Advances and Applications in Mathematical Sciences
Volume 22, Issue 1, November 2022, Pages 261-275

COMMON FIXED POINT THEOREMS FOR SEQUENCE OF MAPPINGS IN GENERALIZED CONE METRIC SPACES

A. SAKILA BHANU, S. CHELLIAH and G. UTHAYA SANKAR

[1]Part time Research Scholar
[2]Associate Professor
The M. D. T. Hindu College
Tirunelveli, Tamilnadu, India - 627 010
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli, India - 627 012
E-mail: aathiraagar@gmail.com
kscmdt@gmail.com

[3]Assistant Professor
Manonmaniam Sundaranar University College
Naduvakurichi, Sankarankovil
Tamilnadu, India - 627 862
E-mail: uthayaganapathy@yahoo.com

Abstract

In this paper, we prove some common fixed point theorems for sequence of mappings in generalized cone metric space.

1. Introduction

In 2007, Huang and Zhang [9] introduced the concept of cone metric space with generalized the concept of the metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for contractive mappings in normal Cone metric space. S-metric space was introduced by Sedghi et al. [19] in 2012 and they generalized fixed point

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords: Generalized Cone metric space, Common fixed point, Sequence of maps.
Received May 27, 2022; Accepted June 1, 2022

theorems in S-metric space. Recently, Ozgur and Tas [13] have studied integral type contractive conditions in S-metric space. In 2017, Dhamodharan and Krishnakumar [11] introduced the concept of cone S –Metric space and prove fixed point theorems for contractive mappings. In this paper, we prove some common fixed point theorems for sequence of mappings in generalized cone metric space.

2. Preliminaries

Definition 2.1. Let E be a real Banach space and let P be a subset of E. P is said to be a cone iff:

(1) P is non-empty, closed and $P \neq \{0\}$,

(2) $ax + by \in P$ $\forall x, y \in P$ and a and b are non-negative real numbers

(3) $x \in P$ and $-x \in P$ implies $x = 0$.

Given a cone $P \subset E$, a partial ordering $\leq$ on E with respect to P defined by $x \leq y$ iff $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{int} P$, where $\text{int} P$ is the interior of P.

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering by P. Then the cone P is called normal if there is a number $K > 0$ such that, for all $x, y \in E, 0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive K number satisfying the above condition is called the normal constant of P.

Definition 2.2. Let E be a real Banach space, then P a cone in E with $\text{int} P$ is nonempty, and $\leq$ is partial ordering with respect to P. Let $X \neq \phi$ and let $d : X \times X \rightarrow E$ mapping such that

(1) $d(u, v) \geq 0$ $\forall u, v \in X$ and $d(u, v) = 0$ iff $u = v$

(2) $d(u, v) = d(v, u)$ $\forall, u, v \in X$

(3) $d(u, v) \leq d(u, w) + d(w, v)$ $\forall u, v, w \in X$,

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3. Let $X \neq \varphi$ be any set and $S : X^3 \to [0, \infty)$ be a function satisfying the conditions for all $x, y, z, a \in X$.

(1) $S(x, y, z) \geq 0$.

(2) $S(x, y, z) = 0$ if and only if $x = y = z$.

(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the function S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Example 2.4. Let X be a non empty set, d be the ordinary metric on X, then $S(u, v, w) = d(u, v) + d(v, w)$ is an S-metric on X.

Lemma 2.5. *Let* (X, S) *be a S-metric space. Then* $S(x, x, y) = S(y, y, x)$.

Lemma 2.6. *Let* (X, S) *be a S-metric space. Then, for all* $x, y, z \in X$, *we have*

$$2S(x, x, y) + S(y, y, z) \geq S(x, x, z)$$

$$2S(x, x, y) + S(z, z, y) \geq S(x, x, z)$$

Lemma 2.7. *Let* (X, S) *be a S-metric space. Then, for all* $x, y, z \in X$ *it follows that*:

1. $S(x, x, y) \geq S(x, y, y)$
2. $S(x, x, y) \geq S(x, y, x)$
3. $S(x, x, z) + S(y, y, z) \geq S(x, y, z)$
4. $S(x, x, y) + S(z, z, y) \geq S(x, y, z)$
5. $S(y, y, x) + S(x, x, z) \geq S(x, y, z)$
6. $\frac{3}{2}[S(y, y, z) + S(y, y, x)] \geq S(x, x, z)$.

