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SEMICIRCULAR-LIKE AND SEMICIRCULAR ELEMENTS INDUCED BY *p*-ADIC ANALYTIC STRUCTURES AND C^*-ALGEBRAS

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Abstract

In this paper, we study semicircular-like elements, and semicircular elements induced by arbitrarily fixed unital C^*- probability spaces, and p-adic analytic structures over p-adic number fields $\mathbb{Q}_p$, for primes p.

1. Introduction

In this paper, we study weighted-semicircular elements and semicircular elements induced by p-adic analysis over the p-adic number fields $\mathbb{Q}_p$, for $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of all primes in the set $\mathbb{N}$ of all natural numbers. The main purpose of this paper is to expand, or to generalize the constructions of semicircular-like laws and the semicircular law obtained from "non-traditional" free-probability-theoretic senses of [12] and [8], to those from "traditional" free-probability-theoretic senses by tensoring p-adic-analytic structures with unital C^*- probability spaces.

In other words, we are interested in universalized constructions of weighted semicircular elements and corresponding semicircular elements

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over p-adic analysis on $\mathbb{Q}_p$, for $p \in \mathcal{P}$. The constructions of such operators, themselves, are one of the main results of this paper; we also consider free distributions of free reduced words in our weighted-semicircular, and semicircular elements.

Our main results illustrate close connections among number theory, operator theory, operator algebra theory, representation theory, dynamical system and quantum statistics, via free probability theory.

1.1. Remark: Non-Traditional & Traditional Free-Probabilistic Approaches. In the beginning of Introduction, we mentioned about "non-traditional senses of free probability theory." Note that the (usual, or traditional) free probability theory provides noncommutative operator-algebraic version of measure theory and statistics (e.g., [10], [12], [23] through [29], and [31] through [35]). But the $*$-algebra $\mathcal{M}_p$ and corresponding C^*-algebra M_p in the sense of [8], [11] and [12] are "commutative," (Also, see Sections 3, 4 and 5 below), and hence, they have commutative functional analysis (determined up to suitable linear functionals on them). We applied free probability-theoretic "methods," "tools," and "concepts" to study such analysis on these algebras. Remark that, under such "non-traditional" senses, free probability theory well-covers commutative operator-algebraic cases of [8], [9], [11] and [12], however, freeness on commutative structures are trivial (which is not interesting in free-probability-theoretic operator-algebraic point of view); but, in earlier works, we were only interested in operators assigning semicircular-like laws and the semicircular law which can be nicely obtained-and-explained by free-probabilistic settings and language. So, we used concepts and terminology from free probability theory there "non-traditionally." But, also remark that such non-traditional approaches become traditional under free product in [8] and [11], and under tensor product on W^*-algebras in [9].

In this paper, with help of (non-traditional) free-probability-theoretic approaches of [8], [9], [11] and [12], we work on (traditional) free-probability-theoretic structures under tensor products on Banach $*$-algebras.

1.2. Preview and Motivation. Relations between number theory and operator algebra theory have been studied in various different approaches (e.g., [2], [10], [13], [14], [15], [18], [19], [20], [21], [22] and [30]). We cannot help emphasizing that, in particular, there are close connections between primes and operators. For example, in [10], we studied operator theory on Hecke algebras $\mathcal{H}(GL_2(\mathbb{Q}_p))$, where $GL_2(X)$ mean the general linear groups consisting of all invertible (2×2)-matrices over X, via representation theory and (traditional) free probability theory.

In [12], the author and Jorgensen construct weighted-semicircular elements, and corresponding semicircular elements in a certain Banach $*$-algebra $\mathfrak{LS}_p$ induced from the $*$-algebra $\mathcal{M}_p$ consisting of all measurable functions on a p-adic number fields $\mathbb{Q}_p$, for any fixed prime $p \in \mathcal{P}$. For a fixed prime p, one can obtain $|\mathbb{Z}|$-many weighted-semicircular elements $Q_{p,j}$ in certain Banach $*$-probability spaces $\mathfrak{LS}_p(j)$, for all $j \in \mathbb{Z}$. By doing suitable scalar-multiples on $Q_{p,j}$'s, we obtained corresponding semicircular elements $\Theta_{p,j}$'s in $\mathfrak{LS}_p(j)$, for all $j \in \mathbb{Z}$.

In [8], the author constructed the free product Banach $*$-probability space $\mathfrak{LS}$ of the system $\{\mathfrak{LS}_p(j)\}_{p \in \mathcal{P}, j \in \mathbb{Z}}$, over both primes and integers (which is a "traditional" free-probability-theoretic structure), and studied weighted-semicircular elements $Q_{p,j}$'s, and semicircular elements $\Theta_{p,j}$'s in $\mathfrak{LS}$, as free generators of $\mathfrak{LS}$. The free distributions of free reduced words generated by $Q_{p,j}$'s and $\Theta_{p,j}$'s were characterized there.

As an application of [8], in [11], we studied free stochastic calculus on $\mathfrak{LS}$ under the free stochastic motions (or free stochastic processes) determined by the weighted-semicircular laws and the semicircular law of [8].

In this paper, we will extend the frameworks of [8] under tensor products (in the traditional free-probability-theoretic senses), and generalize the results.

1.3. Overview. In Sections 2, we briey introduce backgrounds of our works: free probability theory, and p-adic analysis.

Our (non-traditional) free-probabilistic models on $\mathcal{M}_p$ is established and considered in Sections 3. And then, in Section 4, we construct suitable Hilbert-space representations of $\mathcal{M}_p$, preserving the free-distributional data of Section 3 implying number-theoretic information. Under representation, corresponding C^*-algebras M_p and corresponding (non-traditional) C^*-probabilistic structures are studied in Section 5.

In Section 6, we introduce terminology and concepts about free products which will be used in Sections 7, 8, 9 and 10.

In Section 7, from non-traditional C^*-probabilistic structures of Section 5, we construct-and-study (traditional) C^*-probability spaces induced by tensor products with a unital C^*-probability space (A, ψ) and certain C^*-subalgebras of M_p's determined by both free probability on (A, ψ), and p-adic analytic free-probabilistic structures of Section 5.

In Section 8, we construct our weighted-semicircular elements implying free-distributions from both (A, ψ) and p-adic analysis. Under additional conditions, semicircular elements are naturally constructed from our weighted-semicircular elements. Free distributions for weighted-semicircularity and semicircularity are studied there. And the main results of Section 8 are generalized under free product in Section 9.

In Section 10, the free-distributional data of free reduced words and free sums generated by our (weighted)-semicircular elements are computed.

2. Preliminaries

In this section, we briefly introduce backgrounds of our proceeding works. For more about pure number-theoretic motivation and background, see [16] and [17].

2.1. Free Probability. Readers can check fundamental analytic-and-combinatorial free probability from [26] and [35]. Free probability is understood as the noncommutative operator-algebraic version of classical probability theory and statistics. The classical independence is replaced to be the freeness, by replacing measures to linear functionals. It has various applications not only in pure mathematics (e.g., [23], [24], [25], [27], [28], [29], [32], [33] and [34]), but also in related fields ([1], [3] through [12]).

Here, we will use combinatorial free probability theory of Speicher (e.g., [26]). Especially, in the text, without introducing detailed definitions and combinatorial backgrounds, free moments and free cumulants will be computed. However, some important concepts will be introduced precisely, e.g., see Section 6 and Section 8.1.

2.2. Calculus on $\mathbb{Q}_p$. For more about p-adic analysis and Adelic analysis, see [31]. Let $p \in \mathcal{P}$ be a prime, and let $\mathbb{Q}$ be the set of all rational numbers. Define a non-Archimedean norm $|\cdot|_p$ on $\mathbb{Q}$ by

$$|x|_p = \left| p^k \frac{a}{b} \right|_p = \frac{1}{p^k},$$

whenever $x = p^k \frac{a}{b}$, where $k, a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$. For instance,

$$\left| \frac{8}{3} \right|_2 = \left| 2^3 \cdot \frac{1}{3} \right|_2 = \frac{1}{2^3} = \frac{1}{8},$$

and

$$\left| \frac{8}{3} \right|_3 = \left| 3^{-1} \cdot 8 \right|_3 = \frac{1}{3^{-1}} = 3,$$

and

$$\left| \frac{8}{3} \right|_q = 1, \text{ whenever } q \in \mathcal{P} \setminus \{2, 3\}.$$

The p-adic number field $\mathbb{Q}_p$ is the maximal p-norm closure in $\mathbb{Q}$. So, for the norm topology, the set $\mathbb{Q}_p$ forms a Banach space (e.g., [31]).

All elements x of $\mathbb{Q}_p$ are expressed by

$$x = \sum_{k=-N}^{\infty} x_k p^k, \text{ with } x_k \in \{0, 1, \ldots, p-1\},$$

for $N \in \mathbb{N}$, decomposed by

$$x = \sum_{l=-N}^{-1} x_l p^l + \sum_{k=0}^{\infty} x_k p^k.$$

If $x = \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$, then we call x, a p-adic integer. Remark that, $x \in \mathbb{Q}_p$ is a p-adic integer, if and only if $| x |_p \leq 1$. So, by collecting all p-adic integers in $\mathbb{Q}_p$, one can define the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : | x |_p \leq 1\}.$$

Under the p-adic addition and the p-adic multiplication of [31], this Banach space $\mathbb{Q}_p$ forms a ring algebraically, i.e., $\mathbb{Q}_p$ is a Banach ring.

Also, one can understand the Banach ring $\mathbb{Q}_p$ as a measure space,

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

where $\sigma(\mathbb{Q}_p)$ is the σ-algebra of $\mathbb{Q}_p$, consisting of all μ_p- measurable subsets, where μ_p is the left-and-right additive invariant Haar measure on $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1.$$

If we define

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{2.2.1}$$

for all $k \in \mathbb{Z}$, then these μ_p- measurable subsets $U_k's$ of (2.2.1) satisfy

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} U_k,$$

and

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \text{ for all } k \in \mathbb{Z},$$

and

$$\ldots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \ldots. \tag{2.2.2}$$

In fact, the family $\{U_k\}_{k \in \mathbb{Z}}$ forms a basis of the Banach topology for $\mathbb{Q}_p$ (e.g., [31]).

Define now subsets ∂_k of $\mathbb{Q}_p$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{2.2.3}$$

We call such μ_p- measurable subsets ∂_k, the k-th boundaries (of U_k) in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. By (2.2.2) and (2.2.3), one obtains that

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} \partial_k, \tag{2.2.4}$$

and

$$\mu_p(\partial_k) = \mu_p(U_k) - \mu_p(U_{k+1}) = \frac{1}{p^k} - \frac{1}{p^{k+1}},$$

for all $k \in \mathbb{Z}$, where $\sqcup$ means the disjoint union.

Now, let $\mathcal{M}_p$ be the (pure-algebraic) algebra,

$$\mathcal{M}_p = \mathbb{C}[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}], \tag{2.2.5}$$

where χ_S are the usual characteristic functions of μ_p- measurable subsets S of $\mathbb{Q}_p$.

i.e., the algebra $\mathcal{M}_p$ is the algebra consisting of μ_p- measurable functions. So, $f \in \mathcal{M}_p$, if and only if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S, \text{ with } t_S \in \mathbb{C}, \tag{2.2.5}$$

where Σ is the finite sum.

Then the set $\mathcal{M}_p$ of (2.2.5) forms a $*$-algebra over $\mathbb{C}$. Indeed, this algebra $\mathcal{M}_p$ has the adjoint,

$$\left(\sum_{S \in \sigma(G_p)} t_S \chi_S \right)^* \stackrel{def}{=\!=} \sum_{S \in \sigma(G_p)} \overline{t_S} \chi_S,$$

where $t_S \in \mathbb{C}$, having its conjugate $\overline{t_S}$ in $\mathbb{C}$.

Let $f \in \mathcal{M}_p$ be in the sense of (2.2.5)'. Then one can define the p-adic integral of f by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S). \tag{2.2.6}$$

Note that, by (2.2.4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a unique subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{2.2.7}$$

satisfying

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \int_{\mathbb{Q}_p} \sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} d\mu_p$$

$$= \sum_{j \in \Lambda_S} \mu_p(S \cap \partial_j)$$

by (2.2.6)

$$\leq \sum_{j \in \Lambda_S} \mu_p(\partial_j) = \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{2.2.8}$$

by (2.2.4), where Λ_S is in the sense of (2.2.7).

More precisely, one can get the following proposition.

Proposition 2.1. *Let* $S \in \sigma(\mathbb{Q}_p)$ *and let* $\chi_S \in \mathcal{M}_p$. *Then there exist* $r_j \in \mathbb{R}$, *such that*

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S,$$

and

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right). \tag{2.2.9}$$

Proof. The existence of r_j and the p-adic integration in (2.2.9) is guaranteed by (2.2.6), (2.2.7) and (2.2.8).

3. Analysis on $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and $\mathbb{Q}_p$, the corresponding p-adic number field, and let $\mathcal{M}_p$ be the $*$-algebra (2.2.5). In this section, we establish a suitable (non-traditional, in the sense of Section 1.1,) free-probabilistic model on $\mathcal{M}_p$.

