + \frac{1}{2}\pi^2 t^2 - \frac{8}{15}\pi^2 t^{\frac{5}{2}} + \cdots\cdots\cdots)$$

$$= 1 - \sum_{i=0}^{\infty} \frac{(-1)^i (\pi t)^{\frac{i}{2}}}{\Gamma(1 + \frac{i}{2})}$$

$$= 1 - E_{\frac{1}{2}}(-\sqrt{\pi t})$$

where $E_\alpha(Z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha + 1)}$, $(\alpha > 0)$ is the Mittag-Leffler function of one parameter α.

Hence $f(t) = 1 - e^{\pi t} erfc(\sqrt{t})$, which is the exact solution.

Example 5.5. Consider the Abel integral equation of second kind as follows [15]

$$f(t) = \sqrt{t} - \pi t + 2\int_0^t \frac{f(u)}{\sqrt{t - u}}\, du, \text{ with exact solution } \sqrt{t}. \tag{20}$$

The solution of the equation (20) is given by

$$f(t) = e^{\pi\tau^2 t} \int [\frac{d}{dt}\{g(t) + \tau\int_0^t \frac{g(u)}{\sqrt{t-u}}\, du\}] e^{-\pi\tau^2 t} dt + ce^{\pi\tau^2 t},$$

where $\tau = 2$ and $g(t) = \sqrt{t} - \pi t$.

The solution of the equation (20) is given by

$$f(t) = e^{4\pi t}\int [\frac{d}{dt}\{\sqrt{t} - \pi t) + 2\int_0^t \frac{(\sqrt{u} - \pi u)}{\sqrt{t-u}}\,du\}]e^{-4\pi t}dt + ce^{4\pi t}$$

$$= e^{4\pi t}\int [\frac{d}{dt}\{\sqrt{t} - \pi t) + 2(\frac{\pi t}{2} - \frac{4}{3}\pi t^{\frac{3}{2}})\}]e^{-4\pi t}dt + ce^{4\pi t}$$

$$= e^{4\pi t}\int (\frac{1}{2\sqrt{t}} - 4\pi\sqrt{t})e^{-4\pi t}dt + ce^{4\pi t}$$

$$= e^{4\pi t}(e^{-4\pi t}\sqrt{t}) + ce^{4\pi t}$$

$$= \sqrt{t} + ce^{4\pi t}. \tag{21}$$

Putting $t = 0$ in equation (20) and (21), we have $c = 0$.

Hence $f(t) = \sqrt{t}$, which is the exact solution.

Example 5.6. Consider the Abel integral equation of second kind as follows [15]

$$f(t) = \frac{1}{2} - \sqrt{t} + \int_0^t \frac{f(u)}{\sqrt{t-u}}\,du, \text{ with exact solution } \frac{1}{2}. \tag{22}$$

6. Conclusion

The main aim of present work is to introduce an accurate and efficient method for solving the Abel integral equation of the second kind. The approach of the method is based on finding a computational formula for the solution of Abel integral equation of the second kind by using the Laplace transform method and Leibnitz linear differential equation.

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