Definition 2.8. Let E be a real Banach space, then P a cone in E with $\operatorname{int} P$ is nonempty, and $\leq$ is partial ordering with respect to P. Let $X \neq \phi$

and let $S : X^3 \to E$ satisfy the following conditions

(1) $S(u, v, w) \geq 0$

(2) $S(u, v, w) = 0$ if and only if $u = v = w$.

(3) $S(u, v, w) \leq S(u, u, a) + S(v, v, a) + S(w, w, a)$, for all $u, v, w, a \in X$.

Then the function S is said to be a cone S-metric on X and (X, S) is called a cone S-metric space.

Example 2.9. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq R^2$, $X = R$ and d be the ordinary metric on X. Then $S : X^3 \to E$ defined by $S(u, v, w) = (d(u, w) + d(v, w), \alpha(d(u, w) + d(v, w))), (\alpha > 0)$ is a cone S-metric on X.

Definition 2.10. Let (X, S) be a cone S-metric space.

(1) A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(u_n, u_n, u) << c$ for each $c \in E$, $0 << c$. We denote this by $\lim_{n\to\infty} u_n = u$ or $\lim_{n\to\infty} S(u_n, u_n, u) = 0$.

(2) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $m, n \to \infty$. That is, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(u_n, u_n, u_m) << c$ for each $c \in E$, $0 << c$.

The cone S-metric space (X, S) is called complete if every Cauchy sequence is convergent.

3. Main Results

Theorem 3.1. *Let (X, S) be a cone S-Metric space which is complete and let P a normal cone with K as normal constant. Let T_n be a sequence of mappings from X to X satisfying the condition*

$$S(T_i x, T_i x, T_j y) \leq aS(x, x, y) + bS(x, T_i x, T_j y) + cS(y, T_i x, T_j y)$$

for all $i \neq j$ *and* $\forall x, y \in X$, *where* $a, b, c > 0$ *and* $a + 2b + c < 1$. *Then* $\{T_n\}$ *has a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary element in X.

The sequence $\{x_n\}$ in X defined by $x_{n+1} = T_{n+1}x_n$, for $n = 0, 1, \ldots$

Now

$$S(x_{n+1}, x_{n+1}, x_{n+2}) = S(T_{n+1}x_n, T_{n+1}x_n, T_{n+2}x_{n+1})$$

$$\leq aS(x_n, x_n, x_{n+1}) + bS(x_n, T_{n+1}x_n, T_{n+2}x_{n+1}) + cS(x_{n+1}, T_{n+1}x_n, T_{n+2}x_{n+1})$$

$$= aS(x_n, x_n, x_{n+1}) + bS(x_n, x_{n+1}, x_{n+2}) + cS(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\leq aS(x_n, x_n, x_{n+1}) + b[S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1})]$$

$$+ cS(x_{n+1}, x_{n+1}, x_{n+2})$$

$$= (a + b)S(x_n, x_n, x_{n+1}) + bS(x_{n+2}, x_{n+2}, x_{n+1}) + cS(x_{n+1}, x_{n+1}, x_{n+2})$$

$$= (a + b)S(x_n, x_n, x_{n+1}) + (b + c)S(x_{n+1}, x_{n+1}, x_{n+2})$$

Therefore,

$$[1 - (b + c)]S(x_{n+1}, x_{n+1}, x_{n+2}) \leq (a + b)S(x_n, x_n, x_{n+1})$$

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \frac{a + b}{1 - (b + c)} S(x_n, x_n, x_{n+1})$$

$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq hS(x_n, x_n, x_{n+1})$, where $h = \frac{a + b}{1 - b - c} < 1$ as $a + 2b + c < 1$.