Let U_k and ∂_k be in the sense of (2.2.1), respectively, (2.2.3) in $\mathbb{Q}_p$, i.e.,

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z},$$

and

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{3.1}$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by the p-adic integration (2.2.6),

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f \, d\mu_p, \text{ for all } f \in \mathcal{M}_p. \tag{3.2}$$

Then, by (3.2), one naturally obtain that

$$\varphi_p(\chi_{U_j}) = \frac{1}{p^j}, \text{ and } \varphi_p(\chi_{\partial_j}) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

for all $j \in \mathbb{Z}$, by (2.2.2) and (2.2.4).

Definition 3.1. The pair $(\mathcal{M}_p, \varphi_p)$ is called the p-adic free probability space, for $p \in \mathcal{P}$, where φ_p is the linear functional (3.2) on $\mathcal{M}_p$.

Let ∂_k be the k-th boundary of (3.1) in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1, k_2} \chi_{\partial_{k_1}},$$

and

$$\begin{aligned} \varphi_p(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}) &= \delta_{k_1, k_2} \varphi_p(\chi_{\partial_{k_1}}) \\ &= \delta_{k_1, k_2} \left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}} \right), \end{aligned} \tag{3.3}$$

where δ means the Kronecker delta.

Proposition 3.1. *Let* $(j_1, \ldots, j_N) \in \mathbb{Z}^N$, *for* $N \in \mathbb{N}$. *Then*

$$\prod_{l=1}^{N} \chi_{\partial_{j_l}} = \delta_{(j_1, \ldots, j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p,$$

and

$$\varphi_p\left(\prod_{l=1}^{N}\chi_{\partial_{j_l}}\right) = \delta_{(j_1, \ldots, j_N)}\left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}}\right), \tag{3.4}$$

where

$$\delta_{(j_1,\ldots,j_N)} = \left(\prod_{l=1}^{N-1}\chi_{\partial_{j_l, j_{l+1}}}\right) = (\delta_{j_N, j_1}).$$

Proof. The proof of (3.4) is done by induction on (3.3).

Thus, one can get that, for any $S \in \sigma(\mathbb{Q}_p)$,

$$\varphi_p(\chi_S) = \sum_{j\in\Lambda_S} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{3.5}$$

by (3.4), where $0 \leq r_j \leq 1$ and Λ_S are in the sense of (2.2.9), for all $j \in \mathbb{Z}$. Also, if $S_1, S_2 \in \sigma(\mathbb{Q}_p)$, then

$$\begin{aligned}\chi_{S_1}\chi_{S_2} &= \left(\sum_{k\in\Lambda_{S_1}}\chi_{S_1\cap\partial_k}\right)\left(\sum_{k\in\Lambda_{S_2}}\chi_{S_2\cap\partial_j}\right)\\ &= \sum_{(k,j)\in\Lambda_{S_1}\times\Lambda_{S_2}}(\chi_{S_1\cap\partial_k}\chi_{S_2\cap\partial_j})\\ &= \sum_{(k,j)\in\Lambda_{S_1}\times\Lambda_{S_2}}\delta_{k,j}\chi_{(S_1\cap S_2)\cap\partial_j}\\ &= \sum_{j\in\Lambda_{S_1,S_2}}\chi_{(S_1\cap S_2)\cap\partial_j}, \end{aligned} \tag{3.6}$$

where

$$\Lambda_{S_1,S_2} = \Lambda_{S_1} \cap \Lambda_{S_2}.$$

By (3.5) and (3.6), one can get that there exist $w_j \in \mathbb{R}$, such that

$$0 \leq w_j \leq 1, \text{ for all } j \in \Lambda_{S_1,S_2},$$

and

$$\varphi_p(\chi_{S_1}\chi_{S_2}) = \sum_{j \in \Lambda_{S_1, S_2}} w_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \tag{3.7}$$

for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. In (3.7), definitely, if Λ_{S_1, S_2} is empty, then $\varphi_p(\chi_{S_1}\chi_{S_2}) = 0$.

Proposition 3.2. *Let* $S_l \in \sigma(\mathbb{Q}_p)$, *and let* $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, *for* $l = 1, \ldots, N$, *for* $N \in \mathbb{N}$. *Let*

$$\Lambda_{S_1, \ldots, S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} *are in the sense of* (2.2.7), *for* $l = 1, \ldots, N$. *Then there exist* $r_j \in \mathbb{R}$, *such that*

$$0 \le r_j \le 1 \text{ in } \mathbb{R}, \text{ for } j \in \Lambda_{S_1, \ldots, S_N},$$

and

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l} \right) = \sum_{j \in \Lambda_{S_1, \ldots, S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right). \tag{3.8}$$

Proof. The proof of (3.8) is done by induction on (3.7).

4. Representations of $(\mathcal{M}_p, \varphi_p)$

Fix a prime p in $\mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the p-adic free probability space. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2-space of $\mathbb{Q}_p$,

$$H_p \underset{=\!=}{def} L^2(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p) = L^2(\mathbb{Q}_p) \tag{4.1}$$

over $\mathbb{C}$, consisting of all square-integrable μ_p-measurable functions on $\mathbb{Q}_p$. Then this L^2-space is a well-defined Hilbert space equipped with its inner product $<\ ,\ >_2$,

$$\langle h_1, h_2 \rangle_2 \underset{=\!=}{def} \int_{\mathbb{Q}_p} h_1 h_2^* d\mu_p, \tag{4.2}$$

for all $h_1, h_2 \in H_p$.

The L^2-space H_p of (4.1) is the $\| \cdot \|_2$-norm completion in $\mathcal{M}_p$, where

$$\| f \|_2 \underset{=}{\underline{def}} \sqrt{\langle f, f \rangle_2}, \text{ for all } f \in H_p,$$

where $<, >_2$ is the inner product (4.2) on H_p.

Definition 4.1. We call the Hilbert space H_p of (4.1), the p-adic Hilbert space.

By the definition (4.1) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an algebra-action α^p,

$$\alpha^p(f)(h) = fh, \text{ for all } h \in H_p, \tag{4.3}$$

for all $f \in \mathcal{M}_p$. i.e., for any $f \in \mathcal{M}_p$, the image $\alpha^p(f)$ is a multiplication operator on H_p with its symbol f contained in the operator algebra $B(H_p)$ of all bounded linear operators on H_p.

Notation Denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{M}_p$. Also, for convenience, denote $\alpha_{\chi_S}^p$ simply by α_S^p, for all $S \in \sigma(\mathbb{Q}_p)$. For instance,

$$\alpha_{U_k}^p = \alpha_{\chi_{U_k}}^p = \alpha^p(\chi_{U_k}),$$

and

$$\alpha_{\partial_k}^p = \alpha_{\chi_{\partial_k}}^p = \alpha^p(\chi_{\partial_k}),$$

for all $k \in \mathbb{Z}$, where U_k and ∂_k are in the sense of (3.1), for all $k \in \mathbb{Z}$.

By the definition (4.3), the linear morphism α^p is a well-determined $*$-algebra action of $\mathcal{M}_p$ acting on H_p, equivalently, it is a well-defined $*$-homomorphism from $\mathcal{M}_p$ into $B(H_p)$; Indeed,

$$\begin{aligned} \alpha_{f_1, f_2}^p(h) &= f_1 f_2 h = f_1(f_2 h) \\ &= f_1(\alpha_{f_2}^p(h)) = \alpha_{f_1}^p \alpha_{f_2}^p(h), \end{aligned} \tag{4.4}$$

for all $h \in H_p$, for all $f_1, f_2 \in \mathcal{M}_p$; and

$$\langle \alpha_f^p(h_1), h_2 \rangle_2 = \langle fh_1, h_2 \rangle_2 = \int_{\mathbb{Q}_p} fh_1 h_2^* d\mu_p$$

$$= \int_{\mathbb{Q}_p} h_1 f h_2^* d\mu_p = \int_{\mathbb{Q}_p} h_1 (h_2 f^*)^* d\mu_p$$

$$= \int_{\mathbb{Q}_p} h_1 (f^* h_2)^* d\mu_p = \langle h_1, \alpha_{f^*}^p(h_2) \rangle_2, \tag{4.5}$$

for all $h_1, h_2 \in H_p$, for all $f \in \mathcal{M}_p$.

Therefore, one obtains the following proposition.

Proposition 4.1. *The linear morphism* α^p *of (4.3) is a well-defined* $*$-*algebra action of* $\mathcal{M}_p$ *acting on* H_p. *Equivalently, the pair* (H_p, α^p) *is a well-determined Hilbert-space representation of* $\mathcal{M}_p$.

Proof. By (4.4) and (4.5), the morphism α^p of (4.3) is a $*$-homomorphism from $\mathcal{M}_p$ to $B(H_p)$. So, the pair (H_p, α^p) is a Hilbert-space representation of $\mathcal{M}_p$.

Definition 4.2. The Hilbert-space representation (H_p, α^p) is said to be the p-adic representation of $\mathcal{M}_p$.

Depending on the p-adic representation (H_p, α^p) of $\mathcal{M}_p$, one can construct the C^*-algebra M_p in the operator algebra $B(H_p)$.

Definition 4.3. Let M_p be the operator-norm closure of $\mathcal{M}_p$ in the operator algebra $B(H_p)$, i.e.,

$$M_p \underset{=\!=}{\underline{def}} \overline{\alpha^p(\mathcal{M}_p)} = \overline{\mathbb{C}[\alpha_f^p : f \in \mathcal{M}_p]} \text{ in } B(H_p), \tag{4.6}$$

where $\overline{X}$ mean the operator-norm closures of subsets X of $B(H_p)$. Then this C^*-subalgebra M_p of $B(H_p)$ is called the p-adic C^*-algebra of $(\mathcal{M}_p, \varphi_p)$.

5. Functional Analysis on M_p

Throughout this section, let's fix a prime $p \in \mathcal{P}$, and let $(\mathcal{M}_p, \varphi_p)$ be the corresponding p-adic free probability space. And let (H_p, α^p) be the p-adic representation of $\mathcal{M}_p$, and let M_p be the corresponding p-adic C^*-algebra (4.6) of $(\mathcal{M}_p, \varphi_p)$. We here consider suitable (non-traditional, in the sense of Section 1.1,) free-probabilistic models on M_p. In particular, we are interested in a system $\{\varphi_j^p\}_{j \in \mathbb{Z}}$ of linear functionals on M_p, determined by the j-th boundaries $\{\partial_j\}_{j \in \mathbb{Z}}$ of $\mathbb{Q}_p$.

Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi_j^p(a) \underline{\underline{def}} \langle \alpha_a^p(\chi_{\partial_j}), \chi_{\partial_j} \rangle_2, \text{ for all } a \in M_p, \tag{5.1}$$

for all $j \in \mathbb{Z}$, where $\langle \, , \rangle_2$ is the inner product (4.2) on the p-adic Hilbert space H_p of (4.1).

First, remark that, if $a \in M_p$, then

$$a = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \, \chi_S \text{ in } M_p,$$

where Σ is finite or infinite (limit of finite) sum (s) under C^*-topology of M_p.

Definition 5.1. Let $j \in \mathbb{Z}$, and let φ_j^p be the linear functional (5.1) on the p-adic C^*-algebra M_p. Then the pair (M_p, φ_j^p) is said to be the j-th p-adic C^*-probability space.

So, one can get the system

$$\{(M_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of the j-th p-adic C^*-probability spaces.

Now, fix $j \in \mathbb{Z}$, and take the corresponding j-th p-adic C^*-probability space (M_p, φ_j^p). For $S \in \sigma(\mathbb{Q}_p)$, and a generating operator $\alpha_S^p \in M_p$, one has that

$$\begin{aligned}\varphi_j^p(\alpha_S^p) &= \langle \alpha_S^p(\chi_{\partial_j}), \chi_{\partial_j} \rangle_2 = \langle \chi_{S \cap \partial_j}, \chi_{\partial_j} \rangle_2 \\ &= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} \chi_{\partial_j}^* d\mu_p = \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} \chi_{\partial_j} d\mu_p \\ &= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} d\mu_p = \mu_p(S \cap \partial_j) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),\end{aligned} \tag{5.2}$$

for some $0 \leq r_S \leq 1$ in $\mathbb{R}$.

More precisely, we obtain the following theorem.

Theorem 5.1. *Let* $S_l \in \sigma(\mathbb{Q}_p)$, *and* $\alpha_{S_l}^p = \alpha^p(\chi_{S_l}) \in (M_p, \varphi_j^p)$, *for a fixed* $j \in \mathbb{Z}$, *for* $l = 1, \ldots, N$, *for* $N \in \mathbb{N}$. *Then there exists* $r_{(S_1, \ldots, S_N)} \in \mathbb{R}$, *such that*

$$0 \leq r_{(S_1, \ldots, S_N)} \leq 1 \text{ in } \mathbb{R},$$

and

$$\varphi_j\left(\left(\prod_{l=1}^{N} \alpha_{S_l}^p\right)^n\right) = r_{(S_1, \ldots, S_N)}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{5.4}$$

for all $n \in \mathbb{N}$.

Proof. The formula (5.4) is obtained by (3.8) and (5.2). See [9] for details.

The above free-distributional data (5.4) characterizes the "joint" free distributions of finitely many projections $\alpha_{S_1}^p, \ldots, \alpha_{S_N}^p$ in the j-th C^*-probability space (M_p, φ_j^p), for $j \in \mathbb{Z}$.

6. Free Product ∗-Probability Spaces

Before proceeding our works we here introduce concepts and terms used below. More above free products, and free product ∗-probability spaces, see [26] and [35].

