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TRIPLE LAPLACE TRANSFORMS AND ITS PROPERTIES

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Abstract

Integral transforms play most important role in development of modern science and engineering. Theory of Laplace transform of function of a single variable, its properties and applications are mostly available in literature, but a very little work is available on the multiple Laplace transform, its properties and applications. This paper deals with the extension of the Double Laplace transform by Lokenath Debnath. Here we developed Triple Laplace transforms and its properties.

1. Introduction, Definitions and Standard Results

Linear integral transform such as Laplace transforms and Fourier transforms are powerful tools to solve many engineering and technological problems. Therefore, latest advancements of these transforms have vital importance. Laplace transform is developed by the great French mathematician Laplace (1749-1827). British electrical engineer, Oliver Heaviside (1850-1925) made the Laplace transform very popular by applying it to solve ordinary differential equations of electrical circuits and systems. They also developed powerful and effective tool to solve telegraph equation and second order hyperbolic partial differential equation with constant coefficients, [1], [10], [11], [12]. Laplace transform has many applications in

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various fields such as Mechanics, Electrical circuit, Beam problems, Heat conduction, Wave equation, Transmission lines, Signals and systems, Control systems, Communication systems, Hydrodynamics, Solar systems. It can be used for encryption and decryption [2], [3], [4], [5], [6], [7], [8], [9]. At present there is very extensive literature available or no work available on the multiple Laplace transform of several variables. So, the purpose of this paper is to study triple transforms and its properties.

1.1. The Laplace Transform: If $f(t)$ is a function defined for all positive values of t then the Laplace Transform of $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt, \tag{1.1}$$

provided that the integral exists. Here the parameter s is a real or complex number.

The corresponding inverse Laplace transform is $L^{-1}\{F(s)\} = f(t)$. Applications of Laplace transform are found in [1-12].

1.2. The Double Laplace Transform: The Double Laplace transform of a function $f(x, y)$ defined in the first quadrant of the xy plane is defined by the double integral in the form

$$L_2\{f(x, y)\} = \bar{\bar{f}}(p, q) = F(p, q)$$

$$= L\{\{L\{f(x, y); x \to p\}; y \to q\}\}$$

$$= L\{\{\bar{f}(p, y); y \to q\}\}$$

$$= \int_0^\infty \int_0^\infty e^{-(px+qy)} f(x, y) dx dy \tag{1.2}$$

provided that the integral exists where we follow Lokenath Debnath [11] to denote double Laplace transform. The corresponding inverse double Laplace transform is defined by

$$L_2^{-1}\{\bar{\bar{f}}(p, q)\} = f(x, y)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} dp \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{qy} dq F(p, q) dq,$$

Where $\bar{\bar{f}}(p, q)$ must be analytic for all (p, q) in the region defined by $\operatorname{Re} p \geq 0, \operatorname{Re} q \geq 0$ for some c, d are real constants to be chosen suitably.

Theory of double Laplace transform is recently developed by Loknath Debenath, [11] the work is extended here.

1.3. The Triple Laplace Transform: The Triple Laplace transform of a function $f(x, y, z)$ of three variables x, y, z defined in the first quadrant of the xyz plane is defined by the triple integral in the form

$$L_3\{f(x, y, z)\} = \bar{\bar{\bar{f}}}(p, q, r) = F(p, q, r)$$

$$= L\{L\{L\{f(x, y, z); x \to p\}; y \to q\}; z \to r\}$$

$$= L\{L\{\bar{f}(p, y, z); y \to q\}; z \to r\}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+rz)} f(x, y, z) dxdydz, \tag{2.1}$$

provided that the integral exists where we follow Lokenath Debnath [11] to denote double Laplace transform. The corresponding inverse triple Laplace transform is defined by

$$L_3^{-1}\{\bar{\bar{\bar{f}}}(p, q, r)\} = f(x, y, z)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} dp \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{qy} dq \frac{1}{2\pi i} \int_{e-i\infty}^{e+i\infty} e^{rz} F(p, q, r) dr,$$

Where $\bar{\bar{\bar{f}}}(p, q, r)$ must be analytic for all (p, q, r) in the region defined by $\operatorname{Re} p \geq 0, \operatorname{Re} q \geq 0, \operatorname{Re} r \geq 0,$ for some c, d, e are real constants to be chosen suitably.

It is very obvious that triple Laplace transform and corresponding inverse triple transforms are linear transforms.

2. Some Standard functions and Their Triple Laplace Transforms

We assume that a, b, c are real numbers and $i = \sqrt{-1}$. Here we consider some standard functions and we derive the corresponding triple Laplace transforms.

2.1. If $f(x, y, z) = 1$ for $x > 0$, $y > 0$, $z > 0$ then using definition of triple Laplace transform it follows that $L_3\{1\} = \frac{1}{pqr}$.

2.2. If $f(x, y, z) = e^{ax+by+cz}$ then

$$L_3\{f(x, y, z)\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+rz)} e^{(ax+by+cz)} dxdydz$$

$$= \int_0^\infty e^{-(p-a)x} dx \int_0^\infty e^{-(q-b)y} dy \int_0^\infty e^{-(r-c)z} dz$$

$$= \frac{1}{p-a} \frac{1}{q-b} \frac{1}{r-c}.$$

Thus $L_3\{e^{(ax+by+cz)}\} = \frac{1}{p-a} \frac{1}{q-b} \frac{1}{r-c}$.