Similarly,

$$S(x_{n+2}, x_{n+2}, x_{n+3}) \leq hS(x_{n+1}, x_{n+1}, x_{n+2})$$

Thus

$$S(x_n, x_n, x_{n+1}) \leq hS(x_{n-1}, x_{n-1}, x_n)$$

$$\leq h^2 S(x_{n-2}, x_{n-1}, x_{n-1})$$

$$\vdots$$

$$\leq h^n S(x_0, x_0, x_1)$$

$\to 0$ as $n \to \infty$

Now

$$S(x_n, x_n, x_m) \leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)$$

$$\leq 2S(x_n, x_n, x_{n+1}) + \ldots + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_{m-1}, x_{m-1}, x_m)$$

$\to 0$ as $m, n \to \infty$ we get

Therefore, $S(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$

So the sequence $\{x_n\}$ is Cauchy.

Since (X, S) is complete, sequence $\{x_n\}$ converges to $x \in X$.

Now

$$S(T_m x, T_m x, x) = \lim_{n \to \infty} S(T_m x, T_m x, x_{n+2})$$

$$= \lim_{n \to \infty} S(T_m x, T_m x, T_{n+2} x_{n+1})$$

$$\leq \lim_{n \to \infty} \{aS(x, x, x_{n+1}) + bS(x, T_m x, T_{n+2} x_{n+1}) + cS(x_{n+1}, T_m x, T_{n+2} x_{n+1})\}$$

$$= \lim_{n \to \infty} \{aS(x, x, x_{n+1}) + bS(x, T_m x, x_{n+2}) + cS(x_{n+1}, T_m x, x_{n+2})\}$$

$$= aS(x, x, x) + bS(x, T_m x, x) + cS(x, T_m x, x)$$

$$\leq bS(x, x, T_m x) + cS(x, x, T_m x)$$

$$= (b + c)S(x, x, T_m x)$$

Therefore, $\| S(T_m x, T_m x, x) \leq (b + c)K \| S(x, x, T_m x) \|$

Since $b + c < 1$, then $S(T_m x, T_m x, x) = 0$.

Which implies that $S(T_m x, T_m x, x) << 0$.

Hence, $T_m x = x$.

Therefore, $T_n x = x$ for all n.

Hence x is a common fixed point of $\{T_n\}$.

Uniqueness: Let $y \neq x$ such that $T_n y = y, \forall n$.

Now consider

$$S(x, x, y) = S(T_i x, T_i x, T_j y)$$

$$\leq aS(x, x, y) + bS(x, T_i x, T_j y) + cS(y, T_i x, T_j y)$$

$$\leq aS(x, x, y) + bS(x, x, y) + cS(y, y, x)$$

$$\leq aS(x, x, y) + bS(x, x, y) + cS(x, x, y)$$

$$= (a + b + c)S(x, x, y)$$

Where $a + b + c < 1$, which implies $S(x, x, y) = 0$. Hence $x = y$.

Theorem 3.2. *Let (X, S) be a cone S-Metric space which is complete and let P a normal cone with K as normal constant. Let T_n be a sequence of mappings from X to X satisfying the condition*

$$S(T_i x, T_i x, T_j y) \leq aS(x, x, y) + \frac{b}{2}[S(x, T_i x, T_j y) + S(y, T_i x, T_j y)]$$

$$+ \frac{c}{2}[S(x, y, T_i x) + S(x, y, T_j y)]$$

for all $i \neq j$ and $\forall x, y \in X$, where $a, b, c > 0$ and $a + \frac{3}{2}b + \frac{3}{2}c < 1$. Then $\{T_n\}$ has a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element in X.

