Advances and Applications in Mathematical Sciences
Volume 21, Issue 10, August 2022, Pages 5853-5863

ON SYNCHRONIZATION OF CIRCULAR SEMI-FLOWER AUTOMATA

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Abstract

It is known that each n-state circular automaton having a non-permutation input letter is synchronizing, if n is a prime number. The aim of this paper is to investigate the synchronization of n-state circular semi-flower automata having a non-permutation input letter, where $n > 1$ is an integer. We first prove that every semi-flower automaton is one-cluster with respect to each input letter. Using a group-theoretic technique we next prove, for an odd integer $n > 3$, that each n-state circular semi-flower automaton containing a cycle of length at most two is synchronizing. We back up with few examples to conclude that not every circular semi-flower automaton is synchronizing.

1. Introduction

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an n-state (complete and deterministic) automaton, where Q is the state set of size n, Σ is finite alphabet with at least two (input) letters, and $\delta : Q \times \Sigma \to Q$ is the transition function. The canonical extension of δ to $Q \times \Sigma^*$ is also denoted by δ. An automaton $\mathcal{A}$ is called synchronizing if there is a word in Σ^*, called a synchronizing word, that sends all its states to a single state.

Several investigations have been done in the area of synchronization of automata. For prime n, Pin [16] proved that each n-state circular automaton containing a non-permutation letter is synchronizing. Perles et al. [15]

2020 Mathematics Subject Classification: 20M35, 68Q45, 68Q70, 68R10.
Keywords: circular automata, semi-flower automata, one-cluster automata, synchronizing automata, Černý conjecture.
Received May 22, 2020; Accepted September 30, 2022

observed that the class of definite automata is a subclass of synchronizing automata. It has also been verified that each strongly connected aperiodic automaton is synchronizing. Problem of synchronization of automata appears to be nontrivial and, even in case of small alphabet size, it is hard to provide a characterization of synchronization of automata.

Besides many applications of synchronizing automata [26], there is a famous conjecture for synchronized automata, known as the Černý conjecture. The Černý conjecture [5] claims that each n-state synchronizing automaton possesses a synchronizing word of length at most $(n-1)^2$. Despite many attempts, the Černý conjecture in the general case is still unsolved. In this context, Pin [16] proved that the Černý conjecture is satisfied for circular automata with a prime number of states. Dubuc [6] proved that all circular automata satisfy the Černý conjecture. Steinberg [24] proved the Černý conjecture for one-cluster automata with prime length cycle. For other special classes of synchronizing automata, the Černý conjecture has also been verified, or sharper bounds have been proven, see for instance [1, 3, 7, 12, 13, 17, 25, 27].

In this paper, we focus our attention on the synchronization of circular semi-flower automata. Circular automata have been studied in various contexts [6, 16]. Semi-flower automata (in short, SFA) have been introduced to study the finitely generated submonoids of a free monoid [10, 19]. Using SFA, the rank and intersection problem of certain finitely generated submonoids of a free monoid have been investigated [11, 19, 20]. SFA have also been studied in several contexts over the last few years, see for instance [9, 18, 21, 22, 23].

The rest of the paper is organized as follows. In the next section, we introduce the notation and briefly give the required background. We investigate the synchronization of circular semi-flower automata in Section 3. We finally conclude the paper in Section 4.

2. Preliminaries and Notation

In this section, we introduce some necessary concepts and fix notation used within this paper. We refer the reader to the standard books [2, 8, 14] for terminology in digraph, group, and automata theory respectively.

We write $|X|$ to denote the size of a nonempty finite set X, and T_n to denote the full transformation monoid of an n-element set. We write argument of a trans-formation $\alpha \in T_n$ on its left so that $x\alpha$ is the value of α at the argument x. The composition of transformations is designated by concatenation, with the leftmost transformation understood to apply first, so that $x(\alpha\beta) = (x\alpha)\beta$.

Let D be a (labeled) digraph with vertex set $V(D)$. A path in D is an alternating finite sequence $v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$ of distinct vertices and (labeled) edges such that, for $1 \leq i \leq k$, the tail and the head of edge e_i are vertices v_{i-1} and v_i, respectively. A path with at least one edge is called a cycle if its initial vertex and terminal vertex are the same. The length of a path is the number of its edges. A k-cycle is a cycle of length k. If there is a path from vertex u to vertex v, then the distance from u to v is the length of shortest path from u to v.