2.3. $L_3\{e^{(ax+by+cz)i}\} = \frac{1}{p-ia} \frac{1}{q-ib} \frac{1}{r-ic}$.

$$= \frac{p+ia}{p^2+a^2} \frac{q+ib}{q^2+b^2} \frac{r+ic}{r^2+c^2}$$

$$= \frac{(pqr - abc - bcp - acq) + i(bpr + aqr + cpq - abc)}{(p^2+a^2)(q^2+b^2)(r^2+c^2)}.$$

2.4. $L_3\{\cos(ax+by+cz)\} = \frac{(pqr - abc - bcp - acq)}{(p^2+a^2)(q^2+b^2)(r^2+c^2)}$.

2.5. $L_3\{\sin(ax+by+cz)\} = \frac{(bpr + aqr + cpq - abc)}{(p^2+a^2)(q^2+b^2)(r^2+c^2)}$.

2.6. $L_3\{\cosh(ax+by+cz)\} = \frac{1}{2} L_3\{e^{(ax+by+cz)} + e^{-(ax+by+cz)}\}$

$$= \frac{1}{2}\left[\left(\frac{1}{p-a} \frac{1}{q-b} \frac{1}{r-c}\right) + \left(\frac{1}{p+a} \frac{1}{q+b} \frac{1}{r+c}\right)\right].$$

2.7. $L_3\{\sinh(ax + by + cz)\} = \frac{1}{2} L_3\{e^{(ax+by+cz)} - e^{-(ax+by+cz)}\}$

$$= \frac{1}{2}\left[\left(\frac{1}{p-a}\,\frac{1}{q-b}\,\frac{1}{r-c}\right) - \left(\frac{1}{p+a}\,\frac{1}{q+b}\,\frac{1}{r+c}\right)\right].$$

2.8. $L_3\{x^l y^m z^n\} = \frac{\Gamma(l+1)}{p^{l+1}} \cdot \frac{\Gamma(m+1)}{q^{m+1}} \cdot \frac{\Gamma(n+1)}{r^{n+1}}$; $l > -1,\ m > -1,\ n > -1$ are real numbers.

2.9. $L_3\{f(x)g(y)h(z)\} = L\{f(x)\} \cdot L\{g(y)\} \cdot L\{h(z)\} = F(p) \cdot G(q) \cdot H(r)$.

3. Main Theorems of Triple Laplace Transforms

For existence of triple Laplace Transform we need following:

Function $f(x, y, z)$ is said to be of exponential order $a_1(> 0)$, $b_1(> 0)$ and $c_1(> 0)$ on $0 \le x \le \infty$, $0 \le y \le \infty$ and $0 \le z \le \infty$, if there exists a positive constant K such that for all $x > X$, $y > Y$ and $z > Z$,

$|f(x, y, z)| \le Ke^{a_1x+b_1y+c_1z}$ and we write

$f(x, y, z) = O(e^{a_1x+b_1y+c_1z})$ as $x \to 0,\ y \to 0,\ z \to 0$.

Or, equivalently

$$\lim_{x\to\infty,\ y\to\infty,\ z\to\infty} e^{\alpha x+\beta y+\gamma z}|f(x, y, z)| = K,$$

$$\lim_{x\to\infty,\ y\to\infty,\ z\to\infty} e^{-(\alpha-a_1)x+(\beta-b_1)y+(\gamma-c_1)z} = 0,\ \alpha > a_1,\ \beta > b_1,\ \gamma > c_1.$$

Now we will study following theorems.

Theorem 3.1. *If the function $f(x, y, z)$ is a continuous function in every finite intervals $(X, 0, 0), (0, Y, 0), (0, 0, Z)$ and of exponential order $\exp(a_1x + b_1y + c_1z)$, then the triple Laplace transform of $f(x, y, z)$ exists for all p, q, r provided* $\operatorname{Re} p > a_1$, $\operatorname{Re} q > b_1$, $\operatorname{Re} r > c_1$.

Theorem 3.2. $L_3\{e^{-(ax+by+cz)}f(x, y, z)\} = F((p + a), (q + b), (r + c))$.

Theorem 3.3. $L_3\{f(ax)g(by)h(cz)\} = \frac{1}{abc} F\left(\frac{p}{a}\right)G\left(\frac{q}{b}\right)H\left(\frac{r}{c}\right)$, $a > 0$, $b > 0$, $c > 0$.

Theorem 3.4. $L_3\{f(x)\} = \frac{1}{qr} F(p)$.

Theorem 3.5.

$$L_3\{f(x + y + z)\} = \frac{1}{p - r}\left[\frac{F(r)}{q - r} - \frac{F(p)}{q - p}\right] = \frac{1}{q - r}\left[\frac{F(r)}{p - r} - \frac{F(q)}{p - q}\right]$$

$$= \frac{1}{q - p}\left[\frac{F(p)}{r - p} - \frac{F(q)}{r - q}\right],\ p \neq q \neq r.$$

Theorem 3.6. $L_3\{f(x - \xi,\ y - \eta,\ z - \tau)\} = e^{-\xi p - \eta q - \tau r} F_3(p,\ q,\ r)$.

