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ENERGY OF EULER TOTIENT CAYLEY GRAPH

M. VENKATA ANUSHA, M. SIVA PARVATHI, K. MANJULA and G. S. SHANMUGA PRIYA

Department of Applied Mathematics
Sri Padmavati Mahila Visvavidyalayam
Tirupati Andhra Pradesh, India
E-mail: anuanusha648@gmail.com
parvathimani2008@gmail.com
manjula.karre77@gmail.com
gspriya720@gmail.com

Abstract

The notation $Z_n = \{0, 1, 2, 3, \ldots, n\}$, where n is a positive integer. Let S be the set of all integers which are less than and relatively prime to n. Then the Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is Z_n and the edge set $E = \{(x, y)/x - y \in S \text{ or } y - x \in S\}$, where φ is Euler totient function.

Energy is nothing but the sum of the absolute values of all eigen values of a graph G and the Matrix Energy of G is defined as the summation of absolute values the of all singular values of a graph G. Motivated this, authors are interested to find the Energy and Matrix Energy of Euler totient Cayley graph. In this paper the Spectrum, Energy, Matrix Energy and Hyper Energy of an Euler totient Cayley graph are discussed.

1. Introduction

In 1878, Cayley introduced a graphic representation of abstract groups and defined $\Gamma_C(G, S)$ in which one color corresponds to each member of S and the vertex set $g \in G$ is joined to $sg \in G$ by an edge of color of S. Babai [3] was redefined the definition and symmetry of Cayley graph by ignoring colors and orientation of the edges, it was denoted by $\Gamma(G, S)$. Cayley graphs

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have applications particularly in design of interconnection networks and rearrangement problems. In the year 1990, Dejter [12] defined multi colored subgraphs of complete Cayley graphs and again in the year 1995, he introduced Unitary Cayley graph [13]. After that some researchers Bierrizbeitia et al., [5] studied various parameters on Unitary Cayley graph.

Euler totient Cayley graph is a Cayley graph associated with Euler totient Function. In 2003, Madhavi [14] introduced Euler totient Cayley graph and studied some basic properties. Euler totient Cayley graph is quite interesting to study because of its structural representation and vibrant properties.

The idea of Energy of a Graph was presented by Gutman [7] in 1978. Graph energy is calculated to identify the Π-electron energy determined inside the Hückel atomic orbital approximation [6, 8]. An incredible assortment of graph energies is being considered in the current numerical science. It can be used to approximate the complete π-electron energy of a molecule [10]. This range-based graph invariant has been quite contemplated in both substance and scientific writing. Gutman [9] established some results related to the energy of trees. The graph is said to be Hyper energetic if its energy is greater than $2n - 2$ and this concept was introduced by Gutman [11] in 1999.

The concept of energy of chemical compounds in the area of chemical graph theory has its own importance. Nikiforov [15] extended the concept of energy of a graph G as matrix energy of G by observing the relationship between eigen values and singular values of an adjacency matrix of a G. Upper and lower bounds on energy of a graph and also asymptotics of energy for almost all graphs are established.

Authors are motivated by these two concepts, Energy and matrix Energy of a Euler totient Cayley graph are discussed and some of the results are developed. The notations used in this paper can be found in [1, 2, 4].

2. Euler Totient Cayley Graph and its Properties

Definition. Let n be a positive integer and $(Z_n \oplus_n)$ is an additive group of integers modulo n. Let S be the set of all positive integers which are relatively prime to n and less than n. That is $S = \{a/1 \le a \le n$ and $\gcd(a, n) = 1\}$. Then $|S| = \varphi(n)$, where φ is Euler totient function. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is $Z_n = \{0, 1, 2, 3, \ldots, n-1\}$ and the edge set $E = \{(x, y)/x - y \in S$ or $y - x \in S\}$.

The following are basic properties of Euler totient Cayley graph studied by Madhavi [14].

Lemma 2.1. *The graph* $G(Z_n, \varphi)$ *is connected and simple.*

Lemma 2.2. *The graph* $G(Z_n, \varphi)$ *is* $\varphi(n)$*- regular and its size is* $\frac{n\varphi(n)}{2}$.

