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STUDY OF LOCAL CONVERGENCE OF NEWTON-LIKE METHODS FOR SOLVING NONLINEAR EQUATIONS

DEEPAK KUMAR and JANAK RAJ SHARMA

Department of Mathematics
Sant Longowal Institute of Engineering and Technology
Longowal-148106, Sangrur, India
E-mail: deepak.babbi@gmail.com
jrshira@yahoo.co.in

Abstract

We present the local convergence analysis of fifth and eighth order methods developed in [22], to approximate a locally-unique solution of a nonlinear equation in Banach space. We also use the hypothesis which is based on the first Fréchet-derivative only. Moreover, our new approach provides computable radius of convergence as well as error bounds on the distances involved and estimates on the uniqueness of the solution based on some functions appearing in these generalized conditions. Such estimates are not provided in the approaches using Taylor expansions of higher order derivatives which may not exist or may be very expensive or impossible to compute. Finally, numerical examples are provided to show that the present results can be applied to solve equations in the cases where earlier results cannot be applied.

Introduction

Let E_1, E_2 be Banach spaces and $S \subseteq E_1$ be closed and convex subset. In this study, we locate a unique solution ξ of the nonlinear equation

$$T(\theta) = 0, \tag{1}$$

where $T : S \subseteq E_1 \to E_2$ is a Fréchet-differentiable operator. In computational and engineering sciences, many problems can be expressed in the form (1.1). See, for example [3, 20, 23]. The solution of these types of problem can be found in closed form only in particular cases. That explains

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why most methods for solving these equations are usually iterative in nature. The study of convergence analysis is important part in the development of an iterative method. Mostly, the convergence domain is very narrow. Therefore, it is important task to expand the convergence domain without adding more hypotheses. Information of the radius of convergence is useful because it explains the degree of difficulty for obtaining initial points. Another important task is to find more precise error estimates on $\| \theta_{n+1} - \theta_n \|$ or $\| \theta_n - \xi \|$. There are many studies in literature which deal with the local and semilocal convergence analysis of iterative methods such as [1, 2, 3, 4, 5, 6, 7, 8, 10, 13, 14, 15, 17, 18, 19, 21].

The most well-known method for finding a simple solution ξ of equation (1) is Newton's method, which is written as:

$$\theta_{n+1} = \theta_n - T'(\theta_n)^{-1} T(\theta_n), \text{ for each } n = 0, 1, 2, \ldots \tag{2}$$

which has quadratic order of convergence. For gaining the higher order of convergence, a number of modified Newton's or Newton-like methods have been proposed in the literature (see [1, 3, 4, 6, 7, 8, 10, 11, 13, 15, 5, 22, 21, 23, 24]) and references therein. In particular, Sharma and Arora [22] have recently developed a method for approximating solution of $T(\theta) = 0$ using the weighted-Newton technique defined for each $n = 0, 1 \ldots$ by

$$y_n = \theta_n - T'(\theta_n)^{-1} T(\theta_n),$$

$$\theta_{n+1} = y_n - \left(\frac{13}{4} I - G\left(\frac{7}{2} I - \frac{5}{2} G\right)\right) T'(\theta_n)^{-1} T(y_n) \tag{3}$$

and

$$y_n = \theta_n - T'(\theta_n)^{-1} T(\theta_n),$$

$$w_n = y_n - \left(\frac{13}{4} I - G\left(\frac{7}{2} I - \frac{5}{4} G\right)\right) T'(\theta_n)^{-1} T(y_n),$$

$$\theta_{n+1} = w_n - \left(\frac{7}{2} I - G\left(4I - \frac{3}{2} G\right)\right) T'(\theta_n)^{-1} T(w_n), \tag{4}$$

where $G = T'(\theta_n)^{-1} T'(y_n)$ and I is identity operator on S. In order to prove the order of convergence, Sharma-Arora used Taylor expansions and hypotheses required derivatives upto eighth Fréchet-derivative although only the first order derivative appears in the methods. It is very clear that these hypotheses restrict the applicability of methods to functions that are at least eight times Fréchet-differentiable. As a inspirational example, we define a function g on $S = \left[-\frac{1}{2}, \frac{5}{2}\right]$ by

$$\gamma(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, \ t \neq 0, \\ 0, \ t = 0. \end{cases} \tag{5}$$