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an n-state automaton. We write Σ^* and ε to denote the set of all words over Σ and the empty word, respectively. Each word $x \in \Sigma^*$ has a natural interpretation as a transformation in T_n and we do not distinguish between the word x and its interpretation. A permutation letter is a letter in Σ whose interpretation is a permutation; otherwise, it will be called non-permutation. Automaton $\mathcal{A}$ is called circular if there exists a permutation letter which induces a circular permutation on its state set. The set $M(\mathcal{A}) = \{x \in T_n \mid x \in \Sigma^*\}$ forms a submonoid of the full transformation monoid T_n called the transition monoid of $\mathcal{A}$. An automaton $\mathcal{A}$ is called synchronizing if there exists a word $x \in \Sigma^*$ such that the image of Q under the transformation $x \in M(\mathcal{A})$ is a singleton set.

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton and let $a \in \Sigma$. By denoting states as vertices and transitions as (labeled directed) edges, $\mathcal{A}$ can be represented by (labeled) digraph, denoted by $D(\mathcal{A})$. A path (respectively, cycle) in $\mathcal{A}$ is a path (respectively, a cycle) in $D(\mathcal{A})$. An automaton $\mathcal{A}$ is called semi-flower if all the cycles in $\mathcal{A}$ pass through a common state, say q_0. We will use the terms

a-edge and a-cycle to mean respectively an edge labeled by a, and a cycle whose all edges are labeled by the letter a.

Let G be a finite group with identity e, and let $g \in G$. The order of g is the smallest positive integer t such that $g^t = e$. The cyclic subgroup generated by g is denoted by $\langle g \rangle$. If $G = \langle g \rangle$ for some $g \in G$, then G is called cyclic, and g the generator of G. Note that any two finite cyclic groups of the same order are isomorphic. In finite group G, the order of each element of G divides the order of G [8, Theorem 7.1]. Therefore, if $|G|$ is odd, then the order of each element of G is also odd. The set of all invertible elements of a monoid is called its group of units.

3. Main Results

In this section, we investigate the synchronization of circular semi-flower automata (in short, CSFA). In order to investigate the synchronization of CSFA, we first recall the notion of one-cluster automata introduced by Béal and Perrin in [4].

An Automaton $\mathcal{A} = (Q, \Sigma, \delta)$ is one-cluster with respect to input letter $a \in \Sigma$ if the sub graph of $D(\mathcal{A})$ obtained by considering only the edges labeled by a is connected.

Theorem 3.1. *Every semi-flower automaton is a one-cluster automaton with respect to each letter*.

Proof. Let $\mathcal{A} = (Q, \Sigma, \delta)$ be a semi-flower automaton and let $b \in \Sigma$. Suppose that $\mathcal{R}$ is the sub graph of $D(\mathcal{A})$ obtained by considering only the edges labeled by b.

Since the state set is finite and each state has exactly one outgoing edge in $\mathcal{R}$, we see each connected components of $\mathcal{R}$ contains a unique cycle. Since $\mathcal{A}$ is an SFA, all the cycles in the sub graph $\mathcal{R}$ must pass through q_0. That means q_0 belongs to all the connected components of $\mathcal{R}$ and subsequently the sub graph $\mathcal{R}$ is connected. Then, since b is arbitrary, we can conclude the proof. □

The converse of Theorem 3.1 is not necessarily true as it is shown in the following example 3.2.

Example 3.2. Consider the 6-state automaton $\mathcal{A}$ over alphabet $\sum = \{a, b\}$ given in the Figure 1. For any input letter, we see that $\mathcal{A}$ is a one-cluster automaton with respect to that letter. Moreover, $\mathcal{A}$ is not a semi-flower automaton since $\mathcal{A}$ contains no state such that all its cycles pass through that state.

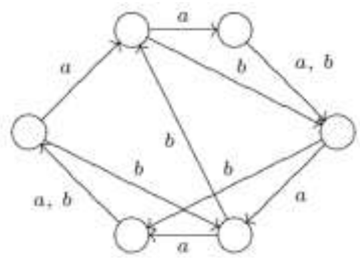

Figure 1. A one-cluster automaton which is not a semi-flower automaton.

In view of Theorem 3.1, we gee that every semi-flower automaton contains a unique a-cycle for each input letter $a \in \Sigma$.

Notation 3.3. Let $\mathcal{A}$ be a semi-flower automaton and let $b \in \Sigma$. We write C_b to denote the unique b-cycle in $\mathcal{A}$. Moreover, the set of states in the b-cycle C_b is denoted by $V(C_b)$.

We next recall the notion of the level of an automaton from [4].

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton and let $b \in \Sigma$. The sub graph of $D(\mathcal{A})$ obtained by considering only the edges labeled by b is a disjoint union of connected components called b-clusters. Since each state has exactly one outgoing edge in this sub graph, each b-cluster contains a unique cycle, called an b-cycle, with trees attached to the cycle at their root. The level of a state q in a b-cluster is defined as the distance between q and the root of the tree containing q. If q belongs to the cycle, its level is defined as zero. The level of A is defined as the maximal level of its states; see Figure 2.