Lemma 2.3. *If n is prime, the graph* $G(Z_n, \varphi)$ *is complete graph.*

Lemma 2.4. *If n is even, the graph* $G(Z_n, \varphi)$ *is bipartite.*

Lemma 2.5. *If* $n \ge 3$, *the graph* $G(Z_n, \varphi)$ *is Eulerian.*

Lemma 2.6. *The graph* $G(Z_n, \varphi)$ *is Hamiltonian.*

3. Energy and Matrix Energy of Euler Totient Cayley Graph

Let $G(Z_n, \varphi)$ be Euler totient Cayley graph with n vertices. Let $A = (a_{ij})$ be the adjacency matrix of $G(Z_n, \varphi)$ is defined by its entries as $a_{ij} = 1$, if two vertices are adjacent in $G(Z_n, \varphi)$ and 0 otherwise and $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \ldots \ge \lambda_n$ are the eigen values of $A(G(Z_n, \varphi))$. All eigen values with their corresponding multiplicities is the spectrum of $G(Z_n, \varphi)$. The Energy of the graph is the sum of the absolute values of the eigen values of $G(Z_n, \varphi)$. That is $\varepsilon(G(Z_n, \varphi)) = \sum_{i=1}^{n} |\lambda_i|$.

Let $A(G(Z_n, \varphi))A(G(Z_n, \varphi))'$ is a positive semi definite matrix, where $A(G(Z_n, \varphi))'$ is the transpose of $A(G(Z_n, \varphi))$. Let the singular values of $A(G(Z_n, \varphi))$ be $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ which are the square root values of eigen values of $A(G(Z_n, \varphi))A(G(Z_n, \varphi))'$ and these are taken in non-increasing order $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_n$. The Matrix Energy of $G(Z_n, \varphi)$ is denoted by $\varepsilon_m(G(Z_n, \varphi))$ and is defined as the summation of absolute values of singular values of $G(Z_n, \varphi)$. That is $\varepsilon_m (G(Z_n, \varphi)) = \sum_{i=1}^{n} | \sigma_i |$.

Theorem 3.1. *The energy of* $G(Z_n, \varphi)$ *is* $2\varphi(p)$, *where p is prime.*

Proof. Consider an Euler totient Cayley graph $G(Z_n, \varphi)$, with vertex set $V = \{0, 1, 2, \ldots, p-1\}$, where p is prime. The adjacency matrix of $G(Z_n, \varphi)$ is

$$A(G(Z_n, \varphi)) = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{pmatrix}_{p \times p}.$$

The characteristic equation of the above matrix is

$$(\lambda + 1)^{(p-1)}(\lambda - (p-1)) = 0.$$

The eigen values are -1 and $(p-1)$ and the corresponding multiplicities are $(p-1)$ and 1. Therefore, the spectrum of the graph $G(Z_n, \varphi)$ is $\begin{pmatrix} -1 & p-1 \\ p-1 & 1 \end{pmatrix}$.

Then $\varepsilon(G(Z_p, \varphi)) = \sum_{i=1}^{n} | \lambda_i | = | -1 | (p-1) + | (p-1) | (1) = 2(p-1)$.

Thus $\varepsilon(G(Z_p, \varphi)) = 2\varphi(p)$, where $\varphi(p) = (p-1)$, p be a prime.

Corollary 3.2. *For every prime p, the matrix energy of* $G(Z_p, \varphi)$ *is* $2\varphi(p)$.

Proof. Consider an Euler totient Cayley graph $G(Z_p, \varphi)$ with vertex set $V = \{0, 1, 2, \ldots, p-1\}$, where p is prime.

From Theorem 3.1, we have

$$A(G(Z_p, \varphi)) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{p\times p}.$$

The transpose of the $A(G(Z_n, \varphi))$ is

$$A(G(Z_p, \varphi))' = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{p\times p}.$$

Then $A(G(Z_p, \varphi))A(G(Z_p, \varphi))' = \begin{pmatrix} n-1 & n-2 & n-2 & \dots & n-2 \\ n-2 & n-1 & n-2 & \dots & n-2 \\ n-2 & n-2 & n-1 & \dots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & n-2 & \dots & n-1 \end{pmatrix}_{p\times p}$. The characteristic equation is $(\sigma-1)^{(p-1)}(\sigma-(p-1)^2) = 0$, where σ denotes the eigen value of $A(G(Z_p, \varphi))A(G(Z_p, \varphi))'$ and the singular value of $A(G(Z_p, \varphi))$.

Then the singular values are 1 and $(p-1)$ and the corresponding multiplicities are $(p-1)$ and 1.