We have that

$$\gamma'(t) = 3t^2 \ln t^2 + 5t^4 - 4t^3 + 2t^2,$$

$$\gamma''(t) = 6t \ln t^2 + 20t^3 - 12t^2 + 10t$$

and

$$\gamma'''(t) = 6 \ln t^2 + 60t^2 - 24t + 22.$$

Then, γ''' is unbounded on S. Notice also that the proofs of convergence use Taylor expansions. In the present study, we study the local convergence of the methods (3) and (4) using the hypotheses only on the first Frechet-derivative taking advantage of the Lipschitz continuity of the first Frechet-derivative. Notice that our results are presented in the more general setting of a Banach space.

We summarize the contents of the paper. In section 2, the resident meeting with radius of convergence, computable slip bounds after that exclusivity fallout of methods (3) as well as (4) is presented. Finally, taking part in portion 3, numerical examples are performed en route for verify the conjectural results.

Local Convergence Analysis

We discuss the local convergence analysis of methods (3) and (4). Let $N_0 > 0, N > 0, B \geq 1$ be given parameters. It is suitable for the local convergence analysis that follows to produce some functions and parameters.

We define function $\phi_1(t)$ on interval $\left[0, \frac{1}{N_0}\right)$ by

$$\phi_1(t) = \frac{Nt}{2(1 - N_0 t)},$$

and parameter

$$r_1 = \frac{2}{2N_0 + N} < \frac{1}{N_0}. \tag{6}$$

Then, we have that $\phi_1(r_1) = 1$ and $0 \leq \phi_1(t) \leq 1$ for each $t \in [0, r_1)$. Moreover, define the functions $\phi_2(t)$ and $\psi_2(t)$ on interval $[0, r_1)$ by

$$\phi_2(t) = \left(1 + \frac{B}{1 - N_0 t}\left(\frac{13}{4} - \frac{B}{1 - N_0 t}\left(\frac{7}{2} - \frac{5}{4}\frac{B}{1 - N_0 t}\right)\right)\right)\phi_1(t)$$

and

$$\psi_2(t) = \phi_2(t) - 1.$$

We have that $\psi_2(0) = -1 < 0$ and $\psi_2(r_1) > 0$. It follows from the intermediate theorem that function ψ_2 has zeros in the interval $(0, r_1)$. Denote by r_2 the smallest such zero. Finally define functions $\phi_3(t)$ and $\psi_3(t)$ on the interval $[0, r_2)$ by

$$\phi_3(t) = \left(1 + \frac{B}{1 - N_0 t}\left(\frac{7}{2} - \frac{B}{1 - N_0 t}\left(4 - \frac{3}{2}\frac{B}{1 - N_0 t}\right)\right)\right)\phi_2(t)$$

and

$$\psi_3(t) = \phi_3(t) - 1.$$

We have that $\psi_3(0) = -1 < 0$ and $\psi_3(r_2) > 0$. It follows from the intermediate theorem that function ψ_3 has zeros in the interval $(0, r_2)$. Denote by r_3 the smallest such zero of function ψ_3 on interval $[0, r_2)$. Set:

$$r = \min\{r_i\},\ i = 1, 2, 3. \tag{7}$$

Then we have

$$0 < r \leq r_1. \tag{8}$$

Then, for each $t \in [0, r)$.

$$0 \leq \phi_1(t) \leq 1, \tag{9}$$

$$0 \leq \phi_1(t) \leq 1 \tag{10}$$

and

$$0 \leq \phi_3(t) \leq 1. \tag{11}$$

Net $H(\nu, \rho)$ and $\overline{H}(\nu, \rho)$ be the open and closed balls in E_1, respectively with center $v \in E_1$ and of radius $\rho > 0$. Net also $\mathcal{N}(E_1, E_2)$ be the set of bounded linear operators between E_1 and E_2.

