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t-PEBBLING NUMBER AND 2t-PEBBLING PROPERTY FOR THE DELETED INDEPENDENT EDGES OF SOME GRAPHS

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Abstract

A pebbling transformation is considered as the shifting of two pebbles from a vertex and the insertion of one of them onto an adjacent vertex, given a configuration of pebbles on the vertices of a connected graph G. The t-pebbling number, $f_t(G)$, of a simple connected graph G is the

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smallest positive integer such that a sequence of pebbling moves can transfer t-pebbles to any target vertex for any distribution of $f_t(G)$, pebbles on G's vertices. In this paper, we discuss the pebbling number, t-pebbling number and 2t-pebbling property for F_n^{-f} graphs obtained from F_n by deleting f independent edges.

1. Introduction

Pebbling, introduced by Lagarias and Saks, has sparked a lot of interest. Chung [5] was the first to put it into the literature, and many others have followed suit, including Hulbert, who published an overview of graph pebbling [6]. Since Hulbert's survey first appeared in graph pebbling, a lot has happened. Graph pebbling has been an important instrument for the conveyance of consumable resources for the past 30 years. Throughout this paper, we have considered a simple connected graph G. Let us denote G's vertex and edge sets as $V(G)$ and $E(G)$, respectively. Consider a graph with a fixed number of pebbles at each vertex. One pebble is taken away and the other is placed on an neighbouring vertex when two pebbles are removed from a vertex. This process is known as a pebbling move. The pebbling number of a vertex v in a graph G is the smallest number $f(G, v)$ that allows us to shift a pebble to v using continuous pebbling move, regardless of where these pebbles are located on G's vertices. In [1] the pebbling number is defined as "The pebble number, $f(G)$, is the maximum $f(G, v)$ over all the vertices of a graph".

A. Lourdusamy et al. in [1, 2] state that "The t-pebbling number of v in G is the lowest positive number, $f_t(G, v)$, that can move t pebbles to v in all pebble configurations. The t-pebbling number $f_t(G)$ of a simple connected graph G is the maximum of $f_t(G, v)$ over all vertices v".

A. Lourdusamy et al. in [1] describe "When the total number of pebbles on a graph G at the beginning of pebble move is $2f_t(G) - q + 1$, and q is the number of vertices with at least one pebble, we can move $2t$-pebbles to the target. In this case, we say that the graph G has the $2t$-pebbling property".

Lourdusamy et al. [7], [1], [2], [3] showed that the n-cube graphs, the complete graphs, the even cycle graphs, the complete r-partite graphs, the star graphs, the fan graphs and the wheel graphs have the $2t$-pebbling property. The aim of this paper is to deal with t-pebbling number and $2t$-pebbling property for the deleted independent edges of fan graph f_n^{-f}.

2. Preliminaries

For the basic definition of graph theory, the reader can refer to [2].

Definition 2.1. Consider the path $P_{n-1} : x_1, \ldots, x_{n-1}$. Then add an extra vertex x_0 to the path P_{n-1} and connect it to all of the path's vertices. The graph obtained by this process is called fan graph and it is denoted by F_n. Considering the fan graph F_n, we construct the f_n^{-f} graph by deleting f independent edges. For convenience, we always assume that $V(F_n^{-f}) = x_1, x_2, \ldots, x_{n-1}, x_0$ and the independent edges of f_n^{-f} are $x_1, x_2, x_3, x_4, \ldots, x_{n-2}x_{n-1}$ if n is odd and $x_1, x_2, x_3, x_4, \ldots, x_{n-3}x_{n-2}$ if n is even.

Note 2.1. If n is even, then the graph f_n^{-f} will have $\left\lfloor \frac{n-1}{2} \right\rfloor$ edges and one singleton vertex x_1 on path P_{n-1} of the fan graph after deleting the f independent edges. All the edges and the singleton vertex will have a common vertex x_0. Hence, the edge set is

$$E(F_n^{-f}) = \{x_0x_j, x_2x_3, \ldots, x_{n-2}x_{n-1}\} \text{ where } j = 1, 2, \ldots, n-1.$$

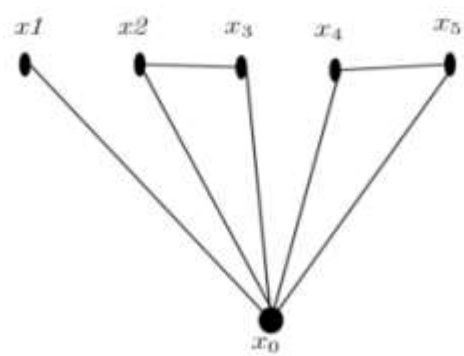

Figure 1. n is even.