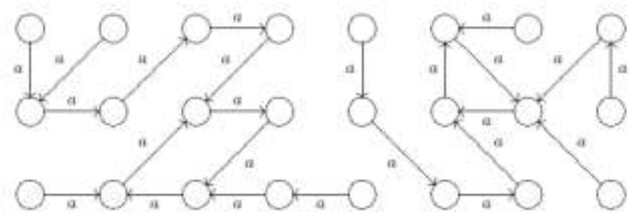

Figure 2. An automaton of the level 5.

Notation 3.4. We denote the level of an automaton by l.

We now have the following theorem.

Theorem 3.5. *Let $\mathcal{A}$ be a semi-flower automaton. If a letter $b \in \Sigma$ has a singleton cycle in C_b, then $\mathcal{A}$ is synchronizing.*

Proof. Notice that $Qb^l = V(C)$. Since $|V(C_b)| = 1$, it follows that $|Qb^l| = 1$. That means $\mathcal{A}$ has a synchronizing word b^l and so $\mathcal{A}$ is synchronizing.

We now recall a useful result from [22].

Theorem 3.6 ([23])**.** *Let $\mathcal{A}$ be a semi-flower automaton and let $a, b \in \Sigma$.*

(i) *If a is a permutation, then a induces a circular permutation on Q.*

(ii) *If a and b are permutations, then permutations induced by a and b are the same.*

Unless otherwise stated, in what follows, $\mathcal{A}$ denotes an n-state circular semi-flower automaton. In view of Theorem 3.6, there is a unique circular permutation induced by letters. For the rest of the paper, we fix the following regarding $\mathcal{A}$. Assume that a letter $a \in \Sigma$ induces the circular permutation, and accordingly $q_0, q_1, \ldots, q_{n-1}$ is the cyclic ordering of Q with respect to a. We write G to denote the cyclic subgroup of $M(\mathcal{A})$ generated by the permutation a.

For completeness of the paper, we state the following necessary result from [23].

Remark 3.7 ([24])**.** Let $\mathcal{A}$ be an n-state circular semi-flower automaton. Then

(i) $|G| = n$;

(ii) G is the group of units of $M(\mathcal{A})$.

By using Lagrange's theorem [8, Theorem 7.1], one of the most fundamental theorems in finite group theory, we now prove the following theorem.

Theorem 3.8. *For an odd positive integer* $n \geq 3$, *let* $\mathcal{A}$ *be an* n*-state circular semi-flower automaton. If a letter* $b \in \Sigma$ *has a cycle* C_b *of length two, then* $\mathcal{A}$ *is synchronizing*.

Proof. Note that $q_0 \in V(C_b)$. As $|V(C_b)| = 2$, let $q_m (1 \leq m \leq n-1)$ be another state such that $q_m \in V(C_b)$ and so $V(C_b) = \{q_0, q_m\}$. It is easy to see that

$$Qb^l = V(C_b) = \{q_0, q_m\}.$$

Observe that $q_m b = q_0$ and $q_0 b = q_m$. It follows that $q_m b^2 = q_0$, and therefore

$$q_0 b^{2l} = q_0 \Rightarrow q_0 b^{1+2l} = q_0 b^{2l} = q_m.$$

Also, $q_m b^2 = q_m$ and so

$$q_m b^{2l} = q_m \Rightarrow q_m b^{1+2l} = q_0 b^{2l} = q_0.$$

This shows the function b^{1+2l} maps Q into $\{q_0, q_m\}$ and swaps the states q_0 and q_m.

Observe that $q_m a^{(n-m)} = q_0$. We now consider the sequence $\langle w_k \rangle$ of words, where $w_k := a^{k(n-m)} b^{1+2l}$. From the suffix word b^{1+2l}, it is obvious that

$$\{q_0, q_m\} w_k \subseteq \{q_0, q_m\}$$

for each positive integer k. If there exists a k such that $\{q_0, q_m\}w_k$ is a singleton, then the automaton $\mathcal{A}$ is synchronizing and hence we are done. Otherwise, assume that $\{q_0, q_m\}w_k = \{q_0, q_m\}$ for each positive integer k. Recall that $w_1 = a^{(n-m)}b^{1+2l}$ and $q_m a^{(n-m)} = q_0$. Then we see that

$q_m a^{(n-m)}b^{1+2l} = q_0 b^{1+2l} = q_m$, and therefore $q_0 a^{(n-m)}b^{1+2l} = q_0$.