Let (A_k, φ_k) be arbitrary (topological) $*$-probability spaces of (topological) $*$-algebras A_k, and (bounded) linear functionals φ_k, for $k \in \Delta$, where Δ is an arbitrary countable (finite or infinite) index set.

The free product $*$-algebra A,

$$A = \underset{l \in \Delta}{*} A_l$$

is the $*$-algebra generated by the noncommutative reduced words in $\bigcup_{l=1}^{N} A_l$, having the free product linear functional,

$$\varphi = \underset{l \in \Delta}{*} \varphi_l.$$

This (topological) $*$-algebra A is understood as a (Banach) vector space,

$$\mathbb{C} \oplus \left(\overset{\infty}{\underset{n=1}{\oplus}} \left(\underset{(i_1, \ldots, i_n) \in alt(\Delta^n)}{\oplus} \left(\overset{n}{\underset{k=1}{\otimes}} A_{i_k}^{o} \right) \right) \right)$$

With

$$A_{i_k}^{o} = A_{i_k} \ominus \mathbb{C}, \text{ for all } k = 1, \ldots, n, \tag{6.1}$$

where

$$alt(\Delta^n) = \left\{ (i_1, \ldots, i_n) \,\middle|\, \begin{array}{r} (i_1, \ldots, i_n) \in \Delta^n \\ i_1 \neq i_2, i_2 \neq i_3, \\ \ldots, i_{n-1} \neq i_n \end{array} \right\},$$

for all $n \in \mathbb{N}$, and where $\oplus$ is the (topological) direct product; and $\otimes$ is the (topological) tensor product of (Banach) vector spaces.

In particular, if an element $a \in A$ is a free "reduced" word,

$$a = \prod_{l=1}^{n} a_{i_l} \text{ in } A, \tag{6.2}$$

then one can understand a as an equivalent "vector" $\overset{n}{\underset{l=1}{\otimes}} a_{i_l}$ in the vector space A of (6.1), contained in a direct summand, $\overset{n}{\underset{k=1}{\otimes}} A_{i_k}^{o}$.

Or this free reduced word a of (6.2) can be regarded as an "operator" in the (topological) $*$-subalgebra

$$\underset{k=1}{\overset{n}{\otimes_{\mathbb{C}}}} A_{i_k} = \mathbb{C} \oplus \left(\underset{k=1}{\overset{n}{\otimes}} A_{i_k}^o \right),$$

of A (up to $*$-isomorphisms), where $\otimes_{\mathbb{C}}$ means the (topological) tensor product of (topological) $*$-algebras.

7. On C^*-Subalgebras $\mathfrak{S}_p$ of M_p

In this section, from our non-traditional free-probabilistic structures, we construct traditional free-probabilistic structures, and then establish foundations to study semicircular-like laws and the corresponding semicircular law in following sections.

7.1. C^*-Subalgebras $\mathfrak{S}_p$ of M_p. Let M_p be the p-adic C^*-algebra for $p \in \mathcal{P}$. Take operators

$$P_{p,j} = \alpha_{\partial_j}^p \in M_p, \tag{7.1.1}$$

for all $j \in \mathbb{Z}$, for $p \in \mathcal{P}$.

As we have seen, these operators $P_{p,j}$ are projections on the p-adic Hilbert space H_p in M_p, i.e.,

$$P_{p,j}^* = P_{p,j} = P_{p,j}^2,$$

moreover,

$$P_{p,j_1} P_{p,j_2} = \delta_{j_1,j_2} P_{p,j_1}, \tag{7.1.1}$$

for all $p \in \mathcal{P}$, and $j, j_1, j_2 \in \mathbb{Z}$. We now restrict our interests to these projections $P_{p,j}$ of (7.1.1).

Definition 7.1. Fix $p \in \mathcal{P}$, and let $\mathfrak{S}_p$ be the C^*-subalgebra

$$\mathfrak{S}_p = C^*(\{P_{p,j}\}_{j \in \mathbb{Z}}) = \overline{\mathbb{C}[\{P_{p,j}\}_{j \in \mathbb{Z}}]} \text{ of } M_p, \tag{7.1.2}$$

where $P_{p,j}$ are projections (7.1.1), for all $j \in \mathbb{Z}$. We call this C^*-subalgebra $\mathfrak{S}_p$, the p-adic projection (C^*)- subalgebra of M_p.

The p-adic projection subalgebra $\mathfrak{S}_p$ satisfies the following structure theorem in M_p.

Proposition 7.1. *Let* $\mathfrak{S}_p$ *be the p-adic projection subalgebra* (7.1.2) *of the p-adic* C^**-algebra* M_p. *Then*

$$\mathfrak{S}_p \overset{*\text{-iso}}{=\!=} \underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot P_{p,j}) \overset{*\text{-iso}}{=\!=} \mathbb{C}^{\oplus |\mathbb{Z}|}, \tag{7.1.3}$$

in M_p.

Proof. The isomorphism theorem (7.1.3) is proven by (7.1.1) and (7.1.2).

By the definition (7.1.2) of the p-adic projection subalgebras $\mathfrak{S}_p$, one can get the (non-traditional) C^*-probability spaces

$$(\mathfrak{S}_p, \varphi_j^p), \text{ for } p \in \mathcal{P} \text{ and } j \in \mathbb{Z}. \tag{7.1.4}$$

as a free-probabilistic sub-structures of (M_p, φ_j^p)'s where the linear functional φ_j^p of (7.1.4) mean the restrictions $\varphi_j^p |_{\mathfrak{S}_p}$ on $\mathfrak{S}_p$ of the linear functionals φ_j^p of (5.1) on M_p, for all $j \in \mathbb{Z}$, and $p \in \mathcal{P}$.

Notation. For convenience, we denote the C^*-probability spaces $(\mathfrak{S}_p, \varphi_j^p)$ of (7.1.4) by $\mathfrak{S}_{p,j}$, for all $p \in \mathcal{P}$, and $j \in \mathbb{Z}$.

Proposition 7.2. *Let* $\mathfrak{S}_{p,j}$ *be the* C^**-probability space* (7.1.4) *for* $p \in \mathcal{P}$, $j \in \mathbb{Z}$, *and let* $P_{p,k}$ *be the generating projections* (7.1.1) *of* $\mathfrak{S}_{p,j}$, *for all* $k \in \mathbb{N}$. *Then*

$$\varphi_j^p((P_{p,k})^n) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \textit{ for all } n \in \mathbb{N}, \tag{7.1.5}$$

for all $k \in \mathbb{Z}$.

Proof. The formula (7.1.5) is proven by (5.3) and (5.4).

Let ϕ be the Euler totient function, the arithmetic function $\phi : \mathbb{N} \to \mathbb{C}$ defined by

$$\phi(n) = \left| \left\{ k \in \mathbb{N} \,\middle|\, \begin{array}{c} 1 \leq k \leq n \text{ in } \mathbb{N}, \\ and \ \gcd(k, n) = 1 \end{array} \right\} \right|, \tag{7.1.6}$$

for all $n \in \mathbb{N}$, where $|Y|$ mean the cardinalities of sets Y, and $\gcd(\ ,\ )$ is the greatest common divisor. It is well-known that

$$\phi(p) = p - 1 = p\left(1 - \frac{1}{p}\right), \text{ for all } p \in \mathcal{P}.$$

So, the above formula (7.1.5) can be re-formulated to be

$$\varphi_j^p(P_{p,k}) = \delta_{j,k}\left(\frac{\phi(p)}{p^{j+1}}\right), \text{ for all } p \in \mathcal{P},\ j \in \mathbb{Z}. \tag{7.1.7}$$

7.2. On Tensor Product Banach $*$-Algebras $A \oplus_{\mathbb{C}} \mathfrak{S}_p$. Throughout this section, let's fix $p \in \mathcal{P}$, and a unital C^*-probability space (A, ψ), satisfying

$$\psi(1_A) = 1,$$

where 1_A is the unit (or the multiplication-identity) of A, satisfying $1_A a = a = a 1_A$, for all $a \in A$.

Define now the tensor product C^*-algebra $\mathfrak{S}_p^A$ by

$$\mathfrak{S}_p^A \ \underline{\underline{def}}\ A \otimes_{\mathbb{C}} \mathfrak{S}_p, \tag{7.2.1}$$

where A is the C^*-algebra from the given C^*-probability space (A, ψ) and $\mathfrak{S}_p$ is the p-adic projection subalgebra (7.1.2) of the p-adic C^*-algebra M_p.

Define linear functionals $\psi_{p,j}$ on the C^*-algebra $\mathfrak{S}_p^A$ of (7.2.1) by bounded linear transformations from $\mathfrak{S}_p^A$ into $\mathbb{C}$, satisfying

$$\psi_{p,j}(a \otimes P_{p,k}) = \varphi_j^p(\psi(a)P_{p,k}), \tag{7.2.2}$$

for all $a \in (A, \psi)$, and $j, k \in \mathbb{Z}$, where $P_{p,k} = \alpha^{p}_{\partial_k}$ are the generating projections (7.1.1) of $\mathfrak{S}_p$.

The linear functionals $\psi_{p,j}$' s of (7.2.2) are indeed well-defined on $\mathfrak{S}_p^A$. Thus, the pairs

$$\mathfrak{S}_{p,j}^A \underline{\underline{denote}} (\mathfrak{S}_{p,j}^A, \psi_{p,j}), \tag{7.2.3}$$

form well-defined (traditional) C^*-probability spaces, for all $j \in \mathbb{Z}$.

Definition 7.2. Let $\mathfrak{S}_p^A$ be the tensor product C^*-algebra (7.2.1) for a fixed $p \in \mathcal{P}$, and let $\psi_{p,j}$ be linear functionals (7.2.2), for all $j \in \mathbb{Z}$. Then $\mathfrak{S}_p^A$ is called the A-tensor p-adic projection $(C^*\text{-})$ algebra for $p \in \mathcal{P}$. The C^*-probability spaces

$$\mathfrak{S}_p^A \ \underline{\underline{denote}} (\mathfrak{S}_p^A, \psi_{p,j}) \text{ of} \tag{7.2.3}$$

are said to be the A-tensor j-th p-adic $(C^*\text{-})$ probability spaces (induced by $\mathfrak{S}_p$), for all $j \in \mathbb{Z}$.

Let $\mathfrak{S}_{p,j}^A$ be the A-tensor j-th p-adic probability space (7.2.3) induced by $\mathfrak{S}_p$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Observe that

$$\psi_{p,j}(a \otimes P_{p,k}) = \varphi_j^p(\psi(a)P_{p,k})$$

by (7.2.2)

$$= \psi(a)\left(\delta_{j,k}\frac{\phi(p)}{p^{j+1}}\right) = \delta_{j,k}\psi(a)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{7.2.4}$$

by (7.1.5) and (7.1.7), for all $a \in (A, \psi)$, and $k, j \in \mathbb{Z}$.

Proposition 7.3. *Let* $T_{p,k}^a = a \otimes P_{p,k}$ *be an operator of the A-tensor j-th p-adic probability space* $\mathfrak{S}_{p,j}^A$, *for* $a \in (A, \psi)$, *and* $k \in \mathbb{Z}$. *Then*

$$\psi_{p,j}((T_{p,j}^a)^n) = \delta_{j,k}\psi(a^n)\frac{\phi(p)}{p^{j+1}}$$

$$= \delta_{j,k}\psi(a^n)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{7.2.5}$$

for all $n \in \mathbb{N}$.

Proof. Let $T_{p,k}^{a} = a \otimes P_{p,k} \in \mathfrak{S}_{p,j}^{A}$ as above. Then

$$(T_{p,k}^{a})^{n} = a^{n} \otimes P_{p,k}^{n} = a^{n} \otimes P_{p,k},$$

for all $n \in \mathbb{N}$.

So, we obtain that

$$\psi_{p,j}((T_{p,k}^{a})^{n}) = \psi_{p,j}(a^{n} \otimes P_{p,k}) = \delta_{j,k} \frac{\psi(a^{n})\phi(p)}{p^{j+1}}$$

$$= \delta_{j,k}\psi(a^{n})\left(\frac{1}{p^{j}} - \frac{1}{p^{j+1}}\right),$$

by (7.2.4), for all $n \in \mathbb{N}$.

Therefore, the above free-distributional data (7.2.5) holds.

7.3. Certain Banach $*$-Probability Spaces Induced by $\mathfrak{S}_{p,j}^{A}$. In this section, we fix $p \in \mathcal{P}$, and $j \in \mathbb{Z}$, and a unital C^{*}-probability space (A, ψ). Let

$$\mathfrak{S}_{p,j}^{A} = (\mathfrak{S}_{p}^{A}, \psi_{p,j})$$

be the A-tensor j-th p-adic probability space (7.2.3). Also, we let

$$T_{p,k}^{a} \underline{\underline{denote}}\, a \otimes P_{p,k}, \text{ with } P_{p,k} = \alpha_{\partial_{k}}^{p}, \tag{7.3.1}$$

for all $a \in (A, \psi)$, $k \in \mathbb{Z}$.

Let's now focus on the A-tensor p-adic projection algebra $\mathfrak{S}_{p}^{A} = A \otimes_{\mathbb{C}} \mathfrak{S}_{p}$, and define linear morphisms c_{p}^{A} and a_{p}^{A} "acting on $\mathfrak{S}_{p}^{A}$," by the bounded linear transformations satisfying

$$c_{p}^{A}(T_{p,k}^{a}) = T_{p,k+1}^{a} = a \otimes P_{p,k+1},$$

and

$$a_{p}^{A}(T_{p,k}^{a}) = T_{p,k-1}^{a} = a \otimes P_{p,k-1}, \tag{7.3.2}$$

for all generating operators $T_{p,k}^{a}$ of (7.3.1) in $\mathfrak{S}_{p}^{A}$.