Next, we discuss the behavior of method (4) in the point of view of local convergence analysis by using the preceding notations.

Theorem 2.1. *Let* $T : S \subseteq E_1 \to E_2$ *be a Frechet-differentiable operator. Assume that there exist* $\xi \in S, N > 0, N_0 > 0$ *and* $B \geq 1$ *such that for each* $x, y \in S$

$$T(\xi) = 0, \; T'(\xi)^{-1} \in \mathcal{N}(E_2, E_1), \tag{12}$$

$$\| T'(\xi)^{-1}(T'(\theta) - T'(\xi)) \| \leq N_0 \| \theta - \xi \|, \tag{13}$$

$$\| T'(\xi)^{-1}(T'(\theta) - T'(y)) \| \leq N \| \theta - y \|, \tag{14}$$

$$\| T'(\xi)^{-1} T'(\theta) \| \leq B, \tag{15}$$

and

$$\overline{H}(\xi, r) \subset S, \tag{16}$$

where the radius r *is defined by* (7). *Then, the sequence* $\{\theta_n\}$ *which is generated by method* (4) *for* $\theta_0 \in H(\xi, r) - \{\xi\}$ *is well defined, lies in* $H(\xi, r)$ *for each* $n = 0, 1, \ldots$ *and approaches to* ξ. *Moreover, the following error bounds*

$$\| y_n - \xi \| \leq \phi_1(\| \theta_n - \xi \|) \| \theta_n - \xi \| < \| \theta_n - \xi \| < r, \tag{17}$$

$$\| w_n - \xi \| \leq \phi_2(\| \theta_n - \xi \|) \| \theta_n - \xi \| < \| \theta_n - \xi \| < r \tag{18}$$

and

$$\| \theta_{n+1} - \xi \| \leq \phi_3(\| \theta_n - \xi \|) \| \theta_n - \xi \|, \tag{19}$$

are satisfied and the "ϕ*" function is discuss in advance. Furthermore, for* $P \in \left[r, \frac{2}{N_0}\right)$ *the limit point* ξ *is the unique solution of equation* $P(\theta) = 0$ *in* $\overline{H}(\xi, P) \cap S.$

Proof. We shall prove the estimates (17)-(19) by using mathematical induction. From (6), (13) and the hypotheses $\theta_0 \in H(\xi, r) - \{\xi\}$, we get that

$$\| T'(\xi)^{-1}(T(\theta_0) - T(\xi)) \| \leq N_0 \| \theta_0 - \xi \| < N_0 r < 1. \tag{20}$$

It follows from (20) and the Banach Lemma on invertible operators [3, 18] that $T'(\theta_0)^{-1} \in \mathcal{N}(E_2, E_1)$ and

$$\| T'(\theta_0)^{-1} T'(\xi) \| \leq \frac{1}{1 - N_0 \| \theta_0 - \xi \|} < \frac{1}{1 - N_0 r}. \tag{21}$$

We will prove that y_0 is well defined by the first step of method (4) for $n = 0.$ Then, we have by equations (6), (9), (14) and (21) that

$$\| y_n - \xi \| \leq \| \theta_0 - \xi - T'(\theta_0)^{-1} T(\theta_0) \|$$

$$\leq \| T'(\theta_0)^{-1} T'(\xi) \| \| \int_0^1 T'(\xi)^{-1} [T'(\xi + \tau(\theta_0 - \xi)) - T'(\theta_0)] (\theta_0 - \xi)] \| d\tau$$