The sequence $\{x_n\}$ in X defined by $x_{n+1} = T_{n+1}x_n$, for $n = 0, 1, \ldots$

Now

$$S(x_{n+1}, x_{n+1}, x_{n+2}) = S(T_{n+1}x_n, T_{n+1}x_n, T_{n+2}x_{n+1})$$

$$\leq aS(x_n, x_n, x_{n+1}) + \frac{b}{2}[S(x_n, T_{n+1}x_n, T_{n+2}x_{n+1}) + S(x_{n+1}, T_{n+1}x_n, T_{n+2}x_{n+1})]$$

$$+ \frac{c}{2}[S(x_n, x_{n+1}, T_{n+1}x_n) + S(x_n, x_{n+1}, T_{n+2}x_{n+1})]$$

$$= aS(x_n, x_n, x_{n+1}) + \frac{b}{2}[S(x_n, x_{n+1}, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})]$$

$$+ \frac{c}{2}[S(x_n, x_{n+1}, x_{n+1}) + S(x_n, x_{n+1}, x_{n+2})]$$

$$= aS(x_n, x_n, x_{n+1}) + \frac{b+c}{2}[S(x_n, x_{n+1}, x_{n+2})] + \frac{b}{2}S(x_{n+1}, x_{n+1}, x_{n+2})$$

$$+ \frac{c}{2}S(x_n, x_{n+1}, x_{n+1})$$

$$\left(1 - \frac{b}{2}\right)S(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\leq aS(x_n, x_n, x_{n+1}) + \frac{b+c}{2}[S(x_n, x_{n+1}, x_{n+2})] + \frac{c}{2}S(x_n, x_n, x_{n+1})$$

$$\leq \left(a + \frac{c}{2}\right)S(x_n, x_n, x_{n+1}) + \frac{b+c}{2}[S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1})]$$

$$= \left(a + \frac{b}{2} + c\right)S(x_n, x_n, x_{n+1}) + \frac{b+c}{2}[S(x_{n+2}, x_{n+2}, x_{n+1})]$$

$$\leq \left(a + \frac{b}{2} + c\right)S(x_n, x_n, x_{n+1}) + \frac{b+c}{2}[S(x_{n+1}, x_{n+1}, x_{n+2})]$$

Therefore,

$$\left(1 - b - \frac{c}{2}\right)S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \left(a + \frac{b}{2} + c\right)S(x_n, x_n, x_{n+1})$$

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \frac{2a + b + 2c}{2 - 2b - c}S(x_n, x_n, x_{n+1})$$

$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq hS(x_n, x_n, x_{n+1})$, where $h = \frac{2a + b + 2c}{2 - 2b - c} < 1$ as $a + \frac{3}{2}b + \frac{3}{2}c < 1$

Similarly,

$$S(x_{n+2}, x_{n+2}, x_{n+3}) \leq hS(x_{n+1}, x_{n+1}, x_{n+2})$$

Thus

$$S(x_n, x_n, x_{n+1}) \leq hS(x_{n-1}, x_{n-1}, x_n)$$

$$\leq h^2 S(x_{n-2}, x_{n-2}, x_{n-1})$$

$$\vdots$$

$$\leq h^n S(x_0, x_0, x_1)$$

$\to 0$ as $n \to \infty$

Now

$$S(x_n, x_n, x_m) \leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)$$

$$\leq 2S(x_n, x_n, x_{n+1}) + \ldots + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_{m-1}, x_{m-1}, x_m)$$

$\to 0$ as $m, n \to \infty$ we get

Therefore, $S(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$

So the sequence $\{x_n\}$ is Cauchy.

Since (X, S) is complete, sequence $\{x_n\}$ converges to $x \in X$.

Now

$$S(T_m x, T_m x, x) = \lim_{n \to \infty} S(T_m x, T_m x, x_{n+2})$$

$$= \lim_{n \to \infty} S(T_m x, T_m x, T_{n+2} x_{n+1})$$

$$\leq \lim_{n \to \infty} \{aS(x, x, x_{n+1}) + \frac{b}{2}[S(x, T_m x, T_{n+2} x_{n+1}) + S(x_{n+1}, T_m x, T_{n+2} x_{n+1})]$$

$$+ \frac{c}{2}[S(x, x_{n+1}, T_m x) + S(x, x_{n+1}, T_{n+2} x_{n+1})]$$

$$= \lim_{n \to \infty} \{aS(x, x, x_{n+1}) + \frac{b}{2}[S(x, T_m x, x_{n+2}) + S(x_{n+1}, T_m x, x_{n+2})]$$

