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VARIOUS CONTRACTIONS IN EXTENDED *B*-METRIC SPACE

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Abstract

T. Kamran et al. introduced the concept of extended b-metric space in 2017. He proved Banach contraction principal in extended b-metric space. Main goal of this research is to show the existence and uniqueness of Rus contraction and Pachpatte contraction in extended b-metric space. Our result extends many existing results in fixed point theory.

1. Introduction

Fixed point theory is a well-known subject in the field of mathematics, which is quite helpful to study the existence if fixed point $f(x) = x$, the knowledge of fixed point plays a significant role in many fields such as engineering, Computer applications and Biology. Beside this, the study of fixed point is also helpful to resolve some physical problems related to this world. This concept is also used to find the solution of differential equations and to study the properties of 2-D and 3-D using topological behavior of the objects. One of the most powerful tool in this field is Banach contraction theorem "Let T be a mapping from a complete metric space (X, d) into itself satisfying; $d(Tx, Ty) \leq \alpha d(x, y)$" which was introduced by Banach in 1922 [1]. After that many authors have extended this result in different metric spaces by using different contractions and mappings [2-11]. one of the generalization of metric space is known as b-metric space which was

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introduced by Bakhtin [12], he generalizes Banach contraction principle and proved some existing fixed point theorems using different contraction conditions in b-metric space. Later, Czerwik [13] extended the work in b-metric space. After that several research have been done to prove the existence of fixed point in b-metric space [14-21]. Recently, Kamran et. al. [22] introduced the notion of extended b-metric space which is the generalization of b-metric space.

2. Preliminaries

Following are some useful definitions in extended b-metric space.

Definition 2.1[23]**.** Let A be a non-empty set. A metric space A is a mapping $d : A \times A \to R$ which satisfies the following properties, for all $x, y \in A$

(M1) $d(x, x) = 0$

(M2) $d(x, y) = 0 \Rightarrow x = y$

(M3) $d(x, y) = d(y, x)$

(M4) $d(x, z) \leq d(x, y) + d(y, z)$.

Then (A, d) is called metric space.

Definition 2.2 [24]**.** Let f is a self-map on metric point A then f is said to be contraction mapping if, $d(f(x), f(y)) \leq \gamma d(x, y)$ for all $x, y \in A$.

Remark 1. If f is contraction on A, then it is continuous on A.

Definition 2.3 [25]**.** Let A be a non-empty set and let $s \geq 1$ be a given real number. A function $d : A \times A \to [0, \infty)$ is called b-metric space if for all $x, y, z \in A$, following conditions satisfied:

(M1) $d(x, y) = 0$ if and only if $x = y$;

(M2) $d(x, y) = d(y, x)$

(M3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (A, d) is called a b-metric space.

Note that every metric space is a b-metric space if $s = 1$.

Definition 2.4 [22]. Let A be a non-empty set and $\theta : X \times X \to [1, \infty]$ be a mapping. We define the extended b-metric to the function $d_\theta : X \times X \to [0, \infty]$ that satisfies the following conditions

(M1) $d_\theta(x, y) = 0$ if and only if $x = y$;

(M2) $d_\theta(x, y) = d_\theta(y, x)$

(M3) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The space (A, d_θ) is called an extended b-metric space.

Note that every b-metric space is an extended b-metric space by taking $\theta(x, z) = s \geq 1$ to be a constant function.

Definition 2.5 [22]. Let (A, d_θ) be an extended b-metric space. A sequence $\{x_n\}$ in A is said to be convergent to $x \in A$ if $\lim_{n\to\infty} d_\theta(x_n, x) = 0$ for $n \in N$.

Definition 2.6 [22]. Let (A, d_θ) be an extended b-metric space. A sequence $\{x_n\}$ in A is said to be a Cauchy sequence, if $\lim_{n\to\infty} d_\theta(x_n, x_m) = 0$, for all $n, m \in N$.

Definition 2.7 [22]. extended b-metric space (A, d_θ) is said to be complete if every Cauchy sequence in A is convergent.

3. Main Result

In this section, we extend our result in extended b-metric space and show the existence and uniqueness of some well-known theorems known as Rus contraction theorem [26] and Pachpatte contraction theorem [27].

Theorem 3.1 [Rus contraction]. *Let (A, d_θ) be an extended b-metric space and $M : X \to X$ is a self map satisfying the following condition*

$$d_\theta(Mx, My) \leq \alpha d(x, y) + \beta[d(x, Mx) + d_\theta(y, My)] + \gamma[d_\theta(x, My) + d_\theta(y, Mx)] \quad (1)$$

$\forall x, y \in A$ *and* $\alpha, \beta, \gamma > 0$, *if* $\alpha + 2\beta + 2\gamma < 1$, *then fixed point in A and if* $\alpha + 2\gamma = 1$ *then M has a unique fixed point in A.*

Proof. Let $\{x_n\}$ be a sequence in A defined as $x_0 \in A$, and consider the iterate $Mx_n = x_{n+1}$, if for some n.