Such bounded linear transformations c_p^A and a_p^A of (7.3.2) are well-defined on $\mathfrak{S}_p^A$ by the structure theorem (7.1.3) of the p-adic projection algebra $\mathfrak{S}_p$ (which is the tensor-factor of $\mathfrak{S}_p^A$), and by the definition (7.2.1) of the A-tensor p-adic algebra $\mathfrak{S}_p^A$. Note that such linear morphisms c_p^A and a_p^A can be understood as Banach-space operators by regarding the C^*-algebra $\mathfrak{S}_p^A$ as a Banach space. In other words, c_p^A and a_p^A are contained in the operator space $B(\mathfrak{S}_p^A)$ in the sense of [13].

Remark 7.1. In [8], [11] and [12], we introduced the Banach-space operators c_p and a_p acting on the p-adic projection subalgebra $\mathfrak{S}_p$ defined by

$$c_p(P_{p,k}) = P_{p,k+1}, \text{ and } a_p(P_{p,k}) = P_{p,k-1},$$

for all $k \in \mathbb{Z}$, for a fixed prime p. So, one can/may regard our operators c_p^A and a_p^A of (7.3.2) as the equivalent forms,

$$1_A \otimes c_p, \text{ respectively, } 1_A \otimes a_p$$

on $A \otimes_{\mathbb{C}} \mathfrak{S}_p = \mathfrak{S}_p^A$.

Definition 7.3. Let c_p^A and a_p^A be the operators (7.3.2) on $\mathfrak{S}_p^A$. Then we call them, the p-adic A-creation, respectively, the p-adic A-annihilation on $\mathfrak{S}_p^A$. Define now a new operator l_p^A by

$$l_p^A = c_p^A + a_p^A \text{ on } \mathfrak{S}_p^A. \tag{7.3.3}$$

This operator l_p^A is called the p-adic A-radial operator on $\mathfrak{S}_p^A$.

By (7.3.3), the p-adic A-radial operator l_p^A is regarded as a well-defined Banachspace operator in the operator space $B(\mathfrak{S}_p^A)$, too (e.g., [13]).

Consider that: if $T_{p,k}^{a}$ are in the sense of (7.3.1) in $\mathfrak{S}_p^A$, then

$$c_p^A a_p^A (T_{p,k}^a) = c_p^A (T_{p,k-1}^a) = T_{p,k}^a,$$

and

$$a_p^A c_p^A (T_{p,k}^a) = a_p^A (T_{p,k+1}^a) = T_{p,k}^a,$$

i.e.,

$$c_p^A a_p^A 1_{\mathfrak{S}_p^A} = a_p^A c_p^A \text{ on } \mathfrak{S}_p^A, \tag{7.3.4}$$

where

$$1_{\mathfrak{S}_p^A} = 1_A \otimes 1_{\mathfrak{S}_p} \in B(\mathfrak{S}_p^A),$$

where 1_A is the unit of A, and $1_{\mathfrak{S}_p}$ is the identity operator on $\mathfrak{S}_p$, satisfying

$$1_{\mathfrak{S}_p}(T) = T, \text{ for all } T \in \mathfrak{S}_p.$$

Proposition 7.4. *Let* c_p^A *and* a_p^A *be the p-adic A-creation, respectively, the p-adic A-annihilation of* (7.3.2) *in* $B(\mathfrak{S}_p^A)$. *Then*

$$(c_p^A)^{n_1}(a_p^A)^{n_2} = (a_p^A)^{n_2}(c_p^A)^{n_1},$$

and

$$\begin{aligned}(c_p^A a_p^A)^n &= (c_p^A)^n (a_p^A)^n = 1_{\mathfrak{S}_p^A}\\ &= (a_p^A)^n (c_p^A)^n = (a_p^A c_p^A)^n,\end{aligned} \tag{7.3.5}$$

for all $n_1, n_2, n \in \mathbb{N}$.

Proof. The proof of (7.3.5) is done by (7.3.4).

By (7.3.4) and (7.3.5), one can realize that: if l_p^A is the p-adic A-radial operator on $\mathfrak{S}_p^A$, then

$$(l_p^A)^n = \sum_{k=0}^{n} \binom{n}{k} (c_p^A)^k (a_p^A)^{n-k}, \tag{7.3.6}$$

for all $n \in \mathbb{N}$, with identity:

$$(c_p^A)^0 = 1_{\mathfrak{S}_p^A} = (a_p^A)^0,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for all } k \leq n \leq \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

Define a closed subspace $\mathfrak{L}_p^A$ of the operator space $B(\mathfrak{S}_p^A)$ by

$$\mathfrak{L}_p^{A^n} = \overline{\mathbb{C}[\{l_p^A\}]}^{\|\cdot\|}, \tag{7.3.7}$$

where $\overline{Y}^{\|\cdot\|}$ mean the operator-norm closures of subsets Y of $B(\mathfrak{S}_p^A)$ and $\mathbb{C}[X]$ mean the collection of all multi-variable polynomials in X, naturally closed under addition and multiplication, and where

$$\| X \| = \sup \left\{ \| X(T) \|_{\mathfrak{S}_p^A} \mid \begin{array}{c} T \in \mathfrak{S}_p^A, \text{ such that} \\ \| T \|_{\mathfrak{S}_p^A} = 1 \end{array} \right\},$$

where $\| \cdot \|_{\mathfrak{S}_p^A}$, is the C^*-norm on the p-adic A-tensor projection algebra $\mathfrak{S}_p^A$.

By the definition (7.3.7), this closed subspace $\mathfrak{L}_p^A$ forms a algebra embedded in $B(\mathfrak{S}_p^A)$. Furthermore, if we define an operation

$$\left(\sum_{n=0}^{\infty} t_n (l_p^A)^n \right)^* = \sum_{n=0}^{\infty} \overline{t_n} (l_p^A)^n,$$

on $\mathfrak{L}_p^A$, where $\overline{t_n}$ are the conjugates of t_n in $\mathbb{C}$, then it forms a $*$-algebra over $\mathbb{C}$. i.e., all elements of $\mathfrak{L}_p^A$ are adjointable in the sense of [13]. In conclusion, this topological sub-structure $\mathfrak{L}_p^A$ of (7.3.7) forms a Banach $*$-algebra in the operator space $B(\mathfrak{S}_p^A)$.

Now, define a new tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p^A$, by

$$\mathfrak{L}\mathfrak{S}_p^A \underline{\underline{def}} \mathfrak{L}_p^A \otimes_{\mathbb{C}} \mathfrak{S}_p^A. \tag{7.3.8}$$

Since $\mathfrak{L}_p^A$, is a Banach $*$-algebra, and $\mathfrak{S}_p^A$ is a C^*-algebra, the tensor product $*$- algebra $\mathfrak{LS}_p^A$ of (7.3.8) is a well-defined Banach $*$-algebra under product topology.

Definition 7.4. The tensor product Banach $*$-algebra $\mathfrak{LS}_p^A$ of (7.3.8) is called the A-tensor p-adic radial-projection (Banach-$*$-) algebra (induced by $\mathfrak{S}_p$), for $p \in \mathcal{P}$.

Let $\mathfrak{LS}_p^A$ be the A-tensor p-adic radial-projection algebra (7.3.8), for a fixed prime p. Define a linear morphism

$$E_p^A : \mathfrak{LS}_p^A \to \mathfrak{S}_p^A$$

by the linear transformation satisfying that

$$E_p^A((l_p^A)^n \otimes T_{p,k}^a) = \frac{(p^{j+1})^{n+1}}{\left[\frac{n}{2}\right] + 1}(l_p^A)^n(T_{p,k}^a), \tag{7.3.9}$$

for all generating operators

$$(l_p^A)^n \otimes T_{p,k}^a, \forall n \in \mathbb{N}, a \in (A, \psi), k \in \mathbb{Z},$$

of $\mathfrak{LS}_p^A$, where $T_{p,k}^a$ are in the sense of (7.3.1).

In the definition of (7.3.9), the notation $\left[\frac{n}{2}\right]$ means the minimal integer greater than or equal to $\frac{n}{2}$, for all $n \in \mathbb{N}$.

Note that the linear morphism E_p^A of (7.3.9) is a well-defined surjective bounded linear transformation from $\mathfrak{LS}_p^A$ onto $\mathfrak{S}_p^A$, by (7.1.3), (7.2.1), (7.3.7) and (7.3.8).

Define now linear functionals $\tau_{p,j}^A$ on the A-tensor p-adic radial-projection algebra $\mathfrak{LS}_p$ of $\mathfrak{S}_p$ by

$$\tau_{p,j}^A = \left(\frac{1}{\phi(p)}\psi_{p,j}\right) \circ E_p^A \text{ on } \mathfrak{LS}_p^A, \tag{7.3.10}$$

where $\psi_{p,j}$ are the linear functionals (7.2.2) on $\mathfrak{S}_p^A$, and E_p^A is the surjective bounded linear morphism (7.3.9) from $\mathfrak{L}\mathfrak{S}_p^A$ onto $\mathfrak{S}_p^A$, for all $j \in \mathbb{Z}$.

Thus, the pairs

$$\mathfrak{L}\mathfrak{S}_p^A(j) \underline{\underline{denote}} (\mathfrak{L}\mathfrak{S}_p^A, \tau_{p,j}^A) \tag{7.3.11}$$

form well-defined Banach $*$-probability spaces, for all $j \in \mathbb{Z}$.

Definition 7.5. Let $\mathfrak{L}\mathfrak{S}_p^A$ be the A-tensor p-adic radial-projection algebra, for $p \in \mathcal{P}$, and let $\{\tau_{p,j}^A\}_{j \in \mathbb{Z}}$ be the linear functionals (7.3.10). Then the corresponding Banach $*$-probability spaces $\mathfrak{L}\mathfrak{S}_p^A(j)$ of (7.3.11) are called the j-th filtered A-tensor p-adic (radial-projection Banach $*$-)probability spaces, for all $j \in \mathbb{Z}$.

Let $X_{p,k}^a$ be the generating operators of $\mathfrak{L}\mathfrak{S}_p^A$,

$$X_{p,k}^a = l_p^A \otimes T_{p,k}^a = l_p^A \otimes (a \otimes P_{p,k}) \in \mathfrak{L}\mathfrak{S}_p^A, \tag{7.3.12}$$

for all $a \in (A, \psi)$, $k \in \mathbb{Z}$.

If $X_{p,k}^a$ are the generating operators (7.3.12) of $\mathfrak{L}\mathfrak{S}_p^A$, then

$$\begin{aligned}(X_{p,k}^a)^n &= (l_p^A \otimes T_{p,k}^a)^n = (l_p^A)^n \otimes (T_{p,k}^a)^n \\ &= (l_p^A)^n \otimes (a^n \otimes P_{p,k}) = (l_p^A)^n \otimes T_{p,k}^n,\end{aligned} \tag{7.3.13}$$

for all $n \in \mathbb{N}$.

By (7.3.13), one can get the following free-distributional data.

Theorem 7.5. *Let $X_{p,j}^a = l_p^A \otimes T_{p,j}^a$ be the "j-th" generating operators* (7.3.12) *of the j-th filtered A-tensor p-adic probability space $\mathfrak{L}\mathfrak{S}_p^A(j)$ of* (7.3.11), *for $j \in \mathbb{Z}$. Then*

$$\tau_{p,j}^{A}((X_{p,j}^{a})^{n}) = \left(\omega_{n}(p^{2(j+1)})^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a^{n})), \tag{7.3.14}$$

where

$$\omega_{n} = \begin{cases} 1 & if \;\; n \;\; is \;\; even \\ 0 & if \;\; n \;\; is \;\; odd \end{cases}$$

for all $n \in \mathbb{N}$, *and* c_m *are the m-th Catalan numbers*,

$$c_{m} = \frac{1}{m+1}\binom{2m}{m} = \frac{(2m)!}{m(m+1)!},$$

for all $m \in \mathbb{N}_0$.

Proof. If $X_{p,j}^{a}$ is the j-th generating operator (7.3.12) of $\mathfrak{L}\mathfrak{S}_{p}^{A}$, for $a \in (A, \psi)$, and $j \in \mathbb{Z}$, then

$$\tau_{p,j}^{A}((X_{p,j}^{a})^{n}) = \left(\left(\frac{1}{\phi(p)}\psi_{p,j}\right) \circ E_{p}^{A}\right)((X_{p,j}^{a})^{n})$$

by (7.3.10)

$$= \left(\frac{1}{\phi(p)}\psi_{p,j}\right)(E_{p}^{A}((l_{p}^{A})^{n} \otimes T_{p,j}^{a^{n}}))$$

by (7.3.13)

$$= \left(\frac{1}{\phi(p)}\psi_{p,j}\right)\left(\frac{(p^{j+1})^{n+1}}{\left[\frac{n}{2}\right]+1}(l_{p}^{A})^{n}(T_{p,j}^{a^{n}})\right)$$

by (7.3.9)

$$= \left(\frac{(p^{j+1})^{n+1}}{\left[\frac{n}{2}\right]+1}\right)\left(\left(\frac{1}{\phi(p)}\psi_{p,j}\right)((l_{p}^{A})^{n}(T_{p,j}^{a}))\right)$$

$$= \left(\frac{(p^{j+1})^{n+1}}{\left(\left[\frac{n}{2}\right]+1\right)\phi(p)}\right)\psi_{p,j}\left(\sum_{k=0}^{n}\binom{n}{k}(c_{p}^{A})^{k}(a_{p}^{A})^{n-k}(T_{p,j}^{a^{n}})\right), \tag{7.3.15}$$

by (7.3.6).