$$+ \frac{c}{2}[S(x, x_{n+1}, T_m x) + S(x, x_{n+1}, x_{n+1})]$$

$$aS(x, x, x) + \frac{b}{2}[S(x, T_m x, x) + S(x, T_m x, x)] + \frac{c}{2}[S(x, x, T_m x) + S(x, x, x)]$$

$$= bS(x, T_m x, x) + \frac{c}{2} S(x, x, T_m x)$$

$$\leq bS(x, x, T_m x) + \frac{c}{2} S(x, x, T_m x)$$

$$S(T_m x, T_m x, x) \leq \left(b + \frac{c}{2}\right) S(x, x, T_m x)$$

$$= \left(\frac{2b + c}{2}\right) S(x, x, T_m x)$$

Therefore, $\| S(T_m x, T_m x, x) \| \leq \left(\frac{2b + c}{2}\right) K \| S(x, x, T_m x) \|$

Since $\frac{2b + c}{2} < 1$, then $S(T_m x, T_m x, x) = 0$.

Which implies that $S(T_m x, T_m x, x) << 0$.

Hence, $T_m x = x$.

Therefore, $T_n x = x$ for all n.

Hence x is a common fixed point of $\{T_n\}$.

Uniqueness: Let $y \neq x$ such that $T_n y = y, \forall n$.

Now consider

$$S(x, x, y) = S(T_i x, T_i x, T_j y)$$

$$\leq aS(x, x, y) + \frac{b}{2}[S(x, T_i x, T_j y) + S(y, T_i x, T_j y)]$$

$$+ \frac{c}{2}[S(x, y, T_i x) + S(x, y, T_j y)]$$

$$= aS(x, x, y) + \frac{b}{2}[S(x, x, y) + S(y, x, y)] + \frac{c}{2}[S(x, y, x) + S(x, y, y)]$$

$$\leq aS(x, x, y) + \frac{b}{2}[S(x, x, y) + S(y, y, x)] + \frac{c}{2}[S(x, x, y) + S(x, x, y)]$$

$$\leq aS(x, x, y) + \frac{b}{2}[S(x, x, y) + S(x, x, y)] + cS(x, x, y)$$

$$= (a + b + c)S(x, x, y)$$

Where $a + b + c < 1,$ which implies $S(x, x, y) = 0.$ Hence $x = y.$

Theorem 3.3. *Let* (X, S) *be a cone S-Metric space which is complete and let* P *a normal cone with* K *as normal constant. Let* T_n *be a sequence of mappings from* X *to* X *satisfying the condition*

$$S(T_i x, T_i x, T_j y) \leq aS(x, x, y) + b \max S(x, T_i x, T_j y), S(x, T_i x, T_j y)\}$$

$+c \max \{S(x, x, T_j x), S(x, x, T_j y)\}$ *for all* $i \neq j$ *and* $\forall x, y \in X,$ *where* $a, b, c > 0$ *and* $a + 2b + 2c < 1.$ *Then* $\{T_n\}$ *has a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary element in $X.$

The sequence $\{x_n\}$ in X defined by $x_{n+1} = T_{n+1} x_n,$ for $n = 0, 1, \ldots$

Now

$$S(x_{n+1}, x_{n+1}, x_{n+2}) = S(T_{n+1} x_n, T_{n+1} x_n, T_{n+2} x_{n+1})$$

$$S(x_{n+1}, T_{n+1} x_n, T_{n+2} x_{n+1})\}$$

$$\leq aS(x_n, x_n, x_{n+1}) + b \max \{S(x_n, T_{n+1} x_n, T_{n+2} x_{n+1}),$$

$$+c \max\{S(x_n, x_{n+1}, T_{n+1} x_n), S(x_n, x_{n+1}, T_{n+2} x_{n+1})\}$$

$$= aS(x_n, x_n, x_{n+1})$$

$$+b \max \{S(x_n, x_{n+1}, x_{n+2}), S(x_{n+1}, x_{n+1}, x_{n+2})\}$$

$$+c \max \{S(x_n, x_{n+1}, x_{n+1}), S(x_n, x_{n+1}, x_{n+2})\}$$

$$= aS(x_n, x_n, x_{n+1}) + bS(x_n, x_{n+1}, x_{n+2}) + cS(x_n, x_{n+1}, x_{n+2})$$

$$\leq aS(x_n, x_n, x_{n+1}) + (b+c)\{S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1})\}$$