$$\leq \frac{N \| \theta_0 - \xi \|^2}{2(1 - N_0 \| \ - \xi \|)}$$

$$= \phi_1(\| \theta_0 - \xi \|) \| \theta_0 - \xi \| < \| \theta_0 - \xi \| < r, \tag{22}$$

which shows (17) for $n = 0$ and $y_0 \in H(\xi, r).$

Notice that for each $\tau \in [0, 1]$ and $\| \xi + \tau(\theta_0 - \xi) - \xi \| = \tau \| \theta_0 - \xi \| < r.$ That is $\xi + \tau(\theta_0 - \xi) \in H(\xi, r).$ We can write

$$T(\theta_0) = T(\theta_0) - T(\xi) = \int_0^1 T'(\xi + \tau(\theta_0 - \xi))(\theta_0 - \xi) d\tau. \tag{23}$$

Then using (15) and (22), we get that

$$\| T'(\xi)^{-1} T(\theta_0) \| = \| \int_0^1 T'(\xi)^{-1} T'(\xi + \tau(\theta_0 - \xi))(\theta_0 - \xi) d\tau \|$$

$$\leq B \| \theta_0 - \xi \|. \tag{24}$$

Similarly, we obtain that

$$\| T'(\xi)^{-1} T(y_0) \| \leq B \| y_0 - \xi \|, \tag{25}$$

$$\| T'(\xi)^{-1} T(z_0) \| \leq B \| z_0 - \xi \|. \tag{26}$$

Next, we have linear operator $G = T'(\theta_0)^{-1} T'(y_0)$, by using (15) and (21), we obtain

$$\| G \| = \| T'(\theta_0)^{-1} T'(y_0) \|,$$

$$= \| T'(\theta_0)^{-1} T'(\xi) \| \left\| T'(\xi)^{-1} T'(y_0) \right\|$$

$$\leq \frac{B}{1 - N_0 \| \theta_0 - \xi \|}. \tag{27}$$

Using the second step of method (4), (10), (21), (22), (25) and (27), it follows that

$$\| w_0 - \xi \| \leq \| y_0 - \xi \| + \| \left(\frac{13}{4} I - G \left(\frac{7}{2} I - \frac{5}{4} G \right) \right) \| \| \| T'(\theta_0)^{-1} T'(\xi) \| \| \| T'(\xi)^{-1} T(y_0) \|$$

$$\leq \| y_0 - \xi \| + \| \left(\frac{13}{4} - \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{7}{2} - \frac{5}{4} \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \right) \right) \frac{B \| y_0 - \xi \|}{1 - N_0 \| \theta_0 - \xi \|}$$

$$\leq \left(1 + \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{13}{4} - \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{7}{2} - \frac{5}{4} \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \right) \right) \right) \| y_0 - \xi \|$$

$$\leq \left(1 + \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{13}{4} - \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{7}{2} - \frac{5}{4} \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \right) \right) \right)$$

$$\times \phi_1(\| \theta_0 - \xi \|) \| \theta_0 - \xi \|$$

$$\leq \phi_2(\| \theta_0 - \xi \|) \| \theta_0 - \xi \| < \| \theta_0 - \xi \| < r. \tag{28}$$

From above inequality, it shows (18) for $n = 0$ and $w_0 \in H(\xi, r)$ Then using the equation (6), (11), (26) and (28), we obtain

$$\| \theta_1 - \xi \| \leq \| w_0 - \xi \| + \| \left(\frac{7}{2} I - G\left(4I - \frac{3}{2} G \right) \right) \| \| T'(\theta_0)^{-1} T'(\xi) \| \| T'(\xi)^{-1} T(w_0) \|$$

$$\leq \| w_0 - \xi \| + \left(\frac{7}{2} - \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(4 - \frac{3}{2} \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \right) \right) \frac{B \| w_0 - \xi \|}{1 - N_0 \| \theta_0 - \xi \|}$$

$$\leq \left(1 + \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{7}{2} - \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(4 - \frac{3}{2} \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \right) \right) \right) \| w_0 - \xi \|$$