$Mx_n = x_n$, then x_n is a fixed point. Let $Mx_n \neq x_n$, then using the condition (1), we have

$$d_\theta(x_1, x_2) = d_\theta(Mx_0, Mx_1)$$

$$\leq \alpha d(x_0, x_1) + \beta[d(x_0, Mx_0) + d(x_1, Mx_1)] + \gamma[d(x_0, Mx_1) + d(x_1, Mx_0)]$$

$$d_\theta(x_1, x_2) \leq \alpha d_\theta(x_0, x_1) + \beta[d_\theta(x_0, x_1) + d(x_1, x_2)] + \gamma[d_\theta(x_0, x_2) + d_\theta(x_1, x_1)]$$

$$d_\theta(x_1, x_2) \leq \alpha d_\theta(x_0, x_1) + \beta[d_\theta(x_0, x_1) + d_\theta(x_1, x_2)] + \gamma[d_\theta(x_0, x_1) + d_\theta(x_1, x_2)]$$

$$(1 - \beta - \gamma) d_\theta(x_1, x_2) \leq (\alpha + \beta + \gamma) d_\theta(x_0, x_1)$$

$$d_\theta(x_1, x_2) \leq \frac{(\alpha + \beta + \gamma)}{(1 - \beta - \gamma)} d_\theta(x_0, x_1)$$

$$d_\theta(x_1, x_2) \leq \delta d_\theta(x_0, x_1),$$

where $\delta < 1$ as $\alpha + 2\beta + 2\gamma + 2\gamma < 1$. Proceeding in the similar manner, we have

$$d_\theta(x_2, x_3) \leq \delta d_\theta(x_1, x_2) \leq \delta^2 d_\theta(x_0, x_1)$$

$$\Rightarrow d_\theta(x_n, x_{n+1}) \leq \delta d_\theta(x_{n-1}, x_n) \leq \delta^2 d_\theta(x_{n-2}, x_{n-1}) \leq \delta^3 d_\theta(x_{n-3}, x_{n-2}) \ldots$$

$$\leq \delta^n d_\theta(x_0, x_1).$$

Now applying the triangular inequality, we conclude

$$d_\theta(x_n, x_{n+m}) \leq d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_{n+3}) + \ldots$$

$$+ d_\theta(x_{n+m-1}, x_{n+m})$$

$$d_\theta(x_n, x_{n+m}) \leq (\delta^n + \delta^{n+1} + \delta^{n+2} + \ldots + \delta^{n+m-1}) d_\theta(x_0, x_1)$$

$$d_\theta(x_n, x_{n+m}) \leq \frac{\delta^n}{1 - \delta} d_\theta(x_0, x_1).$$

Since, $0 < \delta < 1$, therefore $\lim_{n \to \infty} \frac{\delta^n}{1 - \delta} d_\theta(x_0, x_1) \to 0 \Rightarrow$ sequence

$\{x_n\}$ is Cauchy sequence in $(A, d_\theta) \Rightarrow$ there exist a limit point c of m, we have

$$M(c) = \lim_{n\to\infty} M(x_n) = \lim_{n\to\infty} x_{n+1} = c \Rightarrow c \text{ is a fixed point for } M.$$

Now we are to prove that c is unique. If possible consider c, c' are two fixed points. Then from (1) we have

$$d_\theta(c, c') = d_\theta(Mc, Mc') \leq \alpha d_\theta(c, c') + \beta[d_\theta(c, Mc) + d_\theta(c', Mc')]$$

$$+\gamma[d_\theta(c, Mc') + d_\theta(c', Mc)]$$

$$d_\theta(c, c') \leq \alpha d_\theta(c, c') + \beta[d_\theta(c, c) + d_\theta(c', c')] + \gamma[d_\theta(c, c') + d_\theta(c', c)]$$

$$d_\theta(c, c') \leq \alpha d_\theta(c, c) + \gamma[d_\theta(c, c') + d_\theta(c', c)]$$

$$d_\theta(c, c') \leq \alpha d_\theta(c, c') + 2\gamma d_\theta(c, c')$$

$$d_\theta(c, c') \leq (\alpha + 2\gamma) d_\theta(c, c')$$

$d_\theta(c, c') \leq d_\theta(c, c')$ since $(\alpha + 2\gamma) = 1 \Rightarrow c = c' \Rightarrow c$ is unique.

Theorem 3.2 [Pachpatte contraction]. *Let (A, d_θ) be a complete digital metric space and map $M : X \to X$ is a self map satisfying the following condition*

$$d_\theta(Mx, My) \leq q \max\left\{d_\theta(x, y), \frac{d_\theta(x, Mx)d(y, My)}{d_\theta(x, y)}, \frac{d_\theta(x, Mx)d(y, My)}{d_\theta(x, y)}, \right.$$

$$\left. \frac{d_\theta(x, Mx)d_\theta(x, My)}{2d_\theta(x, y)}\right\} \cdots \quad (2)$$

for all $x, y \in A$, $x \neq y$ and $q \in (0, 1)$. Then for each $x \in A$, $\{M^n x\}$ converges to the unique fixed point of M.