Observe now that, for any $n \in \mathbb{N}$,

$$(l_p^A)^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} (c_p^A)^k (a_p^A)^{2n-k-1},$$

by (7.3.6), and hence, $(l_p^A)^{2n-1}$ does not contain $1_{\mathfrak{S}_p^A}$ - terms, where $1_{\mathfrak{S}_p^A}$ is in the sense of (7.3.4) and (7.3.5). i.e., the following statement (7.3.16) holds;

(7.3.16) $(l_p^A)^n$ does not contain $1_{\mathfrak{S}_p^A}$ - terms whenever n is odd in $\mathbb{N}$.

Similarly, for any $n \in \mathbb{N}$,

$$(l_p^A)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (c_p^A)^k (a_p^A)^{2n-k}$$

$$= \binom{2n}{n} (c_p^A)^n (a_p^A)^n + [\text{Rest Terms}]$$

$$= \binom{2n}{n} \cdot 1_{\mathfrak{S}_p^A} + [\text{Rest Terms}],$$

by (7.3.4) and (7.3.5), and hence, $(l_p^A)^{2n}$ contains $\binom{2n}{n} \cdot 1_{\mathfrak{S}_p^A}$ term. i.e., the following statement (7.3.17) holds;

(7.3.17) $(l_p^A)^n$ contains $\binom{n}{\frac{n}{2}} \cdot 1_{\mathfrak{S}_p^A}$ - term whenever n is even in $\mathbb{N}$.

So, with help of (7.3.16) and (7.3.17), the formula (7.3.15) goes to

$$\tau_{p,j}^A((X_{p,j}^a)^n) = \left(\frac{(p^{j+1})^{n+1}}{\left(\left[\frac{n}{2}\right]+1\right)\phi(p)} \right) \psi_{p,j} \left(\sum_{k=0}^{n} \binom{n}{k} (c_p^A)^k (a_p^A)^{n-k} (T_{p,j}^{a^n}) \right)$$

$$= \omega_n \left(\frac{(p^{j+1})^{n+1}}{\left(\left[\frac{n}{2}\right]+1\right)\phi(p)} \right) \psi_{p,j} \left(\binom{n}{\frac{n}{2}} (T_{p,j}^{a^n}) + [\text{Rest Terms}](T_{p,j}^{a^n}) \right)$$

by (7.2.5) and (7.3.10), where

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{7.3.18}$$

for all $n \in \mathbb{N}$, and hence,

$$= \omega_n \left(\frac{(p^{j+1})^{n+1}}{\left(\frac{n}{2} + 1\right)\phi(p)} \right) \psi_{p,j} \left(\binom{n}{\frac{n}{2}} (T_{p,j}^{a^n}) \right)$$

by (7.2.5)

$$= \omega_n \left(\frac{(p^{j+1})^{n+1}}{\left(\frac{n}{2} + 1\right)\phi(p)} \right) \left(\frac{\frac{n}{2} + 1}{\frac{n}{2} + 1} \right) \binom{n}{\frac{n}{2}} \varphi_j^p (P_{p,j}) \psi(a^n)$$

$$= \omega_n \left(\frac{(p^{j+1})^{n+1}}{\phi(p)} \right) \left(c_{\frac{n}{2}} \right) \left(\frac{\phi(p)}{p^{j+1}} \right) (\psi(a^n)),$$

where $c_m = \frac{1}{m+1} \binom{2m}{m}$ are the m-th Catalan numbers for all $m \in \mathbb{N}_0$

$$= \left(\omega_n (p^{j+1})^n c_{\frac{n}{2}} \right) (\psi(a^n)),$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (7.3.18). Therefore, the free-distributional data (7.3.14) holds.

The formula (7.3.14) provides a tool to compute free distributions of arbitrary operators in $\mathfrak{LS}_p^A(j)$. More in detail, one can get the following result.

Corollary 7.6. *Let* $X_{p,k}^a$ *be the k-th generating operator* (7.3.12) *of the A-tensor j-th filtered probability space* $\mathfrak{LS}_p^A(j)$, *for* $p \in \mathcal{P}$, *and* $j, k \in \mathbb{Z}$. *Then*

$$\tau_{p,j}^{A}\left((X_{p,k}^{a})^{n}\right) = \delta_{j,k}\left(\omega_{n}\left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a^{n})), \tag{7.3.19}$$

for all $n \in \mathbb{N}$.

Proof. Let $X_{p,k}^{a}$ be in the sense of (7.3.12) in $\mathfrak{L}\mathfrak{S}_{p}^{A}(j)$, for $j, k \in \mathbb{Z}$. If $j = k$ in $\mathbb{Z}$, then one obtains the formula (7.3.19) by (7.3.14). Meanwhile, if $j \neq k$ in $\mathbb{Z}$, then the free moments of $X_{p,k}^{a}$ vanish by (7.2.5) and (7.3.10). Thus, the formula (7.3.19) holds.

8. Weighted-Semicircularity on $\mathfrak{L}\mathfrak{S}_{p}^{A}(j)$

Throughout this section, we fix a unital 8. Weighted-Semicircularity on C^*-probability space 8. Weighted-Semicircularity on (A, ψ) as in Sections 6 and 7. Here, our weighted-semicircular elements and semicircular elements are constructed in the A-tensor j-th filtered p-adic probability spaces 8. Weighted-Semicircularity on $\mathfrak{L}\mathfrak{S}_{p}^{A}(j)$, for all 8. Weighted-Semicircularity on $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

8.1. Semicircular and Weighted-Semicircular Elements. Let 8. Weighted-Semicircularity on (B, φ_B) be an arbitrary topological $*$-probability space (C^*-probability space, or W^*-probability space, or Banach $*$-probability space) equipped with a topological $*$-algebra A(C^*-algebra, resp., W^*-algebra, resp., Banach $*$-algebra), and a (bounded or unbounded) linear functional φ_B on B.

Definition 8.1. Let a be a self-adjoint free random variable in a $*$-probability space (B, φ_B). It is said to be even in (B, φ_B), if all odd free moments of a vanish, i.e.,

$$\varphi(a^{2n-1}) = 0, \text{ for all } n \in \mathbb{N}. \tag{8.1.1}$$

Let a be a "self-adjoint," and "even" free random variable of (A, φ) satisfying (8.1.1). Then a is said to be semicircular in (A, φ), if

$$\varphi(a^{2n}) = c_n, \text{ for all } n \in \mathbb{N}, \tag{8.1.2}$$

where c_n are the n-th Catalan number,

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} = \frac{(2n)!}{n!(n+1)!},$$

for all $n \in \mathbb{N}_0$.

It is well-known that, if $k_n^B(\ldots)$ is the free cumulant on B in terms of a linear functional φ_B (in the sense of [26]), then a self-adjoint free random variable a is semicircular in (B, φ_B), if and only if

$$k_n^B\left(\underbrace{a, a, \ldots\ldots, a}_{n\text{-times}}\right) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise}, \end{cases} \tag{8.1.3}$$

for all $n \in \mathbb{N}$ (e.g., [26]). The above characterization (8.1.3) is obtained by the Mobius inversion of [26].

Thus, the semicircular elements a of (A, φ) can be re-defined by the self-adjoint free random variables satisfying the free-cumulant characterization (8.1.3). We will use the free-moment definition (8.1.2) and the free-cumulant characterization (8.1.3) alternatively below.

Motivated by (8.1.3), one can define the weighted-semicircular elements.

Definition 8.2. Let $a \in (B, \varphi_B)$ be a self-adjoint free random variable. It is said to be weighted-semicircular in (B, φ_B) with its weight t_0 (in short, t_0- semicircular), if there exists

$$t_0 \in \mathbb{C}^+ \setminus \{0\},$$

such that

$$k_n^B\left(\underbrace{a, a, \ldots\ldots, a}_{n\text{-times}}\right) = \begin{cases} t_0 & \text{if } n = 2 \\ 0 & \text{otherwise}, \end{cases} \tag{8.1.4}$$

for all $n \in \mathbb{N}$.

By the Mobius inversion of [26], one can obtain the following free-moment characterization (8.1.5) of the definition (8.1.4). i.e., A self-adjoint free random variable a of (B, φ_B) is t_0-semicircular, if and only if there exists $t_0 \in \mathbb{C}^{\times}$, such that

$$\varphi(a^n) = \omega_n t_0^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{8.1.5}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (7.3.18), and c_m are the m-th Catalan numbers, for all $m \in \mathbb{N}_0$.

Therefore, we will use the free-cumulant definition (8.1.4) and the free-moment characterization (8.1.5) alternatively.

8.2. Weighted-Semicircular Elements in $\mathfrak{L}\mathfrak{S}_p^A(j)$. Fix $p \in \mathcal{P}$, and $j \in \mathbb{Z}$, and let $\mathfrak{L}\mathfrak{S}_p^A(j)$ be the j-th A-tensor p-adic filtered probability space of $\mathfrak{S}_p^A$.

Theorem 8.1. *Let $X_{p,j}^a$ be a j-th generating operator* (7.3.12) *of* $\mathfrak{L}\mathfrak{S}_p^A(j)$. *Assume that a free random variable $a \in (A, \psi)$ satisfies that*

(i) *a is self-adjoint in A,*

(ii) $\psi(a) \in \mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}$ *in* $\mathbb{C}$, *and*

(iii) $\psi(a^n) = \psi(a)^n$, *for all* $n \in \mathbb{N}$.

Then the operator $X_{p,j}^a$ is $(p^{j+1}\psi(a))^2$-*semicircular in* $\mathfrak{L}\mathfrak{S}_p^A(j)$.

Proof. Let $a \in (A, \psi)$ satisfy the above conditions (i), (ii) and (iii), and let $X_{p,j}^a$ be the j-th generating operator of $\mathfrak{L}\mathfrak{S}_p^A(j)$. Then

$$(X_{p,j}^a)^* = (l_p^A \otimes T_{p,j}^a) = (l_p^A)^* \otimes (a \otimes P_{p,j})^*$$

$$= l_p^A \otimes (a^* \otimes P_{p,j}) = l_p^A \otimes T_{p,j}^a = X_{p,j}^a,$$

in $\mathfrak{L}_p^A(j)$, by the self-adjointness condition (i) for $a \in (A, \psi)$. So, $X_{p,j}^a$ is selfadjoint in the A-tensor p-adic radial-projection algebra $\mathfrak{L}\mathfrak{S}_p^A$.

Observe now that, for any $n \in \mathbb{N}$,

$$\tau_{p,j}^A((X_{p,j}^a)^n) = \left(\omega_n (p^{2(j+1)})^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a^n))$$

by (7.3.18) and (7.3.19)

$$= \left(\omega_n (p^{2(j+1)})^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a)^n)$$

$$= \left(\omega_n (p^{2(j+1)})^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a)^n)^{\frac{n}{2}}$$

by the conditions (ii) and (iii) for $a \in (A, \psi)$

$$= \omega_n (p^{2(j+1)}\psi(a)^2)^{\frac{n}{2}} c_{\frac{n}{2}}$$

$$= \omega_n ((p^{j+1}\psi(a))^2)^{\frac{n}{2}} c_{\frac{n}{2}}. \tag{8.2.1}$$

Therefore, by the self-adjointness, and by (8.1.1), (8.1.2) and (8.2.1), this operator $X_{p,j}^a$ is $(p^{j+1}\psi(a))^2$-semicircular in $\mathfrak{L}\mathfrak{S}_p^A(j)$.

By the weighted-semicircularity (8.2.1), one obtains the following corollary.

Corollary 8.2. *Let* $X_{p,j}^{1_A} = l_p^A \otimes (1_A \otimes P_{p,j})$ *be in the sense of* (7.3.12) *in the A-tensor j-th filtered probability space* $\mathfrak{L}\mathfrak{S}_p^A(j)$ *of* $\mathfrak{S}_p^A$, *for* $p \in \mathcal{P}$, $j \in \mathbb{Z}$. *Then* $X_{p,j}^{1_A}$ *is* $p^{2(j+1)}$-*semicircular in* $\mathfrak{L}\mathfrak{S}_p^A(j)$. *i.e.*,

$$X_{p,j}^{1_A} \text{ is } p^{2(j+1)}\text{-semicircular in } \mathfrak{L}\mathfrak{S}_p^A(j), \forall p \in \mathcal{P}, j \in \mathbb{Z}. \tag{8.2.2}$$

Proof. Let $X_{p,j}^{1_A}$ be given as above in $\mathfrak{LS}_p^A(j)$. Then it is self-adjoint in $\mathfrak{LS}_p^A$, because 1_A is self-adjoint in A. Since our fixed C^*-probability space (A, ψ) is assumed to be unital in the sense that $\psi(1_A) = 1$, one has

$$1 = \psi(1_A) \in \mathbb{R}^{\times} \text{ in } \mathbb{C},$$

and

$$\psi(1_A^n) = 1 = \psi(1_A)^n,$$

for all $n \in \mathbb{N}$.