$$\leq \left(1 + \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(\frac{7}{2} - \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \left(4 - \frac{3}{2} \frac{B}{1 - N_0 \| \theta_0 - \xi \|} \right) \right) \right)$$

$$\times \phi_2 (\| \theta_0 - \xi \|) \| \theta_0 - \xi \|$$

$$\leq \phi_3 (\| \theta_0 - \xi \|) \| \theta_0 - \xi \| < \| \theta_0 - \xi \| \tag{29}$$

which proves the (19) for $n = 0$ and $\theta_1 \in H(\xi, r)$. Then, substitute $\theta_0, y_0, z_0, \theta_1$ by $\theta_n, y_n, z_n, \theta_{n+1}$ in the preceding estimates to obtain (17)-(19). Then, from the estimates $\| \theta_{n+1} - \xi \| \leq c \| \theta_n - \xi \| < r$, where $c = \phi_3 (\| \theta_0 - \xi \|) \in [0, 1)$, we deduce that $\lim_{n \to \infty} \theta_n = \xi$ and $\theta_{n+1} \in H(\xi, r)$.

Finally, for uniqueness of the solution, let $R = \int_0^1 T'(v + t(\xi - \beta)) dt$ for some $v \in \overline{H}(\xi, r)$ with $T(v) = 0$. Using (16), we obtain that

$$\| T'(\xi)^{-1} (R - T'(\xi) \| \leq \int_0^1 a_0 \| v + t(\xi - v) - \xi \| dt$$

$$\leq \int_0^1 a_0 (1 - t) \| \xi - v \| dt$$

$$\leq \frac{a_0}{2} P < 1. \tag{30}$$

It follows from (30) that R is invertible. Then, from the estimate $0 = T(\xi) - T(v) = R(\xi - v)$, we conclude that $\xi = v$. □

Remark 2.2

(i) Method (4) remains the same when we prefer the conditions of Theorem 2.1 instead of very suitable conditions used in [22]. Let $\{w_n\}$ be any iterative method. Define the computational order of convergence (COC) [24] by

$$\text{COC} = \log \left\| \frac{w_{n+2} - \xi}{w_{n+1} - \xi} \right\| / \log \left\| \frac{w_{n+1} - \xi}{w_n - \xi} \right\|, \text{ for each } n \in \mathbb{N} \tag{31}$$

and the approximate computational order of convergence (ACOC) [11], by

$$\text{ACOC} = \log \left\| \frac{w_{n+2} - w_{n+1}}{w_{n+1} - w_n} \right\| / \log \left\| \frac{w_{n+1} - w_n}{w_n - w_{n-1}} \right\|, \text{ for each } n \in \mathbb{N} \tag{32}$$

By using the above formula, we are calculated the order of convergence.

(ii) In order to present the corresponding results for method (3), we simply restrict to the definition of functions $\phi_1(t)$, $\phi_2(t)$ and parameters r_1 and r_2. Moreover, we define

$$r = \min\{r_1, r_2\}. \tag{33}$$

Hence, by the illustration of the proof of Theorem 2.1, we reach at the following Theorem.

Theorem 2.3. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied but with r defined by* (33). *Then, the conclusion of Theorem* 2.1 *holds* (*except* (19)) *but with method* (3) *replacing method* (4).

Numerical Examples

In this section, we are trying the hypothetical consequences of Section 2 by thinking about some numerical illustrations as given as follow:

Example 1. The first example we consider is Kepler's equation; $f_1(x) = x - \alpha \sin(x) - K = 0$ where $0 \le \alpha < 1$ and $0 \le K \le \pi$. A numerical study, for different values of K and α, has been performed in [12]. We take values $K = 0.1$ and $\alpha = 0.25$. In this case the solution is $\xi = 0.1332021508\ldots$.