Therefore, the spectrum of the graph $G(Z_p, \varphi)$ is $spec(G(Z_p, \varphi)) = \begin{pmatrix} 1 & p-1 \\ p-1 & 1 \end{pmatrix}$. Then $\varepsilon_m(G(Z_p, \varphi)) = \sum_{i=1}^{n} |\sigma_i| = 2(p-1)$.

Thus $\varepsilon_m(G(Z_n, \varphi)) = 2\varphi(p)$ where $\varphi(p) = (p-1)$.

Theorem 3.3. *The energy of* $G(Z_{p^\alpha}, \varphi)$ *is* $2\varphi(p^\alpha)$, *where p is prime and* $\alpha > 1$.

Proof. Consider the graph $G(Z_{p^\alpha}, \varphi)$, where p is prime and $\alpha > 1$.

Then the vertex set $V = \{0, 1, 2, \ldots, p^{\alpha} - 1\}$.

The adjacency matrix of $G(Z_{p^{\alpha}}, \varphi)$ is

$$A(G(Z_{p^{\alpha}}, \varphi)) = \begin{pmatrix} Q & \ldots & Q \\ \vdots & \ddots & \vdots \\ Q & \ldots & Q \end{pmatrix}_{p^{\alpha} \times p^{\alpha}},$$

where $Q = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{pmatrix}_{p \times p}$ $(p^{\alpha} times)$.

Then the characteristic equation of $A(G(Z_{p^{\alpha}}, \varphi))$ is $(\lambda + p^{\alpha-1})^{(p-1)} (\lambda)^{(p^{\alpha}-p)}(\lambda - (p^{\alpha-1})(p-1)) = 0$, and the eigen values are $\lambda = -p^{\alpha-1}$, 0 and $(p^{\alpha-1} - 1)(p-1)$, their corresponding multiplicities are $(p-1)$, $(p^{\alpha} - p)$ and 1.

Therefore the spectrum of $G(Z_{p^{\alpha}}, \varphi)$, is $\begin{pmatrix} -p^{\alpha-1} & 0 & (p^{\alpha} - p)(p-1) \\ (p-1) & (p^{\alpha} - p) & 1 \end{pmatrix}$. Then the Energy of $G(Z_{p^{\alpha}}, \varphi)$, is

$$\varepsilon(G(Z_{p^{\alpha}}, \varphi)) = \sum_{i=1}^{n} |\ \lambda_i\ |$$

$$= |\ -p^{\alpha-1}\ |(p-1) + |\ 0\ |(p^{\alpha} - p)) + |\ (p^{\alpha-1})(p-1)\ |(1)$$

$$= 2p^{\alpha-1}(p-1).$$

Thus $\varepsilon(G(Z_{p^{\alpha}}, \varphi)) = 2\varphi(p^{\alpha})$, where $\varphi(p^{\alpha}) = p^{\alpha-1}(p-1)$, p is prime.

Corollary 3.4. *The Matrix energy of* $G(Z_{p^{\alpha}}, \varphi)$, *is* $2\varphi(p^{\alpha})$, *where p is prime and* $\alpha > 1$.

Proof. Consider the graph $G(Z_{p^{\alpha}}, \varphi)$, where p is prime and $\alpha > 1$. Then the vertex set $V = \{0, 1, 2, \ldots, p^{\alpha} - 1\}$.

From Theorem 3.3, the adjacency matrix of $G(Z_{p^{\alpha}}, \varphi)$, is

$$A(G(Z_{p^{\alpha}}, \varphi)) = \begin{pmatrix} Q & \ldots & Q \\ \vdots & \ddots & \vdots \\ Q & \ldots & \ldots \end{pmatrix}_{p^{\alpha} \times p^{\alpha}},$$

where

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{pmatrix}_{p \times p} \quad (p^{\alpha} times).$$

Then the transpose of $A(G(Z_{p^{\alpha}}, \varphi))' = \begin{pmatrix} Q & \ldots & Q \\ \vdots & \ddots & \vdots \\ Q & \ldots & \ldots \end{pmatrix}_{p^{\alpha} \times p^{\alpha}}$. And

$$A(G(Z_{p^{\alpha}}, \varphi))A(G(Z_{p^{\alpha}}, \varphi))' = \begin{pmatrix} T & \ldots & T \\ \vdots & \ddots & \vdots \\ T & \ldots & T \end{pmatrix}_{p^{\alpha} \times p^{\alpha}},$$

where $T = \begin{pmatrix} n-p & n-2p & n-2p & \ldots & n-2p \\ n-2p & n-p & n-2p & \ldots & n-2p \\ n-2p & n-2p & n-p & \ldots & n-2p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2p & n-2p & n-2p & \ldots & n-p \end{pmatrix}_{p \times p} \quad (p^{\alpha} times).$