So, by induction, we will have that

$$q_0 a^{k(n-m)}b^{1+2l} = \begin{cases} q_0 & \text{if } k \text{ is odd} \\ q_m & \text{if } k \text{ is even.} \end{cases} \qquad (1)$$

Since, by assumption, n is an odd positive integer and the letter a induces a circular permutation on the state set, by the application of Lagrange's theorem (cf. [8, Theorem 7.1]), the order of $a^{(n-m)}$ is also an odd positive integer, say t. For the positive integer t, we get that $a^{t(n-m)}$ induces the identity transformation on the state set. Hence

$$q_0 a^{t(n-m)}b^{1+2l} = q_0 b^{1+2l} = q_m,$$

which is a contradicting to the previous statement. This completes the proof □

If a circular semi-flower automaton has an even number of states, the following example shows that the above Theorem 3.8 is not necessarily true.

Example 3.9. Consider a 6-state circular semi-flower automaton $\mathcal{A}_1$ over alphabet $\sum = \{a, b\}$ given in the Figure 3. Since the level of each state in the b-cluster is at most one, we have $Qb = V(C_b) = \{q_0, q_3\}$. Note that a induces a circular permutation on the state set. Moreover, the distance between q_0 and q_3 in the a-cluster is three, we see that

$$\{q_0, q_3\}a^i \in \{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2, q_4\}\}$$

for every $i = 1, \ldots, 6$. Since each of the two letters a and b acts as a bijection between sets in $\{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2, q_4\}\}$, this shows that any word in $\sum^*$ cannot send the set $\{q_0, q_3\}$ to a single state. Hence $\mathcal{A}_1$ is not synchronizing.

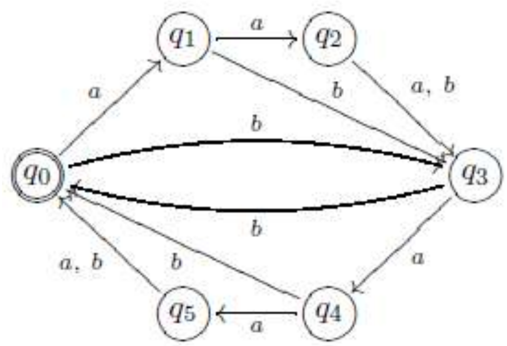

Figure 3. A non-synchronizing 6-state CSFA $\mathcal{A}_1$ with $|V(C_b)| = 2.$

For an odd positive integer $n \geq 3,$ let $\mathcal{A}$ be an n-state circular semi-flower automaton. If C is a 3-cycle in $\mathcal{A}$, then $\mathcal{A}$ is also not necessary synchronizing as shown in the following example.

Example 3.10. Consider a 9-state circular semi-flower automaton $\mathcal{A}_2$ over alphabet $\sum = \{a, b\}$ given in the Figure 4. Since the level of each state in the b-cluster is at most one, we have $Qb = V(C_b) = \{q_0, q_3, q_6\}$. Note that a induces a circular permutation on the state set. Moreover, $d\{q_0, q_3\} = d\{q_3, q_6\} = d\{q_6, q_0\} = 3,$ denotes the distance from u to v in the a-cluster. Therefore

$$\{q_0, q_3, q_6\}a^i \in \{\{q_0, q_3, q_6\}, \{q_1, q_4, q_7\}, \{q_2, q_5, q_8\}\}$$

for every $i = 1, \ldots, 9.$ Since each of the two letters a and b acts as a bijection from S to T, where $S, T \in \{\{q_0, q_3, q_6\}, \{q_1, q_4, q_7\}, \{q_2, q_5, q_8\}\}$. This shows that any word in Σ^* cannot send the set $\{q_0, q_3, q_6\}$ to a single state. Hence $\mathcal{A}_2$ is not synchronizing.

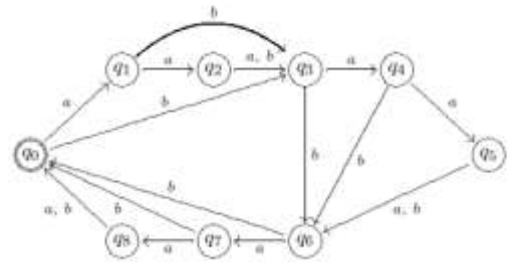

Figure 4. A non-synchronizing 9-state CSFA $\mathcal{A}_2$ with $|V(C_b)| = 3.$

4. Conclusion

This work investigated the synchronization of circular semi-flower automata (CSFA). We proved that every semi-flower automaton is one-cluster with respect to each input letter, and subsequently concluded that every semi-flower automaton containing a 1-cycle is synchronizing. For an odd integer $n > 1,$ using Lagrange's theorem, we next proved that every n-state CSFA containing a 2-cycle is synchronizing. We finally provided examples of non-synchronizing 6-state CSFA and non-synchronizing 9-cycle CSFA containing, respectively, 2-cycle and 3-cycle respectively.

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