Then, we have $L = L_0 = \frac{|\alpha|}{|1-\alpha\cos(\alpha)|} = 0.3324$ and $M = \frac{1+|\alpha|}{|1-\alpha\cos(\alpha)|} = 1.6618.$

Next, we shall try to determine the convergence radius r, so that we have taken the initial approximation from the convergence ball $\overline{H}(\xi, r)$. The parameter r_1 is the zero of $\psi_1(t)$. Then, $\psi_1(t) = \phi_1(t) - 1 = 0$ yields $r_1 \approx 2.00562$. The parameter r_2 is the smallest zero of $\psi_2(t)$ Thus, solving $\psi_2(t) = \varphi_2(t) - 1 = 0$, we get the smallest zero $r_2 \approx 0.837207$ The parameter r_3 is the smallest zero of $\psi_3(t)$ then solving $\psi_3(t) = \phi_3(t) - 1 = 0$, we get the smallest zero $r_3 \approx 0.430107$.

$$r = \min\{r_1, r_2, r_3\} = \min\{2.00562, 0.837207, 0.430107\} = 0.430107.$$

Hence, it follows from Theorem 1 that the methods (3) and (4) converge to α and remain in $\overline{H}(\xi, r)$.

Example 2. Consider isentropic supersonic flow around a sharp expansion corner. The relationship between the Mach number before the corner (i.e. M_1) and after the corner (i.e. M_2) is given by (see [16])

$$\delta = b^{1/2}\left(\tan^{-1}\left(\frac{M_2^2 - 1}{b}\right)^{1/2} - \tan^{-1}\left(\frac{M_1^2 - 1}{b}\right)^{1/2}\right)$$

$$-\left(\tan^{-1}(M_2^2 - 1)^{1/2} - \tan^{-1}(M_1^2 - 1)^{1/2}\right),$$

where $b = \frac{\gamma + 1}{\gamma - 1}$ and γ is the specific heat ratio of the gas.

For a particular case study, we solve the equation for M_2 given that $M_1 = 1.5$, $\gamma = 1.4$ and $\delta = 10^0$. In this case, we have

$$f_3(x) = 6^{1/2}\left(\tan^{-1}\left(\frac{x^2 - 1}{6}\right)^{1/2} - \tan^{-1}\left(\frac{1}{12}\right)^{1/2}\right)$$

$$-\left(\tan^{-1}(x^2 - 1)^{1/2} - \tan^{-1}\left(\frac{1}{2}\right)^{1/2}\right) - \pi/18 = 0,$$

where $x = M_2$. The solution of this problem is $\xi = 1.8409890211\ldots$. Then, we have that

$$L = L_0 = | f_2'(\alpha)^{-1} | \max_{1.2 \le x \le 2} | f_2''(x) | = 1.304650$$

and

$$M = | f_2'(\alpha)^{-1} | \max_{1.2 \le x \le 2} | f_2'(x) | = 1.031902.$$

According to (7), we must compute r_1, r_2 and r_3. The parameter r_1 is the zero of $\psi_1(t)$. Then, $\psi_1(t) = \phi_1(t) - 1 = 0$ yields $r_1 \approx 0.510993$. The parameter r_2 is the smallest zero of $\psi_2(t)$. On solving $\psi_2(t) = \phi_2(t) - 1 = 0$, we get the smallest zero $r_2 \approx 0.322667$. Then solving $\psi_3(t) = \phi_3(t) - 1 = 0$, we get the smallest zero $r_3 \approx 0.223059$. Then, by (7)

$$\begin{aligned} r &= \min\{r_1, r_2, r_3\} = \min\{0.510993, 0.322667, 0.223059\} \\ &= 0.223059. \end{aligned}$$

So, from Theorem 1 it follows that the methods (3) and (4) converge to α and remain in $\overline{H}(\xi, r)$.