Therefore, by (8.2.1), we obtain that

$$\tau_{p,j}^A((X_{p,j}^{1_A})^n) = \omega_n((p^{j+1}\psi(1_A))^2)^{\frac{n}{2}} c_{\frac{n}{2}}$$

$$= \omega_n(p^{2(j+1)})^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$.

So, this operator $X_{p,j}^{1_A}$ is $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_p^A(j)$, and hence, the statement (8.2.2) holds.

The above relation (8.2.2) generalizes the weighted-semicircularity of [8], [11] and [12].

Recall that a linear functional φ_B on an arbitrary topological $*$-algebra B is called a state on B, if φ_B satisfies

$$\varphi_B(b_1 b_2) = \varphi_B(b_1)\varphi_B(b_2), \ \forall b_1, b_2 \in B.$$

By the above theorem, we obtain the following result.

Corollary 8.3. *Let (A, ψ) be a fixed unital C^*-probability space, and assume that ψ is a state on A. Let $a \in (A, \varphi)$ be a self-adjoint free random variable satisfying*

$$\psi(a) \in \mathbb{R}^{\times} \text{ in } \mathbb{C}.$$

Then the j-th generating operator $X_{p,j}^{a}$ is $(p^{j+1}\psi(a))^2$-semicircular in $\mathfrak{LS}_p^A(j)$.

Proof. Let ψ be a state on A. Then one has that

$$\psi(a^n) = \psi(a)^n \text{ in } \mathbb{R}^{\times}, \text{ for all } n \in \mathbb{N}.$$

So, the given self-adjoint free random variable $a \in (A, \psi)$ satisfies the conditions (i), (ii) and (iii) of the above theorem. Therefore, the corresponding operator $X_{p,j}^{a}$ is $(p^{j+1}\psi(a))^2$-semicircular in $\mathfrak{L}\mathfrak{S}_p^A(j)$.

8.3. Semicircular Elements in $\mathfrak{L}\mathfrak{S}_p^A(j)$. Let's fix $p \in \mathcal{P}$, and $j \in \mathbb{Z}$, and let $\mathfrak{L}\mathfrak{S}_p^A(j)$ be the A-tensor j-th filtered probability space of $\mathfrak{S}_p^A$, where (A, ψ) is the fixed unital C^*-probability space.

Theorem 8.4. *Let $X_{p,j}^{a}$ be the j-th generating operator of $\mathfrak{L}\mathfrak{S}_p^A(j)$. Assume that a free random variable $a \in (A, \psi)$ satisfies that*

(i) *a is self-adjoint in A,*

(ii) *$\psi(a) \in \mathbb{R}^{\times}$ in $\mathbb{C}$, and*

(iii) *$\psi(a^n) = \psi(a)^n$, for all $n \in \mathbb{N}$.*

Then the operator

$$Y_{p,j}^{a} = \frac{1}{p^{j+1}\psi(a)} X_{p,j}^{a} \in \mathfrak{L}\mathfrak{S}_p^A \tag{8.3.1}$$

is semicircular in $\mathfrak{L}\mathfrak{S}_p^A(j)$.

Proof. Let $X_{p,j}^{a}$ and $Y_{p,j}^{a}$ be in the sense of (8.3.1) in $\mathfrak{L}\mathfrak{S}_p^A(j)$, where a free random variable $a \in (A, \psi)$ satisfies the above conditions (i), (ii) and (iii). Now, let $k_n^{A,p,j}(\ldots)$ be the free cumulant on $\mathfrak{L}\mathfrak{S}_p^A$ in terms of the linear functional $\tau_{p,j}^{A}$ (in the sense of [26]).

Note and recall that, under the conditions (i), (ii) and (iii), this j-th generating operator $X_{p,j}^{a}$ is $(p^{j+1}\psi(a))^2$-semicircular in $\mathfrak{L}\mathfrak{S}_p^A(j)$ by (8.2.1).

Observe that

$$k_n^{A,p,j}\Big(\underbrace{Y_{p,j}^{a}, Y_{p,j}^{a}, \dots, Y_{p,j}^{a}}_{n\text{-times}}\Big) = \left(\frac{1}{p^{j+1}\psi(a)}\right)^{n} k_n^{A,p,j}(X_{p,j}^{a}, \dots, X_{p,j}^{a})$$

by the bi-module-map property of free cumulants (e.g., [26])

$$= \begin{cases} \left(\frac{1}{p^{j+1}\psi(a)}\right)^{2} k_2^{A,p,j}(X_{p,j}^{a}, X_{p,j}^{a}) = 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

by the $(p^{j+1}\psi(a))^2$-semicircularity (8.2.1) of $X_{p,j}^{a}$

$$= \begin{cases} \left(\frac{1}{p^{j+1}\psi(a)}\right)^{2} (p^{j+1}\psi(a))^2 = 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{8.3.2}$$

for all $n \in \mathbb{N}$. Therefore, by (8.3.2) and (8.1.3), this operator $Y_{p,j}^{a}$ of (8.3.1) is semicircular in $\mathfrak{LS}_p^A(j)$.

The following result is a direct consequence of the above theorem.

Corollary 8.5. *Let* $Y_{p,j}^{1_A} = \frac{1}{p^{j+1}} X_{p,j}^{1_A} \in \mathfrak{LS}_p^A(j)$. *Then* $Y_{p,j}^{1_A}$ *is semicircular in* $\mathfrak{LS}_p^A(j)$.

Proof. Note and recall that $X_{p,j}^{1_A}$ is $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_p^A(j)$ by (8.2.2). So, by the proof of the above theorem, the semicircularity of $Y_{p,j}^{1_A}$ is guaranteed in $\mathfrak{LS}_p^A(j)$.

The above corollary generalizes the semicircularity of [8], [11] and [12].

9. A-Tensor Free Adelic Filterizations $\mathfrak{LS}_A$

In this section, we use free product of Section 6 to construct the generalized or globalized free-probabilistic structures from our A-tensor j-th filtered p-adic probability spaces $\mathfrak{LS}_p^A(j)$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. As before, let

(A, ψ) be a fixed unital C^*-probability space, and let $\mathfrak{L}\mathfrak{S}_p^A$ be the A-tensor p-adic radial-projection algebras, for all $p \in \mathcal{P}$.

Define the free product Banach $*$-probability space,

$$\mathfrak{L}\mathfrak{S}_A = (\mathfrak{L}\mathfrak{S}_A, \tau_A) \underset{=\!=}{\underline{def}} \underset{p \in \mathcal{P}, j \in \mathbb{Z}}{*} \mathfrak{L}\mathfrak{S}_p^A(j)$$

$$= \left(\underset{p \in \mathcal{P}, j \in \mathbb{Z}}{*} \mathfrak{L}\mathfrak{S}_p^A, \underset{p \in \mathcal{P}, j \in \mathbb{Z}}{*} \tau_{p,j}^A \right), \tag{9.1}$$

by the free product $*$-probability space in the sense of Section 6 (e.g., [26] and [35]).

Definition 9.1. We call this free product Banach $*$-probability space $\mathfrak{L}\mathfrak{S}_A \underset{=\!=}{\underline{denote}} (\mathfrak{L}\mathfrak{S}_A, \tau_A)$ of (9.1), the A-tensor free (Adelic) filterization.

In the following, we will use same notations and concepts of Section 6, for instance, minimal free summands, free reduced words, and free sums, etc..

Let (B, φ_B) be an arbitrary topological $*$-probability space, and let $\mathcal{F} = \{b_k\}_{k \in \Lambda}$ be a subset of B, where Λ is an (finite, or infinite) index set. Such a family $\mathcal{F}$ is said to be a free family, if all elements b_k of $\mathcal{F}$ are mutually free from each other in (B, φ_B). i.e., whenever $k_1 \neq k_2$ in Λ, the elements b_{k_1} and b_{k_2} of $\mathcal{F}$ are free in (B, φ_B), if and only if all "mixed" free cumulants of

$$\{b_{k_1}, b_{k_1}^*\} \cup \{b_{k_2}, b_{k_2}^*\}$$

vanish (e.g., see [26]), if and only if the minimal free summands $B[b_{k_1}]$ and $B[b_{k_2}]$ of B, containing b_{k_1}, respectively, b_{k_2}, are "distinct" in B (e.g., see Section 6 above).

Definition 9.2. Suppose $\mathcal{F}$ is a free family of (B, φ_B), and assume that all elements of $\mathcal{F}$ are weighted-semicircular (or semicircular) in (B, φ_B). Then this free family $\mathcal{F}$ is said to be a free weighted-semicircular (resp., semicircular) family.

Let $a \in (A, \psi)$ be a self-adjoint free random variable, and assume a satisfies the additional conditions:

(I) $\psi(a) \in \mathbb{R}^{\times}$ in $\mathbb{C}$, and

(II) $\psi(a^n) = \psi(a)^n$, for all $n \in \mathbb{N}$.

For a self-adjoint element $a \in A$, satisfying the conditions (I) and (II), let's construct the subsets

$$\mathfrak{X}_a = \{X_{p,j}^a \in \mathfrak{LS}_p^A(j) \mid p \in P, \; j \in \mathbb{Z}\},$$

and

$$\mathfrak{Y}_a = \{Y_{p,j}^a = \frac{1}{p^{j+1}\psi(a)} X_{p,j}^a \in \mathfrak{LS}_p^A(j) \mid p \in \mathcal{P}, \; j \in \mathbb{Z} \tag{9.2}$$

in the A-tensor free filterization $\mathfrak{LS}_A$.

Note here that every element $X_{p,j}^a$ (or $Y_{p,j}^a$) of the family $\mathfrak{X}_a$ (resp., $\mathfrak{Y}_a$) of (9.2) is contained in the free block $\mathfrak{LS}_p^A(j)$ in $\mathfrak{LS}_A$, for all $p \in \mathcal{P}, \; j \in \mathbb{Z}$.

Theorem 9.1. *Let* $\mathfrak{LS}_A$, *be our A-tensor free filterization* (9.1), *and let* $\mathfrak{X}_a$ *and* $\mathfrak{Y}_a$ *be the families* (9.2) *in* $\mathfrak{LS}_A$, *where a self-adjoint free random variable* $a \in (A, \psi)$ *satisfies the conditions* (I) *and* (II).

(9.3) *The family* $\mathfrak{X}_a$ *is a free weighted-semicircular family in* $\mathfrak{LS}_A$.

(9.4) *The family* $\mathfrak{Y}_a$ *is a free semicircular family in* $\mathfrak{LS}_A$.

Proof. Let $\mathfrak{X}_a$ be in the sense of (9.2) in the A-tensor free filterization $\mathfrak{LS}_A$ of (9.1). By (9.1) and (9.2), all elements $X_{p,j}^a$ of $\mathfrak{X}_a$ are mutually free from each other in $\mathfrak{LS}_A$, for all $p \in \mathcal{P}, \; j \in \mathbb{Z}$, because these free reduced words $X_{p,j}^a$ with their lengths-1 have their minimal free summand $\mathfrak{LS}_A[X_{p,j}^a]$ which are identical to the free blocks $\mathfrak{LS}_p^A(j)$ of $\mathfrak{LS}_A$, i.e.,

$$\mathfrak{LS}_A[X_{p,j}^a] = \mathfrak{LS}_p^A(j), \text{ for all } p \in \mathcal{P}, \; j \in \mathbb{Z},$$

which are mutually distinct from each other, and hence, mutually free from each other in $\mathfrak{LS}_A$, for all $p \in \mathcal{P}, \; j \in \mathbb{Z}$. So, this family $\mathfrak{X}_a$ forms a free family in $\mathfrak{LS}_A$.

Furthermore, by the conditions (I) and (II) for $a \in (A, \psi)$, one has

$$\tau_A((X_{p,j}^a)^{(n)}) = \tau_A((X_{p,j}^a)^n)$$

$$= \tau_{p,j}^A((X_{p,j}^a)^n)$$

$$= \omega_n((p^{j+1}\psi(a))^2)^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, for all $n \in \mathcal{P}$, $j \in \mathbb{Z}$, by (8.2.1).

So, all elements of $\mathfrak{X}_a$ are weighted-semicircular in $\mathfrak{LS}_A$. Therefore, $\mathfrak{X}_a$ is a free weighted-semicircular family in $\mathfrak{LS}_A$, i.e., the statement (9.3) holds.

Similarly, one can show that the statement (9.4) holds true.

In the proofs of (9.3) and (9.4), we used the notations

$$(X_{p,j}^a)^{(n)}, \text{ and } (X_{p,j}^a)^n, \text{ for } n \in \mathbb{N},$$

introduced in Section 6. Remark the differences between the above two notations; the first one is in $\mathfrak{LS}_A$ by regarding $X_{p,j}^a$ as a free reduced word with length-1, and the second one is in the minimal free summand $\mathfrak{LS}_A[X_{p,j}^a]$ of $\mathfrak{LS}_A$ containing this free reduced word $X_{p,j}^a$ with length-1. As we discussed in Section 6, since $X_{p,j}^a$ is a free reduced word with its "length-1," equivalently, since its minimal free summand is identical to a free block, i.e.,

$$\mathfrak{LS}_A[X_{p,j}^a] = \mathfrak{LS}_p^A(j) \text{ in } \mathfrak{LS}_A,$$

the above two notations $(X_{p,j}^a)^{(n)}$ and $(X_{p,j}^a)^n$ are identified in $\mathfrak{LS}_A$.