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq (a+b+c)S(x_n, x_n, x_{n+1}) \leq (b+c)S(x_{n+2}, x_{n+2}, x_{n+1})$$

$$(1-b-c)S(x_{n+1}, x_{n+1}, x_{n+2}) \leq (a+b+c)S(x_n, x_n, x_{n+1})$$

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \frac{a+b+c}{1-b-c} S(x_n, x_n, x_{n+1})$$

$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq hS(x_n, x_n, x_{n+1})$, where $h = \frac{a+b+c}{1-b-c} < 1$ as $a + 2b + 2c < 1$

Similarly,

$$S(x_{n+2}, x_{n+2}, x_{n+3}) \leq hS(x_{n+1}, x_{n+1}, x_{n+2})$$

Thus

$$S(x_n, x_n, x_{n+1}) \leq hS(x_{n-1}, x_{n-1}, x_n)$$

$$\leq h^2 S(x_{n-2}, x_{n-2}, x_{n-1})$$

$$\vdots$$

$$\leq h^n S(x_0, x_0, x_1)$$

$$\to 0 \text{ as } n \to \infty$$

Now

$$S(x_n, x_n, x_m) \leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)$$

$$\leq 2S(x_n, x_n, x_{n+1}) + \ldots + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_{m-1}, x_{m-1}, x_m)$$

$\to 0$ as $m, n \to \infty$ we get

Therefore, $S(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$

So the sequence $\{x_n\}$ is Cauchy.

Since (X, S) is complete, sequence $\{x_n\}$ converges to $x \in X$.

Now

$$S(T_m x,\ T_m x,\ x) = \lim_{n\to\infty} S(T_m x,\ T_m x,\ x_{n+2})$$

$$= \lim_{n\to\infty} S(T_m x,\ T_m x,\ T_{n+2}x_{n+1})$$

$$\leq \lim_{n\to\infty}\{aS(x,\ x,\ x_{n+1}) + b\max\{S(x,\ T_m x,\ T_{n+2}x_{n+1}),\ S(x_{n+1},\ T_m x,\ T_{n+2}x_{n+1})\}$$

$$+ c\max\{S(x,\ x_{n+1},\ T_m x),\ S(x,\ x_{n+1},\ T_{n+2}x_{n+1})\}$$

$$= \lim_{n\to\infty}\{aS(x,\ x,\ x_{n+1}) + b\max\{S(x,\ T_m x,\ x_{n+2}),\ S(x_{n+1},\ T_m x,\ x_{n+2})\}$$

$$+ c\max\{S(x,\ x_{n+1},\ T_m x),\ S(x,\ x_{n+1},\ x_{n+2})\}$$

$$= aS(x,\ x,\ x) + b\max\{S(x,\ T_m x,\ x),\ S(x,\ T_m x,\ x)\}$$

$$+ c\max S(x,\ x,\ T_m x),\ S(x,\ x,\ x)\}$$

$$= bS(x,\ T_m x,\ x) + cS(x,\ x,\ T_m x)$$

$$\leq bS(x,\ x,\ T_m x) + cS(x,\ x,\ T_m x)$$

$$S(x,\ x,\ T_m x) \leq (b+c)S(x,\ x,\ T_m x)$$

Therefore, $\| S(T_m x,\ T_m x,\ x) \| \leq (b+c)K \| S(x,\ x,\ T_m x) \|$

Since $b + c < 1$, then $S(T_m x,\ T_m x,\ x) = 0$.

Which implies $S(T_m x,\ T_m x,\ x) << 0$.

Hence, $T_m x = x$.

Therefore, $T_n x = x$ for all n.