Example 3. Suppose that $E_1 = E_2 = C[0, 1]$, where $C[0, 1]$ stands for the space of continuous functions defined on $[0, 1]$. We shall use the maximum norm. Let $S = \overline{H}(0, 1)$. Define operator T on S by

$$T(\mu)(x) = \mu(x) - 5\int_0^1 x\tau\mu(\tau)^3 d\tau. \tag{34}$$

From above equation, we have that

$$T'(\mu(\lambda))(x) = \lambda(x) - 15\int_0^1 x\tau\mu(\tau)^2 \lambda(\tau) d\tau, \text{ for each } \lambda \in D.$$

Then, for $\xi = 0$, we have $L_0 = 7.5$, $L = 15$ and $M = 2$. According to (7), we must compute r_1, r_2 and r_3. The parameter r_1 is the zero of $\psi_1(t)$. Then, $\psi_1(t) = \phi_1(t) - 1 = 0$ yields $r_1 \approx 0.0666667$. The parameter r_2 is the smallest zero of $\psi_2(t)$. On solving $\psi_2(t) = \phi_2(t) - 1 = 0$, we get the smallest zero

$r_2 \approx 0.020233$. Then solving $\psi_3(t) = \phi_3(t) - 1 = 0$, we get the smallest zero $r_3 \approx 0.00679559$. Then, by (7)

$$r = \min\{r_1, r_2, r_3\} = \min\{0.0666667, 0.020233, 0.00679559\}$$
$$= 0.00679559.$$

So, from Theorem 1 it follows that the methods (3) and (4) converge to α and remain in $\overline{H}(\xi, r)$.

Example 4. The Vander Waal equation of state for a vapor is (see [16])

$$\left(P + \frac{a}{V^2}\right)(V - b) - RT, \tag{35}$$

where P is the pressure $(Pa - N/m^2)$, V is specific volume (m^3/kg), T is the temperature (K), R is the gas constant (J/kgK) and a and b are empirical constants. Consider water vapor, for which $R = 461.495 J/kgK$, $a = 1703.28 Pa(m^3/kg)$, $b = 0.00169099(m^3/kg)$, $P = 10{,}000 kPa$ and $T = 800K$. Equation (35) can be rearranged into the form

$$PV^3 - (Pb + RT)V^2 + aV - ab = 0. \tag{36}$$

The solution of this problem is $\xi = 0.337822\ldots$. Then we have that

$$L = L_0 = |f_2'(\xi)^{-1}| \max 0 \le x \le 1 |f_2''(x)| = 17.5162$$

and

$$M = |f_2'(\xi)^{-1}| \max 0 \le x \le 1 |f_2'(x)| = 8.75857.$$

According to (7), we must compute r_1, r_2 and r_3. The parameter r_1 is the zero of $\psi_1(t)$. Then, $\psi_1(t) = \phi_1(t) - 1 = 0$ yields $r_1 \approx 0.038060$. The parameter r_2 is the smallest zero of $\psi_2(t)$. On solving $\psi_2(t) = \phi_2(t) - 1 = 0$, we get the smallest zero $r_2 \approx 0.000187339$. Then solving $\psi_3(t) = \phi_3(t) - 1 = 0$, we get the smallest zero $r_3 \approx 0.00000025937$. Then, by (7)

$$r = \min\{r_1, r_2, r_3\} = \min\{0.0338060, 0.000187339, 0.00000025937\}$$
$$= 0.00000025937.$$

So, from Theorem 1 it follows that the methods (3) and (4) converge to α and remain in $\overline{H}(\xi, r)$.

Example 5. Now, we return to the example (5) given in the introduction of this paper. In this case, the value of parameters are as $\xi = 1$, $L_0 = L = 146.66290$ and $M = 2$. The parameters r_1, r_2 and r_3 are given as:

$$r = \min \{r_1, r_2, r_3\} = \min \{0.00454557, 0.0014722, 0.00055541\}$$

$$= 0.00055541.$$

So, from Theorem 1 it follows that the methods (3) and (4) converge to ξ and remain in $\overline{H}(\xi, r)$.

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