Theorem 3.7. *Convolution theorem*:

$$L_3\{(f *** g)(x,\ y,\ z)\} = F_3(p,\ q,\ r) \cdot G_3(p,\ q,\ r),$$

where

$$(f *** g)(x,\ y,\ z) = \int_0^x \int_0^y \int_0^z f(x - \xi,\ y - \eta,\ z - \tau) d\xi d\eta d\tau.$$

It can be easily followed from definition of convolution that it satisfied following properties:

a. Associative: $[f *** (g *** h)](x,\ y,\ z) = [(f *** g) *** h](x,\ y,\ z)$.

b. Distributive: $[f *** (ag + bh)](x,\ y,\ z) = a(f *** g)(x,\ y,\ z) + b(f *** h)$ $\cdot(x,\ y,\ z)$

c. Identity: $(f *** \delta)(x,\ y,\ z) = (\delta *** f)(x,\ y,\ z) = f(x,\ y,\ z)$, here $\delta(x,\ y,\ z)$ is Dirac delta function of $x,\ y,\ z$.

Recently some results on triple Laplace transforms are found in [13], [14], [15], [16].

4. Illustrative Examples

Here we will discuss illustrative examples based on results obtained in sections 2 and 3.

4.1. Using 2.3 we have $L_3\{e^{(x+y+z)}\} = \frac{1}{(p-1)(q-1)(r-1)}$. In particular $L_3\{e^x\} = \frac{1}{qr(p-1)}$ which can also be obtained from Theorem 3.4.

4.2. Using 2.3 we have $L_3\{e^{(x+y+z)i}\} = \frac{(pqr-1-p-q)+i(pr+qr+pq-1)}{(p^2+1)(q^2+1)(r^2+1)}$

4.3. Using 2.4 we have $L_3\{\cos(x+y+z)\} = \frac{(pqr-1-p-q)}{(p^2+1)(q^2+1)(r^2+1)}$ and $L_3\{\cos x\} = \frac{p}{qr(p^2+1)}$ which can also be obtained from Theorem 3.4. Also, using Theorem 3.2

$$L_3\{e^{-x}\cos x\} = \frac{p+1}{qr(p+2p+2)}.$$

4.4. Using 2.5 we have $L_3\{\sin(x+y+z)\} = \frac{(pr+qr+pq-1)}{(p^2+1)(q^2+1)(r^2+1)}$ and $L_3\{\sin x\} = \frac{a}{qr(p^2+1)}$ which can also be obtained from Theorem 3.4. Also, using Theorem 3.3 $L_3\{\sin 2x\} = \frac{4a}{qr(p^2+4)}$.

4.5. Using 2.6 we have $L_3\{e^{ix}\} = \frac{p+ia}{qr(p^2+1)}$ which can also be obtained from 4.3 and 4.4 as well as from Theorem 3.4.

4.6. Using 2.6 and 2.7 we have

$$L_3\{\cosh(x+y+z) = \frac{1}{2}\left[\left(\frac{1}{(p-1)(q-1)(r-1)}\right)+\left(\frac{1}{(p+1)(q+1)(r+1)}\right)\right],$$

$$L_3\{\sinh(x+y+z) = \frac{1}{2}\left[\left(\frac{1}{(p-1)(q-1)(r-1)}\right)-\left(\frac{1}{(p+1)(q+1)(r+1)}\right)\right].$$

In particular,

$$L_3\{\cosh x\} = \frac{p}{qr(p^2-1)} \text{ and } L_3\{\sinh x\} = \frac{1}{qr(p^2-1)}.$$

4.7. Using 2.8 and 2.9, we have

$$L_3\{x^{\frac{1}{2}}y^{\frac{1}{2}}\} = \frac{\pi}{4(pq)^{\frac{3}{2}}}.$$

4.8. Using 2.8 and 2.9 and Theorem 3.2, we have $L_3\{e^{-(x+y+z)}xy^2z^3\}$

$$= \frac{12}{(p+1)^2(q+1)^3(r+1)^4}.$$

5. Concluding Remark

Triple Laplace transform, corresponding theorems and properties are discussed in this paper. Theory developed in this paper can be applied to solve first order partial differential equations of three variables, partial differential equations such as wave equation, heat equation, Laplace equation. These results can further be applicable to solve integral equations, as well as problems in cryptography, [2], [3], [4], [5], [6], [7], [9]. Theory of Extension of prefunctions discussed in [8] can also be used for further extension and applications. Results of triple Laplace transforms obtained in this paper can be applied to solve problems in biomedical signals like Phonocardiograph (PCG) to improve readability of signals. Also, it has applications in signal and image processing techniques like Electroencephalograph (ECG) and Magnetic resonance Images (MRI). Extension of results of this paper for emerging problems as mentioned above will be discussed in the subsequent work.

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