If n is odd, then the graph f_n^{-f} will have $\left\lfloor \frac{n-1}{3} \right\rfloor$ edges and two singleton vertices x_1 and x_{n-1} on P_{n-1} path of the fan graph after deleting the f independent edges. The edge set of $E(F_n^{-f}) = \{x_0x_j, x_2x_3, \ldots, x_{n-3}x_{n-2}\}$ where $j = 1, 2, \ldots, n-1$.

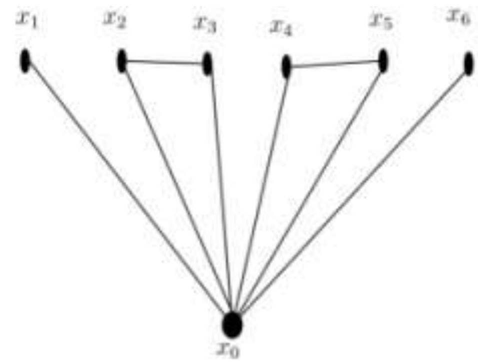

Figure 2. n is odd.

The following theorems and definitions are used to prove the theorems on section 3.

Theorem 2.1 [7]**.** *The t-pebbling number of a vertex v in a simple connected graph with n vertices where* $n \geq 4$ *and a vertex v with the property of* $d(v) = n-1$ *is* $f_t(G, v) = 2t + n - 2$.

Theorem 2.2 [7]**.** *The t-pebbling number of a path* P_n *is* $f_t(P_n) = t^{2n-1}$.

Notation 2.1. Let us denote $p(x)$ as the number of pebbles placed on the vertex x. This paper sometimes uses the destination vertex interchangeably with the target vertex. Throughout the paper, we use w to denote the target vertex.

Remark 2.1. Consider graph G, which has a pebble configuration on its vertices. From G, we select a destination vertex w. We can easily shift a pebble to w if $p(w) = 1$ or $p(s) = 2$, where $ws \in E(G)$. When w is the destination vertex, we consider that $p(s) \leq 1$ and $p(w) = 0$ for all $ws \in E(G)$ because if the destination vertex has a pebble then there is nothing to discuss.

3. The Pebbling number for f_n^{-f} graphs

Theorem 3.1. *Let* f_n^{-f} *be a simple connected graph. Then* $f(F_n^{-f})$ *is* $n+1$ *where* $n \geq 4$.

Proof. Let the vertices of f_n^{-f} be $\{x_0, x_1, x_2, \ldots, x_{n-1}\}$ and the edge set be $\{x_0 x_j, x_2 x_3, \ldots, x_{n-2,} x_{n-1}\}$ where $j = 1, 2, \ldots, n-1$ for n is even and the edge set be $\{x_0 x_j, x_2 x_3, \ldots, x_{n-3}, x_{n-2}\}$ for n is odd. When n is even, the graph f_n^{-f} has $\left\lfloor \frac{n-1}{2} \right\rfloor$ deleted edges, $\left\lfloor \frac{n-1}{2} \right\rfloor$ pairs of vertices, a singleton vertex and all these vertices have x_0 as a common neighbour. When n is odd, we have $\left\lfloor \frac{n-1}{3} \right\rfloor$ pairs of vertices, two singleton vertices, and all these vertices have x_0 as their common neighbour. To prove the necessary part, suppose we are given with n pebbles and fix x_1 as the target. By placing $p(x_1) = p(x_0) = 0$, $p(x_{n-1}) = 3$ and one pebble each on the remaining vertices, we cannot reach w. Hence, the pebbling number of $f(F_n^{-f}) \geq n+1$.

Now we prove $f(F_n^{-f}) \geq n+1$. Let C be any configuration of $n+1$ pebbles.

Case 1. Let w be x_0. Let $p(x_0) = 0$. If $p(x_j) \geq 2$ where $j = 1, 2, \ldots, n-1$, then we can shift a pebble to w.

Case 2. Let $w = x_l$, where $l = 1, 2, \ldots, n-1$. Let $p(x_l) = 0$.

Subcase 2.1. If $p(x_0) \geq 2$ or $p(x_j) \geq 4$ for $j = 1, 2, \ldots, n-1$, then we can transfer a pebble to w using one of these transmitting subgraphs $\{x_0, x_l\}$ or $\{x_j, x_0, x_l\}$.

Subcase 2.2. If $p(x_0) \geq 1$ or $p(x_j) \leq 3$ for all $j = 1, 2, \ldots, n-1$, there exists some $x_k (k \neq j)$ with $p(x_k) \geq 3$. So $\{x_j, x_0, x_l\}$ gives a transmitting subgraph.