Hence x is a common fixed point of $\{T_n\}$.

Uniqueness. Let $y \neq x$ such that $T_n y = y,\ \forall\, n$.

Now consider

$$S(x,\ x,\ y) = S(T_i x,\ T_i x,\ T_j x)$$

$$\leq aS(x,\ x,\ y) + b\max\{S(x,\ T_i x,\ T_j y),\ S(y,\ T_i x,\ T_j y)\}$$

$$+c \max \{S(x, y, T_i x), S(x, y, T_j y)\}$$

$$= aS(x, x, y) + b \max \{S(x, x, y), S(y, x, y)\}$$

$$+c \max \{S(x, x, y), S(x, x, y)\}$$

$$\leq aS(x, x, y) + b \max \{S(x, x, y), S(y, y, x)\}$$

$$+c \max \{S(x, x, y), S(x, x, y)\}$$

$$\leq aS(x, x, y) + b \max \{S(x, x, y), S(x, x, y)\} + cS(x, x, y)$$

$$= aS(x, x, y) + bS(x, x, y) + cS(x, x, y)$$

$$= (a + b + c)S(x, x, y).$$

Where $a + b + c < 1,$ which implies $S(x, x, y) = 0.$ Hence $x = y.$

References

[1] J. M. Afra, Fixed point type theorem in S-metric spaces, Middle-East J. Sci. Res. 22 (2014), 864-869.

[2] J. M. Afra, Fixed point type theorem for weak contraction in S-metric spaces, Int. J. Res. Rev. Appl. Sci. 22 (2015), 11-14.

[3] J. M. Afra, Double contraction in S-metric spaces, Int. J. Math. Anal. 9 (2015), 117-125.

[4] A. H. Ansari, S. Chandok, N. Hussin and L. Paunovic, Fixed points of weak contractions in regular cone metric spaces via new function, J. Adv. Math. Stud. 9(1) (2016), 72-82.

[5] P. Chouhan and N. Malviya, A common unique fixed point theorem for expansive type mappings in S-metric spaces, Int. Math. Forum 8 (2013), 1287-1293.

[6] B. C. Dhage, Generalized metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[7] A. Gupta, Cyclic contraction on cone S-metric space, International Journal of Analysis and Applications 3(2) (2013), 119-130.

[8] V. Gupta and R. Deep, Some coupled fixed point theorems in partially ordered S-metric spaces, Miskolc Math. Notes 16 (2015), 181-194.

[9] L. G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[10] R. Krishnakumar and D. Dhamodharan, Fixed point theorems in normal cone metric space, International J. of Math. Sci. Engg. Appls. 10(III) 213-224.

[11] R. Krishnakumar and D. Dhamodharan, Cone S-Metric Space and Fixed point theorems of Contractive Mappings, Annals of Pure and Applied Mathematics DOI:http://dx.doi.org/10.22457/apam.v14n2a5.

[12] J. K. Kim, S. Sedghi and N. Shobkolaei, Common fixed point theorems for the R-weakly commuting mappings in S-metric spaces, J. Comput. Anal. Appl. 19 (2015), 751-759.

[13] N. Y. Ozgur and N. Tas, Some fixed point theorems on S-metric spaces, Mat. Vesnik, 69(1) (2017), 39-52.

[14] M. U. Rahman, M. Sarwar and M. U. Rahman, Fixed point results of Altman integral type mappings in S-metric spaces, Int. J. Anal. Appl. 10 (2016), 58-63.

[15] Sh. Rezapour and R. Hamlbarani, Some notes on the paper' cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345 (2008), 719-724.

[16] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces Mat. Vesn. 64(3) (2012), 258-266.

[17] S. Sedghi, N. V. Dung, Fixed point theorems on S-metric spaces, Mat. Vesnik 66 (2014), 113-124.

[18] S. Sedghi, K. P. R. Rao and N. Shobe, Common fixed point theorems for six weakly compatible mappings in D*-metric spaces, Int. J. Math. Sci. 6 (2007), 225-237.

[19] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik 64 (2012), 258-266.