Then the characteristic equation is

$$(\sigma + p^{\alpha-1})^2)^{(p-1)}(\sigma)^{(p^{\alpha}-p)}(\sigma - (p^{\alpha-1})(p-1)^2) = 0.$$

So, the singular values are $p^{\alpha-1}$, 0 and $(p^{\alpha-1} - 1)(p-1)$, their corresponding multiplicities are $(p-1)$, $(p^{\alpha} - p)$ and 1.

Therefore the spectrum of the graph $G(Z_{p^{\alpha}}, \varphi)$ is

$\begin{pmatrix} p^{\alpha-1} & 0 & (p^{\alpha}-p)(p-1) \\ (p-1) & (p^{\alpha}-p) & 1 \end{pmatrix}$. Then the Matrix energy of $G(Z_{p^{\alpha}}, \varphi)$ is

$$\varepsilon_m(G(Z_{p^{\alpha}}, \varphi)) = \sum_{i=1}^{n} |\ \sigma_i\ |$$

$$= |\ p^{\alpha-1}\ |(p-1) + |\ 0\ |(p^{\alpha}-p)) + |\ (p^{\alpha-1})(p-1)\ |(1)$$

$$= 2p^{\alpha-1}(p-1).$$

Thus, $\varepsilon_m(G(Z_{p^{\alpha}}, \varphi)) = 2\varphi(p^{\alpha})$ where $\varphi(p^{\alpha}) = p^{\alpha-1}(p-1)$.

Theorem 3.5. *The energy of* $G(Z_{2p}, \varphi)$ *is* $4(p-1)$, *where p is prime.*

Proof. Consider an Euler totient Cayley graph $G(Z_{2p}, \varphi)$ with vertex set $V = \{0, 1, 2, \ldots, 2p-1\}$, where p is prime.

The adjacency matrix of $G(Z_{2p}, \varphi)$ is $A(G(Z_{2p}, \varphi)) = \begin{pmatrix} R & S \\ S & R \end{pmatrix}_{2p\times 2p}$,

where $R = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{pmatrix}_{p\times p}$ and $S = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{pmatrix}_{p\times p}$.

The characteristic equation of $A(G(Z_{2p}, \varphi))$ is

$$(\lambda + (p-1))(\lambda+1)^{(p-1)}(\lambda-1)^{(p-1)}(\lambda-(p-1)) = 0,$$

and the eigen values are $\lambda = -(p-1), -1, 1$ and $(p-1)$, their corresponding multiplicities are $1, (p-1), (p-1)$ and 1.

Therefore, the spectrum of the graph $G(Z_{2p}, \varphi)$ is $\begin{pmatrix} -(p-1) & -1 & 1 & (p-1) \\ 1 & (p-1) & (p-1) & 1 \end{pmatrix}$.

Then the Energy of $G(Z_{2p}, \varphi)$ is

$$\varepsilon(G(Z_{2p}, \varphi)) = \sum_{i=1}^{n} |\lambda_i|$$

$$= |-(p-1)|(1) + |-1|((p-1)) + |1|((p-1)) + |(p-1)|(1)$$

$$= 4(p-1).$$

Corollary 3.6. *The matrix energy of* $G(Z_{2p}, \varphi)$ *is* $4(p-1)$, *where p is prime.*

Proof. From Theorem 3.5., $A(G(Z_{2p}, \varphi)) = \begin{pmatrix} R & S \\ S & R \end{pmatrix}_{2p\times 2p}$.

The transpose of the above matrix is $A(G(Z_{2p}, \varphi))' = \begin{pmatrix} R & S \\ S & R \end{pmatrix}_{2p\times 2p}$.