Subcase 2.3. Let $p(x_0) = 0$ and $p(x_j) \geq 3$ where $j = \{1, 2, \ldots, n-1\}$, then there exists at least one vertex x_k with $p(x_k) \geq 2$. Further, if there exists at least 2 vertices x_k and x_m with $2 \leq p(x_k) \leq 3$, and $2 \leq p(x_m) \leq 3$, then a pebble can be shifted from x_k to x_0. So $\{x_m, x_0, x_l\}$ is a transmitting subgraph.

Case 3. Suppose there exists 2 pebbles adjacent to the target vertex of x_1 or x_0 we can reach the target. Hence, $f(F_n^{-f})$ is $n+1$, for both even and odd number of vertices of F_n^{-f}.

4. The t-Pebbling number for F_n^{-f} graphs

Theorem 4.1. *The t-pebbling number of* F_n^{-f} *is* $f_t(F_n^{-f}) = 4t + n - 3$.

Proof. Let us use the induction method on t to prove the theorem. The t-pebbling number for the vertex x_0 be $f_t(F_n^{-f}, x_0) = 2t + n - 2$. Since, $d(x_0) = n - 1$, thus, by Theorem 2.1 the result holds. Now let us find the t-pebbling number for x_1. Let $p(x_1) = 0$, $p(x_{n-1}) = 4t - 1$ and $p(x_i) = 1$ for each $i = 2, 3, \ldots, n-2$. Then we cannot reach w. Thus, $f_t(F_n^{-f}) \geq 4t + n - 3$.

Now let us prove $f_t(F_n^{-f}) \leq 4t + n - 3$. For $t = 1$, the result is true by Theorem 3.1. Let us assume that the result is true $1 < t' < t$.

Let $w = x_k$, where $k \in \{1, 2, \ldots, n-1\}$, and $p(w) = 0$.

Case 1. If $p(x_0) \geq 2$, we can transfer one pebble to the destination w. Then $4t + n - 3 - p(x_0)$ pebbles on the vertices of F_n^{-f}. After moving one pebble to the target using $4t + n - 3$ pebbles, there will be at least $(4t + n - 3) - (4 + n - 3) = 4(t-1) + n - 3 = f_{t-1}(F_n^{-f})$ pebbles left on the vertices of F_n^{-f}. Thus, $t - 1$ additional pebbles can be shifted to w using induction on t.

Case 2. Let $p(x_0) \leq 1$. If there exists some $x_j (j \neq k)$ and $p(x_j) \geq 2$ then we can transfer a pebble to x_0 using two pebbles of x_j. Which is sufficient to shift a pebble to w. Then using $4t + n - 3 - p(x_0) - p(x_j)$ pebbles $t - 1$ more pebbles can be shifted to w.

Case 3. If $p(x_k) = l$ where $1 \le l \le t - l$, then $p(F_n^{-f} - x_k) = (4t + n - 3) - l \ge 4(t - l) + n - 3$. By placing $4(t - l) + n - 3$ pebbles on remaining vertices, we can shift $t - l$ more pebbles to w. Hence, we have shifted $t - l + l = t$ pebbles to w by induction on t.

Thus, $f_t(F_n^{-f}) \ge 4t + n - 3$.

Theorem 4.2. *The graph F_n^{-f} satisfies the $2t$ pebbling property.*

Proof. The number of pebbles is denoted by p, while q represents the number of vertices having at least one pebble. Then we have the $2t$ pebbling property as $p \ge 2f_t(F_n^{-f}) - q + 1$. Note that we have $2f_t(F_n^{-f}) - q + 1 = 2(4t + n - 3) - q + 1 = 8t + 2n - 5 - q$ pebbles on the vertices of F_n^{-f}. Let $Q = 8t + 2n - 5$. To move $2t$ pebble to the target w, we use induction on t. For $t = 1$ by Theorem 3.2 holds. We assume $t > 1$ and $p(w) = 0$.

Case 1. Let w be x_0. There are $n - 1$ vertices which have at least one pebble on it and denote as q. Let the initial pebble be at least $Q \ge Q - (n - 1) = Q - n + 1$ pebbles.

Subcase 1.1. Let $n \le 8t - 5$.