Proof. Let $\{x_n\}$ be a sequence in A defined as $x_0 \in A$, and consider the iterate $Mx_n = x_{n+1}$, if for some n. $Mx_n = x_n$, then x_n is a fixed point. Let $Mx_n \neq x_n$, then using the condition (2), we have

$$d_\theta(x_1, x_2) = d_\theta(Mx_0, Mx_1)$$

$$\leq \alpha Max \{d_\theta(x_0, x_1), \frac{d_\theta(x_0, Mx_0)d_\theta(x_1, Mx_1)}{d_\theta(x_0, x_1)}, \frac{d_\theta(x_0, Mx_1)d_\theta(x_1, Mx_0)}{d_\theta(x_0, x_1)},$$

$$\frac{d_\theta(x_0, Mx_0)d_\theta(x_1, Mx_1)}{2d_\theta(x_0, x_1)}\}$$

$$d_\theta(x_1, x_2) = d_\theta(Mx_0, Mx_1)$$

$$\leq \alpha Max \{d_\theta(x_0, x_1), \frac{d_\theta(x_0, x_1)d(x_1, x_2)}{d_\theta(x_0, x_1)}, \frac{d_\theta(x_0, x_2)d(x_1, x_1)}{d_\theta(x_0, x_1)},$$

$$\frac{d_\theta(x_0, x_1)d_\theta(x_1, x_2)}{2d_\theta(x_0, x_1)}\}$$

$$d_\theta(x_1, x_2) = d_\theta(Mx_0, Mx_1) \leq \alpha Max \{d_\theta(x_0, x_1), d_\theta(x_1, x_2), 0, \frac{d_\theta(x_1, x_2)}{d_\theta(x_0, x_1)}\}$$

$$d_\theta(x_1, x_2) = d_\theta(Mx_0, Mx_1) \leq \alpha d_\theta(x_0, x_1)$$

$$d_\theta(x_2, x_3) \leq \alpha d_\theta(x_1, x_2) \leq \alpha^2 d_\theta(x_0, x_1).$$

Proceeding in the similar manner, we have

$$\Rightarrow d_\theta(x_n, x_{n+m}) \leq \alpha d_\theta(x_{n-1}, x_n) \leq \alpha^2 d_\theta(x_{n-2}, x_{n-1}) \leq \alpha^3 d_\theta(x_{n-3}, x_{n-2})$$

$$\leq \alpha^n d_\theta(x_0, x_1).$$

Now applying the triangular inequality, we conclude

$$d_\theta(x_n, x_{n+m}) \leq d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_{n+3}) +$$

$$+ d_\theta(x_{n+m-1}, x_{n+m})$$

$$d_\theta(x_n, x_{n+m}) \leq (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \ldots + \alpha^{n+m-1}) d_\theta(x_0, x_1)$$

$$d_\theta(x_n, x_{n+m}) \leq \frac{\alpha^n}{1-\alpha} d_\theta(x_0, x_1).$$

Since, $0 < \alpha < 1$, therefore $\lim_{n\to\infty} \frac{\alpha^n}{1-\alpha} d_\theta(x_0, x_1) \to 0 \Rightarrow$ sequence $\{x_n\}$ is Cauchy sequence in $(A, d_\theta) \Rightarrow$ there exist a limit point c of M, we have $M(c) = \lim_{n\to\infty} M(x_n) = \lim_{n\to\infty} x_{n+1} = c \Rightarrow c$ is a fixed point for M.

Now we are to prove that c is unique. If possible, consider c, c' are two fixed points. Then from (2) we have

$$d_\theta(c, c') = (d_\theta(Mc, Mc'))$$

$$\leq \alpha \max \{d_\theta(c, c') \frac{d_\theta(c, Mc)d_\theta(c', Mc')}{d_\theta(c, c')}, \frac{d_\theta(c, Mc)d_\theta(c', Mc)}{d_\theta(c, c')}$$

$$\frac{d_\theta(c, Mc)d_\theta(c', Mc')}{2d_\theta(c, c')}\}$$

$$d_\theta(c, c') \leq \alpha \max \{d_\theta(c, c') \frac{d_\theta(c, c)d_\theta(c, c')}{d_\theta(c, c')}, \frac{d_\theta(c, c')d_\theta(c', c)}{d_\theta(c, c')} \frac{d_\theta(c, c)d_\theta(c', c')}{2d_\theta(c, c')}\}$$

$$d_\theta(c, c') \leq \alpha \max \{d_\theta(c, c'), 0, d_\theta(c, c'), 0\}$$

$$d_\theta(c, c') \leq \alpha d_\theta(c, c')$$

$$d_\theta(c, c') \leq d_\theta(c, c') \text{ since } \alpha < 1 \Rightarrow c = c' \Rightarrow c \text{ is unique.}$$

3. Conclusion

In the above sections, we revised the general background of *b*-metric and extended *b*-metric space prove some well-known results in extended *b*-metric space such as Rus contraction and Pachpatte contraction. We hope that our results are helpful to do more research in this field and to prove the uniqueness of fixed point using different contraction conditions.

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