Then $A(G(Z_{2p}, \varphi))A(G(Z_{2p}, \varphi))' = \begin{pmatrix} M & N \\ N & M \end{pmatrix}_{2p\times 2p}$, where,

$$M = \begin{pmatrix} p-2 & p-1 & 0 & \dots & p-1 \\ 0 & p-2 & 0 & \dots & 0 \\ p-1 & 0 & p-2 & \dots & p-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & p-1 & \dots & p-2 \end{pmatrix}_{p\times p}$$

and

$$N = \begin{pmatrix} 0 & p-1 & 0 & \dots & 0 \\ p-1 & 0 & p-1 & \dots & p-1 \\ 0 & p-1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p-1 & 0 & \dots & 0 \end{pmatrix}_{p\times p}.$$

The characteristic equation of $A(G(Z_{2p}, \varphi))A(G(Z_{2p}, \varphi))'$ is $(\sigma-1)^{2(p-1)}(\sigma-(p-1)^2)^2 = 0$, and the singular values of $A(G(Z_{2p}, \varphi))$ are 1, $(p-1)$ and their corresponding multiplicities are $2(p-1)$ and 2.

Therefore, the spectrum of the graph $G(Z_{2p}, \varphi)$ is $\begin{pmatrix} 1 & (p-1) \\ 2(p-1) & 2 \end{pmatrix}$.

Then the matrix energy of $G(Z_{2p}, \varphi)$ is

$$\varepsilon_m(G(Z_{2p}, \varphi)) = \sum_{i=1}^{n} |\sigma_i|$$

$$= |1|(2(p-1)) + |(p-1)1|(2) = 4(p-1).$$

Theorem 3.7. *For a graph* $G(Z_{2p}, \varphi)$ *such that* $n = \prod_{i=1}^{r} p_i$, *where* $p_1, p_2, \ldots, p_i$ *are distinct primes then* $\varepsilon(G(Z_{2p}, \varphi))$ *is* $2^r \prod_{i=1}^{r}(p_i - 1)$.

Proof. Consider the graph $G(Z_{2p}, \varphi)$ with $n = p_1 p_2 \ldots p_i$, where $p_1, p_2, \ldots, p_i$ are distinct primes and the vertex set V be $\{0, 1, 2, \ldots, (p_1, p_2, \ldots, p_i) - 1\}$.

The adjacency matrix of $G(Z_n, \varphi)$ is $A(G(Z_n, \varphi)) = \begin{pmatrix} R & S \\ S & R \end{pmatrix}_{n \times n}$, where $(R)_{\frac{n}{2} \times \frac{n}{2}}$ and $(S)_{\frac{n}{2} \times \frac{n}{2}}$ are the sub matrices of $A(G(Z_n, \varphi))$.

Now the characteristic equation of $A(G(Z_n, \varphi))$ is

$A(G(Z_n, \varphi)) = [\lambda + (p_2 - 1)(p_3 - 1), \ldots, (p_i - 1)]^{(p_1 - 1)}$

$[\lambda + (p_i - 1)]^{(p_1-1)(p_2-1), \ldots, (p_{i-1}-1)}$

$[\lambda + (p_i - 1)]^{(p_1-1)(p_2-1), \ldots, (p_{i-2}-1)(p_i-1)}$

$\vdots$

$[\lambda + (p_{i-(i-2)} - 1)]^{(p_1-1)(p_2-1), \ldots, (p_{i-(i-3)}-1)(p_{i-(i-1)}-1), \ldots, (p_i-1)}$

$\vdots$

$[\lambda + (p_1 - 1)]^{(p_2-1)(p_2-1), \ldots, (p_i-1)}$

$[\lambda - (p_1 - 1)]^{(p_2-1)(p_2-1), \ldots, (p_i-1)}$

$$[\lambda - (p_i - 1)]^{(p_1-1)(p_2-1), \ldots, (p_{i-1}-1)}$$

$$[\lambda - (p_{i-1} - 1)]^{(p_1-1)(p_2-1), \ldots, (p_{i-2}-1)(p_i-1)}$$

$$\vdots$$

$$[\lambda - (p_{i-(i-2)} - 1)]^{(p_1-1)(p_2-1), \ldots, (p_{i-(i-3)}-1)(p_{i-(i-1)}-1), \ldots, (p_i-1)}$$

$$\vdots$$

$$[\lambda + (p_1 - 1)]^{(p_2-1)(p_2-1), \ldots, (p_i-1)}$$

$$\vdots$$

$$[\lambda - (p_2 - 1)(p_3 - 1), \ldots, (p_i - 1)]^{(p_1-1)} = 0.$$

The eigen values of the above equation are, $\lambda = [-(p_2 - 1)(p_3 - 1), \ldots, (p_r - 1)], [-(p_r - 1)], [-(p_{r-1} - 1)], [-(p_{r-(r-2)} - 1)], \ldots, [-(p_1 - 1)], [(p_r - 1)], [(p_{r-1} - 1)], [(p_{r-1} - 1)], \ldots, [(p_{r-(r-2)} - 1)], \ldots, [(p_1 - 1)]$ and $[(p_2 - 1)(p_3 - 1), \ldots, (p_r - 1)]$.

