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GRAPH PRODUCTS ON CYCLIC SUBGROUP GRAPH OF A FINITE GROUP

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Abstract

The key objective of this article is to examine the concept of some graph products on cyclic subgroup graph of a finite group. In addition, we look over some bounds, properties and parameters for graph products, which we discuss in detail.

1. Introduction

Algebraic graph theory is a branch of mathematics in which graphs are constructed from the algebraic structures such as groups, rings etc. J. John Arul Singh and S. Devi have introduced the notion of cyclic subgroup graph of a finite group [5]. In this paper, we discuss some graph products on cyclic subgroup graph of a finite group. Furthermore, we have given some bounds and parameters of graph products on cyclic subgroup graph of a finite group.

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Before entering, let us look into some necessary definitions and notations. The cyclic subgroup graph $\Gamma_z(G)$ for a finite group G is a simple undirected in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. The composition or lexicographic product $G \cdot H$ is a graph whose set of vertices is $V(G) \times V(H)$, with $(u, v) \sim (u', v')$ whenever $u \sim u'$ and $v \sim v'$. The co-normal product of a graph G of order m with the vertex set $V(G) = \{v_1, v_2, \ldots, v_m\}$ and a graph H of order n with vertex set $V(H) = \{u_1, u_2, \ldots, u_n\}$ is the graph $G * H$ with the vertex set $V(G) \times V(H) = \{v_{ij} = (v_i, u_j) : v_i \in V(G)$ and $u_j \in V(H)\}$ the adjacency relation defined as $v_{ij} \sim v_{rs}$ if $v_i \sim v_r$ in G or $u_j \sim u_s$ in H. The tensor product $G \otimes H$ is a graph whose set of vertices is $V(G) \times V(H)$, with (g, h) and (g', h') are adjacent in $G \otimes H$ if and only if g is adjacent to g' and h is adjacent to h'. If G and H are two graphs, then the modular product $G \diamond H$ with the vertex set $V(G) \times V(H)$ in which two vertices are adjacent if (i) $xy \in E(G)$ and $uv \in E(H)$, or (ii) $xy \notin E(G)$ and $uv \in E(H)$. For any graph G, the maximum degree is the largest number of neighbours of a vertex in G, i.e. $\Delta(G) = \max\{\deg v / v \in V(G)\}$ and minimum degree is the smallest number of neighbours of a vertex in G, i.e. $\delta(G) = \min\{\deg v / v \in V(G)\}$. A dominating set for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G. A dominating set D of a graph $G = (V, E)$ is said to be a split dominating set, if the induced subgraph $\langle V \setminus D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set. The chromatic number $\chi(G)$ is the minimum number of colours needed to colour G.

2. Main Results

Theorem 2.1. *Let* $G = \Gamma_z(\mathbb{Z}_{p^2})$ *and* $H = \Gamma_z(\mathbb{Z}_p)$ *be two cyclic subgroup graphs on a finite group. Then*

(i) $(G \cdot H) \cong (G * H)$ *and* $(G \diamond H) \cong (G \otimes H)$

(ii) $\gamma_s(G \cdot H) < \gamma(G \cdot H) < \gamma_s(G \diamond H) \leq \gamma(G \diamond H)$

(iii) $\chi(G \cdot H) \geq \chi(G \diamond H)$

(iv) $\Delta(G \cdot H) \geq \Delta(G \diamond H)$

(v) $\delta(G \diamond H) \leq \delta(G \cdot H)$.

Proof. (i) Let $(G \cdot H)$ and $(G * H)$ be the lexicographic product and co-normal product of cyclic subgroup graphs on a finite group. Here the vertex set belongs to $(G \cdot H)$ is the cartesian product of $V(G) \times V(H)$ and any two vertices (a, b) and (c, d) are adjacent in $(G \cdot H)$ if and only if either a is adjacent with c in G or $a = c$ and b is adjacent with d in H. For, co-normal product u, v is adjacent with (u', v') in $(G * H)$ when u is adjacent with u' in G and (u, v) is adjacent with (u', v') in $(G * H)$ when v is adjacent with v' in H. By computing this, the obtained graph will have same number of elements with same degree which is obviously isomorphic. Hence $(G \cdot H) \cong (G * H)$. The proof is similar for $(G \diamond H) \cong (G \otimes H)$.

(ii) For $(G \cdot H)$, the vertices belongs to $(G \cdot H)$ are adjacent with every other vertex, which forms a complete graph. Clearly, the domination number will be 1. For a split dominating set, the induced subgraph will be disconnected when the dominating set is removed from the graph. But for a complete graph, by removing every vertex to the end which concludes in a single vertex which is K_1. Therefore, the minimum cardinality of the split dominating set will be 0. Now for $(G \diamond H)$, the vertices belongs to $(G \diamond H)$ are adjacent to exactly 2 vertices. By selecting any 2 vertices which is not adjacent to each other makes a dominating set. The minimum cardinality of the dominating set is 2. Now, by removing the dominating set from $(G \diamond H)$ then the obtained graph will be disconnected. Hence the condition holds.