Corollary 9.2. *Let* $\mathfrak{LS}_A$ *be the A-tensor free filterization, and let* $\mathfrak{X}_{1_A}$ *and* $\mathfrak{Y}_{1_A}$ *be in the sense of* (9.2) *in* $\mathfrak{LS}_A$ *where* 1_A *is the unit in* (A, ψ). *Then*

(9.5) *The family* $\mathfrak{X}_{1_A}$ *is a free weighted-semicircular family in* $\mathfrak{LS}_A$.

(9.6) *The family* $\mathfrak{Y}_{1_A}$ *is a free semicircular family in* $\mathfrak{LS}_A$.

Proof. The proofs of the statements (9.5) and (9.6) are done by the general arguments in (9.3), respectively, in (9.4), with help of the weighted-semicircularity (8.2.2) and the semicircularity (8.3.2).

The above corollary generalizes the main results of [8].

10. Free-Probabilistic Information on $\mathfrak{L}\mathfrak{S}_A$

Let (A, ψ) be a fixed unital C^*-probability space, and let

$$\mathfrak{L}\mathfrak{S}_A = (\mathfrak{L}\mathfrak{S}_A, \tau_A)$$

be the A-tensor free Adelic filterization (9.1).

Notation and Assumption 10.1 (in short, NA 10.1 below)**.** In this section, we automatically assume a free random variable $a \in (A, \psi)$ is self-adjoint, and satisfies the additional conditions: $\psi(a) \in \mathbb{R}^{\times}$, and $\psi(a^n) = \psi(a)^n$, for all $n \in \mathbb{N}$. □

Let $\mathfrak{X}_a$ and $\mathfrak{Y}_a$ be the corresponding free weighted-semicircular family (9.3), respectively, free semicircular family (9.4) of $\mathfrak{L}\mathfrak{S}_A$, under NA 10.1. In this section, we consider free reduced words W of $\mathfrak{L}\mathfrak{S}_A$ in $\mathfrak{X}_a \cup \mathfrak{Y}_a$. In particular, we are interested in the free distributions of such operators W in their minimal free summands $\mathfrak{L}\mathfrak{S}_A[W]$, and in $\mathfrak{L}\mathfrak{S}_A$.

10.1. Joint Free Moments of Semicircular Elements. To consider free-probabilistic information on the A-tensor free Adelic filterization $\mathfrak{L}\mathfrak{S}_A$, we first concentrate on studying joint free distribution of mutually free multi semicircular elements $x_1, \ldots, x_N$ of a Banach $*$-probability space $(\mathfrak{X}, \varphi)$ of a Banach $*$-algebra $\mathfrak{X}$, and a linear functional φ on $\mathfrak{X}$, for $N \in \mathbb{N} \setminus \{1\}$.

By the semicircularity of x_l in $(\mathfrak{X}, \varphi)$, the free-moment sequences, and the free-cumulant sequences of x_l are determined to be

$$(0, c_1, 0, c_2, 0, c_3, 0, c_4, \ldots),$$

and

$$(0, 1, 0, 0, 0, 0, \ldots), \tag{10.1.1}$$

respectively, for all $l = 1, \ldots, N$. However, the characterization of the joint free distribution of $x_1, \ldots, x_N$ is not easily obtained.

Note that the joint free cumulants of $x_1, \ldots, x_N$, are characterized by all sums of free cumulants of $x_1, \ldots, x_N$, by the freeness of them (e.g., [26] and [27]). However, computing the joint free moments of $x_1, \ldots, x_N$ is not easy.

Joint free moments consist of the free moments of x_l, for $l = 1, \ldots, N$, and the mixed free moments of $x_1, \ldots, x_N$. Since free moments of x_l are already characterized by their semicircularity (10.1.1), for $l = 1, \ldots, N$, we now focus on computing their "mixed" free moments.

Throughout this section, for any $s \in \mathbb{N}/\{1\}$, we fix an s-tuple I_s,

$$I_s \underline{\underline{denote}} (i_1, \ldots, i_s) \in \{1, \ldots, N\}^s. \tag{10.1.2}$$

(Without loss of generality, one can regard I_s as a mixed s-tuple, below.) For example, one can take

$$I_8 = (1, 1, 3, 2, 4, 2, 2, 1),$$

in $\{1, 2, 3, 4, 5\}^8$.

From the sequence I_s of (10.1.2), define a set

$$[I_s] = \{i_1, i_2, \ldots, i_s\}, \tag{10.1.3}$$

without considering repetition. For instance, if I_8 is given as above, then

$$[I_8] = \{i_1, i_2, \ldots, i_8\},$$

with

$$i_1 = i_2 = i_3 = 1,$$

$$i_4 = i_6 = i_7 = 2$$

$$i_3 = 3, \text{ and } i_5 = 4.$$

without considering repetition; for example, we regard all 1's in I_8 as different elements i_1, i_2 and i_8 in $[I_8]$.

Then from the set $[I_s]$ of (10.1.3), one can define a unique "noncrossing" partition $\pi_{(I_s)}$ of the lattice NC $([I_s])$ such that (i)

$$\forall V = (i_{j_1}, i_{j_2}, \ldots, i_{j_{|V|}}) \in \pi_{(I_s)}, \tag{10.1.4}$$

$$\Leftrightarrow \exists k \in \{1, \ldots, N\}, s.t., i_{j_1} = i_{j_2} = \ldots = i_{j_{|V|}} = k,$$

and (ii) such a partition $\pi_{(I_s)}$ of (i) is "maximal" satisfying (10.1.4), under the partial ordering on NC $([I_8])$.

For example, if I_8 and $[I_8]$ are as above, then there exists a noncrossing partition

$$\pi_{(I_8)} = \{(i_1, i_2, i_8), (i_3), (i_4, i_6, i_7), (i_5)\}$$

$$= \{(1, 1, 1), (3), (2, 2, 2), (4)\},$$

in $NC([I_8])$, satisfying the above conditions (i) and (ii).

Now, suppose $\pi_{(I_s)} \in NC([I_s])$ is the noncrossing partition (10.1.4) over the set $[I_s]$ of (10.1.3), and let

$$\pi_{(I_s)} = \{U_1, \ldots, U_t\},$$

where $t \leq s$ and $U_k \in \pi_{(I_s)}$ are the blocks of (ii), satisfying the condition (i), for $k = 1, \ldots, t$.

Then the partition $\pi_{(I_s)}$ is regarded as the joint partition,

$$\pi_{(I_s)} = 1_{|U_1|} \vee 1_{|U_2|} \vee \ldots \vee 1_{|U_t|}, \tag{10.1.5}$$

where $1_{|U_k|}$ are the maximal elements of $NC(U_k)$, for all $k = 1, \ldots, t$.

Let I_s be in the sense of (10.1.2), and let $x_{i_1}, \ldots, x_{i_s}$ be the corresponding semicircular elements of (A, φ) induced by I_s, without considering repetition in the set $\{x_1, \ldots, x_N\}$ of our fixed mutually free, N-many semicircular elements of (A, φ).

Define a free random variable $X[I_s]$ by

$$X[I_s] \underset{=\!=}{def} \prod_{l=1}^{s} x_{i_1} \in (\mathfrak{X}, \varphi). \tag{10.1.6}$$

If $X[I_s]$ is in the sense of (10.1.6), then

$$\varphi(X[I_s]) = \sum_{\pi \in NC([I_s])} k_\pi$$

by the Mobius inversion of [26] and [27], where k_π are the π-depending free cumulants

$$= \sum_{\pi \in NC([I_s]),\ \pi \leq \pi_{(I_s)}} k_\pi$$

by the mutual-freeness of $x_1, \ldots, x_N$ in $(\mathfrak{X}, \varphi)$, where $\pi_{(I_s)}$ is in the sense of (10.1.4)

$$= \sum_{(\theta_1, \ldots, \theta_t) \in NC(U_1) \times \ldots \times NC(U_t)} k_{\theta_1 \vee \ldots \vee \theta_t}$$

by (10.1.5)

$$= \sum_{(\theta_1, \ldots, \theta_t) \in NC_2(U_1) \times \ldots \times NC_2(U_t)} k_{\theta_1 \vee \ldots \vee \theta_t}$$

$$= \sum_{(\theta_1, \ldots, \theta_t) \in NC_2(U_1) \times \ldots \times NC_2(U_t)} \left(\prod_{l=1}^{t} k_{\theta_l} \right), \tag{10.1.7}$$

by the semicircularity of $x_{i_1}, \ldots, x_{i_s}$ in $(\mathfrak{X}, \varphi)$ where $NC_2(Y)$ is the subset of the noncrossing-partition lattice $NC(Y)$,

$$NC_2(Y) = \{\pi \in NC(Y) : \forall V \in \pi, |V| = 2\}, \tag{10.1.8}$$

over countable finite sets Y.

By (10.1.7) and (10.1.8), it is not difficult to check that, if there exists at least one $k_0 \in \{1, \ldots, t\}$, such that $|U_{k_0}|$ is odd in $\mathbb{N}$, then

$$\varphi(X[I_s]) = 0,$$

where $X[I_s]$ is the free random variable (10.1.6) of $(\mathfrak{X}, \varphi)$.

So, the formula (10.1.7) is non-zero, only if

$$|U_k| \in 2\mathbb{N}, \text{ for all } k = 1, \ldots, t, \tag{10.1.9}$$

where $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$.

Moreover, if the condition (10.1.9) is satisfied, then the summands $k_{\theta_1 \forall \ldots \forall \theta_t}$ of (10.1.7) satisfy that

$$k_{\theta_1 \forall \ldots \theta_t} = \prod_{V \in \theta_1 \vee \ldots \vee \theta_t} k_V = \prod_{V \in \theta_1 \vee \ldots \vee \theta_t} \left(\prod_{i=1}^{t} 1^{\#(\theta_i)} \right) = 1, \tag{10.1.10}$$

by the semicircularity, where $\#(\theta_i)$ are the number of blocks of θ_i, for all $i = 1, \ldots, t$. Therefore, if the condition (10.1.9) holds, then

$$\varphi(X[I_s]) = \sum_{(\theta_1, \ldots, \theta_t) \in NC_2([U_1]) \times \ldots \times NC_2([U_1])} 1$$

$$= |NC_2(U_1) \times \ldots \times NC_2(U_t)|, \tag{10.1.11}$$

by (10.1.7) and (10.1.10), where $|Y|$ mean the cardinalities of sets Y.

Proposition 10.1. *Let* I_s *be an s-tuple* (10.1.2), *and let* $X[I_s] = \prod_{l=1}^{s} x_{i_l}$ *be the corresponding free random variable* (10.1.6) *of* $(\mathfrak{X}, \varphi)$. *If* $[I_s]$ *is the set* (10.1.3), *and if*

$$\pi_{(I_s)} = 1_{|U_1|} \vee \ldots \vee 1_{|U_t|},$$

in the sense of (10.1.4) *and* (10.1.5), *then*

$$\varphi(X[I_s]) = \begin{cases} \prod_{i=1}^{t} c_{\frac{|U_i|}{2}} & \text{if } |U_k| \in 2\mathbb{N}, \\ & \text{for all } k = 1, \ldots, t \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.12}$$

Proof. Under hypothesis,

$$\varphi(X[I_s]) = \begin{cases} | NC_2(U_1) \times \ldots \times NC_2(U_t) | & \text{if } | U_k | \in 2\mathbb{N}, \\ & \text{for all } k = 1, \ldots, t \\ 0 & \text{otherwise}, \end{cases}$$

by (10.1.11).

Recall that, for every countable set X, with $| X | \in 2\mathbb{N}$, the subset

$$NC_2(X) = \{\theta \in NC(X) : \forall V \in \theta, | V | = 2\}$$

is equipotent (or bijective) to the noncrossing-partition lattice $NC\left(\frac{| X |}{2}\right)$ over $\left\{1, \ldots, \frac{| X |}{2}\right\}$. i.e., if $| U_k | \in 2\mathbb{N}$, then

$$| NC_2(U_k) | = \left| NC\left(\frac{| U_k |}{2}\right) \right|, \tag{10.1.13}$$

for all $k = 1, \ldots, t$.

So, the formula (10.1.11) goes to

$$\varphi(X[I_s]) = \begin{cases} \left| NC\left(\frac{| U_1 |}{2}\right) \times \ldots \times NC\left(\frac{| U_t |}{2}\right) \right| & \text{if } | U_k | \in 2\mathbb{N}, \\ & \text{for all } k = 1, \ldots, t \\ 0 & \text{otherwise}. \end{cases}$$

by (10.1.13)

$$= \begin{cases} \prod_{l=1}^{t} \left| NC\left(\frac{| U_l |}{2}\right) \right| & \text{if } | U_l | \in 2\mathbb{N}, \\ & \text{for all } l = 1, \ldots, t \\ 0 & \text{otherwise}. \end{cases}$$

$$= \begin{cases} \prod_{l=1}^{t} c_{\frac{| U_l |}{2}} & \text{if } | U_l | \in 2\mathbb{N}, \\ & \text{for all } l = 1, \ldots, t \\ 0 & \text{otherwise}, \end{cases} \tag{10.1.14}$$

because $| NC(X) | = c_{| X |}$, for all finite sets X (e.g., [8], [11] and [12]). Therefore, the formula (10.1.12) holds by (10.1.14).