The corresponding multiplicities are

$[(p_1 - 1)], [(p_1 - 1)(p_2 - 1), \ldots, (p_{i-1} - 1)], [(p_1 - 1)(p_2 - 1), \ldots, (p_{i-2} - 1)(p_i - 1)], \ldots$ $[(p_1 - 1)(p_2 - 1), \ldots, (p_{i-(i-3)} - 1)(p_{i-(i-1)} - 1), \ldots, (p_i - 1)], \ldots, [(p_2 - 1)(p_2 - 1), \ldots, (p_i - 1)]$, $[(p_1 - 1)(p_2 - 1), \ldots, (p_{i-1} - 1)], [(p_1 - 1)(p_2 - 1), \ldots, (p_{i-2} - 1)(p_i - 1)], \ldots, [(p_1 - 1)$ $(p_2 - 1), \ldots, (p_{i-(i-3)} - 1)(p_{i-(i-1)} - 1), \ldots, (p_i - 1)], \ldots, [(p_2 - 1)(p_2 - 1), \ldots, (p_i - 1)]$ and $[(p_1 - 1)]$.

Hence the energy of the graph $\varepsilon(G(Z_n, \varphi)) = \sum_{i=1}^{n} |\lambda_i|$

$$= |-(p_1 - 1)(p_3 - 1), \ldots, (p_i - 1)| [(p_1 - 1)]$$

$$+ |-(p_i - 1)| [(p_1 - 1), (p_2 - 1), \ldots, (p_{i-1} - 1)]$$

$$+ |-(p_{i-1} - 1)| [(p_1 - 1), (p_2 - 1), \ldots, (p_{i-2} - 1)(p_i - 1)]$$

$+$

$\vdots$

$+| -(p_{i-(i-2)}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-(i-3)}-1)(p_{i-(i-1)}-1),\ldots,(p_i-1)]$

$+$

$\vdots$

$+| -(p_1-1)|[(p_2-1),(p_2-1),\ldots,(p_i-1)]$

$+| -(p_1-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-1}-1)]$

$+| -(p_{i-1}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-2}-1)(p_i-1)]$

$+$

$\vdots$

$+| -(p_{i-(i-2)}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-(i-3)}-1)(p_{i-(i-1)}-1),\ldots,(p_i-1)]$

$+$

$\vdots$

$+|(p_1-1)|[(p_2-1),(p_2-1),\ldots,(p_i-1)]+|(p_2-1)(p_3-1),\ldots,(p_i-1)|[(p_1-1)]$

$= 2^r[(p_1-1)(p_2-1),\ldots,(p_{i-1}-1)(p_i-1)]$

$= 2^r\prod_{i=1}^{r}(p_i-1).$

Observation 3.8. *The graph* $G(Z_n, \varphi)$ *is Hyper energetic if and only if* $n=\prod_{i=1}^{r}p_i, r\geq 3$ *where* $p_1, p_2, \ldots, p_i$ *are distinct primes.*

Proof. In [16], the authors proved that the energy of complete graph is $2n-2$. A graph is said to be Hyper energetic, if the energy of a graph is greater than $2n-2$. By the theorem 3.7., an Euler totient cayley graph $G(Z_n, \varphi)$ such that $n=\prod_{i=1}^{r}p_i$, where $p_1, p_2, \ldots, p_i$ are distinct primes, the energy value is $2^r\prod_{i=1}^{r}(p_i-1)$.

That means $\varepsilon(G(Z_n, \varphi)) > 2n-2$. Therefore the graph $G(Z_n, \varphi)$ such that $n = \prod_{i=1}^{r} p_i$, where $p_1, p_2, \ldots, p_i$ are distinct primes, is Hyper energetic and vice versa.

4. Conclusion

In this paper, authors discussed energy and matrix energy of Euler totient Cayley graphs and also hyper energy of this graph was observed. Authors also developed results related to various energies and degree based indices of Euler totient Cayley graphs.

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