(iii) For a graph, no two adjacent vertices should have the same colour. Now for $(G \cdot H)$, clearly it is a complete graph which has n distinct colours. Then, for $(G \diamond H)$, assign colours for alternative vertices to paint the graph. The minimum colours used to colour the graph is 2. Obviously, $\chi(G \cdot H) \geq \chi(G \diamond H)$.

(iv) Clearly, by case (ii) the maximum degree of $(G \cdot H)$ is greater than the maximum degree of $(G \diamond H)$. The proof is obvious for (v). □

Theorem 2.2. *Let* $G = \Gamma_z(\mathbb{Z}_{p^2})$ *and* $H = \Gamma_z(\mathbb{Z}_p)$ *be two cyclic subgroup graphs on a finite group. Then*

(i) $\gamma(G \cdot H) \leq \gamma(G * H) < \gamma(G \diamond H) \leq \gamma(G \otimes H)$

(ii) $\gamma_s(G \diamond H) \leq \gamma_s(G \otimes H) < \gamma_s(G \cdot H) < \gamma_s(G * H)$

(iii) $\chi(G * H) \geq \chi(G \cdot H) > \chi(G \diamond H) \geq \chi(G \otimes H)$

(iv) $\Delta(G \diamond H) \leq \Delta(G \otimes H) < \Delta(G * H) \leq \Delta(G \cdot H)$

(v) $\delta(G * H) > \delta(G \cdot H) > \delta(G \diamond H) \geq \delta(G \otimes H)$.

Proof. Let p and q be any two distinct primes where $q > p$. The vertex set of GH is $V(GH) = \{(a, a'), (a, b'), (b, a'), (b, b'), (c, b'), (d, a'), (d, b')\}$. Based on the definition of the graph products, adjacency rule will be satisfied.

(i) For $G \cdot H$, the vertex (a, a') dominates rest of its vertices. The domination number will be 1. Now, for $G * H$, the vertices belongs to $G * H$ are adjacent with every other vertex. Clearly, the domination number will be 1. For $G \diamond H$, the vertices $(a, a'), (a, b')$ dominates every other persisting vertices which forms a dominating set. The domination number will be 2. This is similar for $G \otimes H$.

(ii) For $G \diamond H$, when the dominating set is removed, the whole graph will be disconnected, which results to the split dominating set. The split domination number is 2. This case is similar for $G \otimes H$. Now for $G \cdot H$, by removing the dominating set, the graph will not be disconnected. Many vertices are adjacent with every vertices in the graph. The required split domination number will be at least 4. Now for $G * H$, by removing at least six vertices, to make the graph disconnected. Therefore, the split domination number will be at least 6.

(iii) For, $G * H$, a single vertex which dominates every other vertices, so the required colour will be n. This is similar for $G \cdot H$. Now for GH, decompose the vertices into two vertex sets and named as u and v. By colouring the vertices belongs to u with one colour and vertices belongs to v with another colour, which gives a proper colouring of the graph.

(iv) By using case (i) and (ii), the maximum degree and minimum degree can be found and the inequality holds. □

Theorem 2.3. *Let* $G = \Gamma_z(\mathbb{Z}_{p^2})$ *and* $H = \Gamma_z(\mathbb{Z}_{p^2})$ *be two cyclic subgroup graphs on a finite group. Then*

(i) $\gamma(G \cdot H) \leq \gamma(G * H) < \gamma(G \diamond H) \leq \gamma(G \otimes H)$

(ii) $\gamma_s(G \diamond H) \leq \gamma_s(G \otimes H) < \gamma_s(G \cdot H) < \gamma_s(G * H)$

(iii) $\chi(G * H) \geq \chi(G \cdot H) > \chi(G \diamond H) \geq \chi(G \otimes H)$

(iv) $\Delta(G \otimes H) \leq \Delta(G \diamond H) < \Delta(G * H) < \Delta(G \cdot H)$

(v) $\delta(G * H) > \delta(G \cdot H) > \delta(G \otimes H) \geq \delta(G \diamond H)$.

Proof. Let p be any prime number and p, q be any two distinct primes where $q > p$.

The vertex set of GH will be $V(GH) = \{(a, a'), (a, b'), (a, c'), (b, a'), (b, b'), (b, c'), (c, a'), (c, b'), (c, c'), (d, a'), (d, b'), (d, c')\}$. By applying the adjacency rules, graphs will be obtained.