Example 10.1. (1) Let $x_1, \ldots, x_5$ be mutually free semicircular elements of a Banach $*$-probability space $(\mathfrak{X}, \varphi)$, and let

$$X = x_2^3 x_3^4 x_2 x_3^2 \; x_5^4 \in (\mathfrak{X}, \varphi).$$

Then one can determine a 14-tuple I_{14},

$$I_{14} = (i_1, \ldots, i_{14}) = (2, 2, 2, 3, 3, 3, 3, 2, 3, 3, 5, 5, 5, 5).$$

Then I_{10} has its corresponding noncrossing partition

$$\pi_{(I_{10})} = NC(\{i_1, \ldots, i_{10}\}),$$

satisfying

$$\pi_{(I_{10})} = \left\{ \begin{matrix} (i_1, i_2, i_3, i_8), (i_4, i_5, i_6, i_7), \\ (i_9, i_{10}), (i_{11}, i_{12}, i_{13}, i_{14}\} \end{matrix} \right\}.$$

Therefore, by (10.1.12), we have

$$\varphi(X) = c_{\frac{4}{2}} c_{\frac{4}{2}} c_{\frac{2}{2}} c_{\frac{4}{2}} = c_1 c_2^3.$$

(2) Let x_1, x_2, x_3 be mutually free semicircular elements of $(\mathfrak{X}, \varphi)$, and let

$$X = x_1 \, x_2 x_1^3 x_2^2 \in (\mathfrak{X}, \varphi).$$

Then one can take

$$I_7 = (i_1, \ldots, i_7) = (1, 2, 1, 1, 1, 2, 2),$$

having

$$\pi_{(I_7)} = \{(i_1, i_3, i_4, i_5), (i_2), (i_6, i_7)\}.$$

Therefore, $\varphi(X) = 0$, by (10.1.12).

10.2. Free Reduced Words of $\mathfrak{LS}_A$ in $\mathfrak{X}_a$. Let $\mathfrak{LS}_A$ be the A-tensor free filterization (9.1), and let $a \in (A, \psi)$ be a self-adjoint free random variable under NA 10.1. And let $\mathfrak{X}_a$ be the free weighted-semicircular family (9.3) in $\mathfrak{LS}_A$, consisting of $(p^{j+1}\psi(a))^2$-semicircular elements $X_{p,j}^a$ of (7.3.12), contained in the free block $\mathfrak{LS}_p^A(j)$ of $\mathfrak{LS}_A$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Now, let $\mathbb{P} \underset{=}{def} \mathcal{P} \times \mathbb{Z}$, $s \in \mathbb{N}$, and take an s-tuple I_s in $\mathbb{P}^s$,

$$I_s = ((p_{i_1}, j_{i_1}), (p_{i_2}, j_{i_2}), \dots, (p_{i_s}, j_{i_s})), \tag{10.2.1}$$

as in (10.1.1). For convenience, let's take

$$i_l = (p_{i_l}, j_{i_l}) \in \mathbb{P}, \text{ for } l = 1, \dots, s.$$

Then the s-tuple I_s of (10.2.1) can be re-written by

$$I_s = (i_1, i_2, i_s). \tag{10.2.1)'$$

Then, for this s-tuple I_s, one can define the corresponding free random variable

$$X[I_s, a] = \prod_{l=1}^{s} x_{i_l}^a \in \mathfrak{LS}_A,$$

with

$$x_{i_l}^a = X_{p_l, j_l}^a, \text{ for all } l = 1, \dots, s. \tag{10.2.2}$$

By (10.2.1)' and (10.2.2), if $s > 1$, and if I_s is an s-tuple of s-copies of $(p, j)'s$, i.e., if

$$I_s = (i_1, \dots, i_s) = ((p, j), \dots, (p, j)),$$

then

$$X[I_s, a] = (X_{p,j}^a)^s, \text{ for } s \in \mathbb{N},$$

and hence,

$$\begin{aligned}\tau_A(X[I_s, a]) &= \tau_A((X_{p,j}^a)^s) = \tau_{p,j}^A((X_{p,j}^a)^s) \\ &= \omega_s(\psi(a)p^{2(j+1)})^s c_{\frac{s}{2}},\end{aligned} \tag{10.2.3}$$

by (9.3), under NA 10.1.

By (10.2.3), we are interested in the cases where an s-tuple I_s of (10.2.1), or (10.2.1)' is mixed.

Theorem 10.2. *Let* I_s *be an* s*-tuple* (10.2.1), *for* $s \in \mathbb{N} \setminus \{1\}$, *and let* $X[I_s, a] \in \mathfrak{LS}_A$ *be the free random variable* (10.2.2), *where* $a \in (A, \psi)$ *satisfies NA* 10.1. *Then*

$$\tau_A(X[I_s, a]) = \begin{cases} \psi(a^s)\left(\prod_{l=1}^{s} p_l^{j_l+1}\right)\prod_{l=1}^{t} c_{\frac{|U_l|}{2}} & \begin{array}{c} \text{if } \frac{|U_l|}{2} \in \mathbb{N}, \\ \text{for all } l = 1, \ldots, N \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{10.2.4}$$

Proof. Let I_s be given as in (10.2.1), or (10.2.1)'. Then there exists the maximal noncrossing partition $\pi_{(I_s)}$ in the noncrossing-partition lattice

$$NC(\{i_1, i_2, \ldots, i_s\}),$$

satisfying (10.1.4). Assume now that

$$\pi_{(I_s)} = \{U_1, \ldots, U_t\},$$

for some $t \leq s$, as in (10.1.5). Then

$$\tau_A(X[I_s, a]) = \tau_A\left(\prod_{l=1}^{s} x_{i_l}^a\right) = \tau_A\left(\prod_{l=1}^{s} X_{p_l, j_l}^a\right)$$

by (10.2.2)

$$= \tau_A\left(\prod_{l=1}^{s} (p_l^{j_l+1}\psi(a)Y_{p_l, j_l}^a)\right)$$

by (8.3.1), where Y_{p_l, j_l}^a are semicircular elements of $\mathfrak{LS}_A$ by (8.3.2) (under NA 10.1)

$$= \left(\prod_{l=1}^{s} p_l^{j_l+1}\psi(a)\right)\tau_A\left(\prod_{l=1}^{s} Y_{p_l, j_l}^a\right)$$

$$= \left(\prod_{l=1}^{s} p_l^{j_l+1} \right) (\psi(a))^s \left| \prod_{l=1}^{t} NC\left(\frac{|U_l|}{2} \right) \right|$$

by (10.1.12) and (8.3.2)

$$= \begin{cases} \psi(a^s)\left(\prod_{l=1}^{s} p_l^{j_l+1} \right) \prod_{l=1}^{t} c_{\frac{|U_l|}{2}} & \text{if } \frac{|U_l|}{2} \in \mathbb{N}, \\ & \text{for all } l = 1, \ldots, N \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the free-distributional data (10.2.4) holds on $\mathfrak{LS}_A$ (under NA 10.1).

The above theorem characterizes how to compute joint free moments of generating operators of $\mathfrak{LS}_A$ (under NA 10.1).

References

[1] S. Albeverio, P. E. T. Jorgensen and A. M. Paolucci, On Fractional Brownian Motion and Wavelets, Compl. Anal. Oper. Theo. 6(1) (2012), 33-63.

[2] S. Albeverio, P. E. T. Jorgensen and A. M. Paolucci, Multiresolution Wavelet Analysis of Integer Scale Bessel Functions, J. Math. Phy. 48(7) (2007), 073516, 24.

[3] D. Alpay, P. E. T. Jorgensen and D. Levanony, On the Equivalence of Probability Spaces, J. Theo. Prob., (2016) To Appear.

[4] D. Alpay, P. E. T. Jorgensen and D. Kimsey, Moment Problems in an Infinite Number of Variables, Infinit. Dimension. Anal. Quant. Prob. 18(4), 1550024, (2015)

[5] D. Alpay and P. E. T. Jorgensen, Spectral Theory for Gaussian Processes: Reproducing Kernels, Random Functions, Operator Theory 83(2) (2015), 211-229.

[6] D. Alpay and P. E. T. Jorgensen, Spectral Theory for Gaussian Processes: Reproducing Kernels, Boundaries, L_2- Wavelet Generators with Fractional Scales, Numb. Funct. Anal, Optim. 36(10) (2015), 1239-1285.

[7] D. Alpay, P. E. T. Jorgensen and G. Salomon, On Free Stochastic Processes and Their Derivatives, Stochast. Process. Appl. 124(10) (2014), 3392-3411.

[8] I. Cho, Semicircular Families in Free Product Banach $*$-Algebras Induced by p-Adic Number Fields over Primes p, Compl. Anal. Oper. Theo. 11(3) (2017), 507-565.

[9] I. Cho, p-Adic Number Fields Acting on W^*- Probability Spaces, Turkish J. Anal. Numb. Theo. Acad., (2018) To Appear.

[10] I. Cho, Free Probability on Hecke Algebras and Certain Group C^*- Algebras Induced by Hecke Algebras, Opuscula Math. 36(2) (2016), 153-187.

[11] I. Cho, p-Adic Free Stochastic Integrals for p-Adic Weighted-Semicircular Motions Determined by Primes p, Libertas Math. (N.S) 36(2) (2016), 65-110.

[12] I. Cho and P. E. T. Jorgensen, Semicircular Elements Induced by p-Adic Number Fields, Opuscula Math. 35(5) (2017), 665-703.

[13] A. Connes, Noncommutative Geometry, ISBN: 0-12-185860-X, (1994) Academic Press (San Diego, CA).

[14] A. Connes, Hecke Algebras, Type III-Factors, and Phase Transitions with Spontaneous Symmetry Breaking in Number Theory, Selecta Math. (New Series) 1(3) (1995), 411-457.

[15] A. Connes, Trace Formula in Noncommutative Geometry and the Zeroes of the Riemann Zeta Functions, Available at http://www.alainconnes.org/en/download.php,

[16] T. Gillespie, Superposition of Zeroes of Automorphic L-Functions and Functoriality, Univ. of Iowa, (2010) PhD Thesis.

[17] T. Gillespie, Prime Number Theorems for Rankin-Selberg L-Functions over Number Fields, Sci. China Math. 54(1) (2011), 35-46.

[18] P. E. T. Jorgensen, Operators and Representation Theory: Canonical Models for Algebras of Operators Arising in Quantum Mechanics (2nd Ed.), ISBN-13: 978-0586466651, (2008) Dover Publications.

[19] P. E. T. Jorgensen and A. M. Paolucci, q-Frames and Bessel Functions, Numb. Funct. Anal. Optim. 33(7-9) (2012), 1063-1069.

[20] P. E. T. Jorgensen and A. M. Paolucci, Markov Measures and Extended Zeta Functions, J. Appl. Math. Comput. 38(1-2) (2012), 305-323.

[21] P. E. T. Jorgensen and A. M. Paolucci, States on the Cuntz Algebras and p-Adic Random Walks, J. Aust. Math. Soc. 90(2) (2011), 197-211.

[22] P. E. T. Jorgensen and A. M. Paolucci, Wavelets in Mathematical Physics: q-Oscillators, J. Phy. A. 36(23) (2003), 6483-6494.

[23] F. Radulescu, Random Matrices, Amalgamated Free Products and Subfactors of the C^*-Algebra of a Free Group of Nonsingular Index, Invent. Math. 115 (1994), 347-389.

[24] F. Radulescu, Free Group Factors and Hecke Operators, notes taken by N. Ozawa, Proceed, 24-th Conference in Oper. Theo., Theta Advanced Series in Math., (2014) Theta Foundation.

[25] F. Radulescu, Conditional Expectations, Traces, Angles Between Spaces and Representations of the Hecke Algebras, Lib. Math. 33(2) (2013), 65-95.

[26] R. Speicher, Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory, Amer. Math. Soc. Mem. 132(627) (1998).

[27] R. Speicher, A Conceptual Proof of a Basic Result in the Combinatorial Approach to Freeness, Infinit. Dimention. Anal. Quant. Prob. Related Topics 3 (2000), 213-222.

[28] R. Speicher and T. Kemp, Strong Haagerup Inequalities for Free R-Diagonal Elements, J. Funct. Anal. 251(1) (2007), 141-173.

[29] R. Speicher and U. Haagerup, Brown's Spectrial Distribution Measure for R-Diagonal Elements in Finite Von Neumann Algebras, J. Funct. Anal. 176(2) (2000), 331-367.

[30] V. S. Vladimirov, p-Adic Quantum Mechanics, Comm. Math. Phy. 123(4) (1989), 659-676.

[31] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, p-Adic Analysis and Mathematical Physics, Ser. Soviet and East European Math. 1, ISBN: 978-981-02-0880-6, (1994) World Scientific.

[32] D. Voiculescu, Aspects of Free Analysis, Jpn. J. Math. 3(2) (2008), 163-183.

[33] D. Voiculescu, Symmetries Arising from Free Probability Theory, Frontier in Numb. Theo., Phy. & Geo., I. Spriner (Berlin), (2006) 231-243.

[34] D. Voiculescu, Free Probability and the Von Neumann Algebras of Free Groups, Rep. Math. Phy. 55(1) (2005), 127-133.

[35] D. Voiculescu, K. Dykemma and A. Nica, Free Random Variables, CRM Monograph Series, 1 (1992),