Let us consider the configuration with at least $2n + 1$ pebbles. If there exists $(p(x_i), p(x_l)) \ge 3$ where $i, l \in \{1, 2, \ldots, n - 1\}$ and at least one vertex $p(x_j) \ge 5$ where $j \in \{1, 2, \ldots, n - 1\}$, then we can shift a pebble to w. If this configuration does not exist then there exists one j such that $p(x_k) \ge 2$ for every $k \ne j$ and $p(x_j) = 4$. This particular distribution will have $(q - 2)2 + p(x_j) \le (n - 1 - 2)2 + p(x_j) = 2n - 6 + p(x_j) = 2(n - 3) + p(x_j) < 2n + 1$ pebbles which a contradiction to the initial configuration of pebbles. As a result, we can shift two pebbles to w using $p(x_j)$. Placing the remaining pebbles on $n - 2$ vertices, we will have $Q - p(x_j) - (n - 2) = Q - n + 2 - p(x_j) > 2f_{t-1}(F_n^{-f}) - q + 1 \Longrightarrow Q - 2 - q > 2f_{t-1}(F_n^{-f}) - q + 1$ pebbles. Using induction on t, $2(t - 1)$ more pebbles can be transferred to x_0.

Subcase 1.2. $n > 8t - 5$.

Assume that there exist $p(x_j) \geq 5$ where $j \in \{1, 2, \ldots, n-1\}$ or for some k, l with $(p(x_k), p(x_l)) \geq 3$ where $k, l \in \{1, 2, , n-1\}$ then we can shift $2t$ pebbles to w as in subcase 1.1. If we cannot shift 2t pebbles on w, then we assume that there exists $p(x_i) = 4$ where $i \in \{1, 2, \ldots, n-1\}$ Therefore, $p(x_m) \leq 2$ for all $m \neq i$.

For this configuration, we will have exactly two pebbles each on $2t-1$ vertices. Let $q \leq n-1$. If the configuration has maximum $2t-2$ vertices having exactly two pebbles each, then there will be at most $(2t-2)2 + p(x_i)$ $+(q-(2t-1)) = 2t + p(x_i) + (n-1) - 3 = 2t + n + p(x_i) - 4$ pebbles. This is a contradiction. As a result, we will have one vertex with four pebbles and at least $2t-1$ vertices having exactly two pebbles each. Thus, $2t$ pebbles can be shifted to x_0.

Case 2. Let $w = x_l$, where $l \in \{1, 2, \ldots, n-1\}$. Without loss of generality, assume that $(p(x_0), p(x_1)) = 0$. As $q = n-2$, we begin with $Q \geq Q-(n-2) = 8t+n-3$ pebbles.

Subcase 2.1. Let $n \leq 8t-6$. Let us initiate the pebble distribution with minimum $2n+3$ pebbles. If there is one $p(x_k) \geq 9$, where $k \in \{1, 2, \ldots, n-1\}$ or there exist $(p(x_j), p(x_i)) \geq 5$ where $j \neq k, i \neq k$ or $(p(x_i), p(x_j), p(x_l), p(x_m)) \geq 3$ where $i, j, l, m \in \{1, 2, \ldots, n-1\}$ such that $i \neq k, j \neq k, l \neq k, m \neq k$, then we can reach w. Otherwise, we may assume that there is one vertex with 8 pebbles. As a result, the remaining vertices will only have two pebbles each. Hence, there will be at most $(q-1)2+ = 8 = 2q-2+8 = 2q+6$ pebbles. Since, $q = n-2$, there will be $2(n-2)+6 = 2n+2$ pebbles. This is a contradiction. So, by moving 4 pebbles to x_0 we could shift 2 pebbles to w. In this distribution we use only eight pebbles and the remaining pebbles are not used. After this move, we will have $Q-q-8 > 2f_{t-1}(F_n^{-f}) - q + 1$ pebbles which are not been transferred.

By induction, using the remaining pebbles we could shift $2(t-1)$ more pebbles on w.

Subcase 2.2. $n > 8t - 6$.

If there is one $p(x_k) \geq 9$, where $k \in \{1, 2, \ldots, n-1\}$, then we are done. If there are j and i with $p(x_j) \geq 5$ and $p(x_i) \geq 5$, then we reach the destination. If there are $(p(x_i), p(x_j)) \geq 3$, and $p(x_l) \geq 5$, where $i, j, l \in \{1, 2, \ldots, n-1\}$, then we can reach the destination. If there are $(p(x_i), p(x_j), p(x_l)) \geq 3$ where $i, j, l, m \in \{1, 2, \ldots, n-1\}$, and $p(x_m) \geq 3$, then we reach the destination. Suppose not, let there be a vertex with 8 pebbles and $q \leq n-2$. Then, we can put at most two pebbles each on $4t-1$ vertices which are the remaining vertices after the distribution. Suppose, there exist exactly two pebbles each on at most $4t-2$ vertices a then there will be at most $\leq 4t + n + 3$ pebbles which is a contradiction to the initial configuration. Therefore, we have at least 2 pebbles each on $4t-1$ vertices and one vertex with 8 pebbles. Thus, we can shift $2t$ pebbles to x_l.

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