(i) For $G \cdot H$, the vertex (a, a') is adjacent to every other vertices, where the vertex (a, a') dominates every vertices in $G \cdot H$, the domination number will be 1. Similarly, for $G * H$, the domination number is 1. Now, for $G \diamond H$, the vertices $(a, a'), (b, a'), (c, a')$ dominates every persisting vertices. Therefore, the domination number is 3. Similarly, for $G \otimes H$, the domination number is at least 3.

(ii) For $G \diamond H$, by removing the dominating set, the graph will not be disconnected. So, by removing at least 5 dominating vertices, to get split dominating set. Hence, the split domination number will be at least 5. This is similar for $G \otimes H$. Now, for $G \cdot H$, at least 6 vertices required to disconnect the graph. Then the split domination number is at least 6. This is similar for $G * H$.

(iii) As mentioned in case (i), a single vertex (a, a') is adjacent to every other vertices. Clearly, n colours required to paint the graph. This is similar for $G \cdot H$. Now for $G \diamond H$, the vertices $(a, a'), (b, a'), (c, a')$ are adjacent to every other vertices. So, the minimum colours required to colour the graph is 3. This is similar for $G \otimes H$.

(iv) By using the above cases, the maximum degree and minimum degree can be obtained. Hence, the inequality holds. □

Theorem 2.4. *Let* $G = \Gamma_z(\mathbb{Z}_p)$ *and* $G = \Gamma_z(\mathbb{Z}_p)$ *be two cyclic subgroup graphs on a finite group. Then*

(i) $G \diamond H$ *and* $G \otimes H$ *are disconnected.*

(ii) $G * H$ *is connected.*

Proof. Let p be any prime number.

The vertex set of $V(GH) = \{(u, u'), (v, v'), (v, u'), (v, v')\}$.

(i) By applying the adjacency rules for $G \otimes H$ and $G \diamond H$, the resulting graph split into two components, which makes the graph disconnected.

(ii) For $G * H$, every vertex belongs to the vertex set is adjacent with at least two vertices of the proceeding vertices. It is clear that, there is a path in between every vertices which makes the graph connected. □

Theorem 2.5. *Let* $G = \Gamma_z(\mathbb{Z}_{p^2})$ *and* $H = \Gamma_z(\mathbb{Z}_{p^2})$ *be two cyclic subgroup graphs on a finite group. Then*

(i) $\gamma(G * H) < \gamma(G \diamond H) \leq \gamma(G \otimes H)$

(ii) $\gamma_s(G * H) < \gamma_s(G \diamond H) \leq \gamma_s(G \otimes H)$

(iii) $\chi(G * H) > \chi(G \diamond H) \geq \chi(G \otimes H)$

(iv) $\Delta(G \diamond H) \leq \Delta(G \otimes H) < \Delta(G * H)$

(v) $\delta(G \diamond H) \leq \delta(G \otimes H) \leq \delta(G * H)$.

Proof. Let p be any prime number. The vertex set $V(GH) = \{(a, a'), (a, b'), (a, c'), (b, a'), (b, b'), (b, c'), (c, a'), (c, b'), (c, c')\}$. Based on the definition of graph products, adjacency rules will be satisfied.

(i) For $G * H$, the obtained graph is complete. Clearly, domination number will be 1. Now for $(G \diamond H)$, the resulting graph is 4-regular graph. The minimum cardinality of the dominating set is at most 3. The proof is similar for $G \otimes H$.

(ii) For $G * H$, clearly the split domination number is 0. Now, for $(G \diamond H)$, by removing at most 5 vertices from the 4-regular graph to get graph disconnected. Similarly, $\gamma(G \otimes H) \leq 5$.

(iii) For $G * H$, it is clear that n colours required to paint the graph. Hence, $\chi(G * H) = n$. Obviously, $\chi(G \diamond H)$ and $\chi(G \otimes H)$ will be less than $\chi(G * H)$.

(iv) For $G \diamond H$ and $G \otimes H$, the obtained graph is 4-regular graph. Clearly, the maximum degree and minimum degree is 4. Now for $G * H$, the maximum degree and minimum degree will be n. Hence the inequality holds.□

Theorem 2.6. *Let* $G = \Gamma_z(\mathbb{Z}_{pq})$ *and* $H = \Gamma_z(\mathbb{Z}_{pq})$ *be two cyclic subgroup graphs on a finite group. Then*

(i) $\gamma(G * H) < \gamma(G \diamond H) \leq \gamma(G \otimes H)$

(ii) $\gamma_s(G * H) > \gamma_s(G \diamond H) \geq \gamma_s(G \otimes H)$

(iii) $\chi(G \diamond H) \leq \gamma(G \otimes H) < \chi(G * H)$

(iv) $\Delta(G * H) > \Delta(G \diamond H) \geq \Delta(G \otimes H)$

(v) $\delta(G * H) > \delta(G \otimes H) \geq \gamma(G \diamond H)$.

Proof is similar by using above theorem.

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