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STRUCTURE OF CATEGORIES OF DIOPHANTINE 3-TUPLES APROPOS $(2j, 3)$-POLYGONAL NUMBERS

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Abstract

In this paper, multiplicity of Diophantine triples regarding $(2j, 3)$-polygonal numbers consequent from polygonal numbers baptized as decagonal, dodecagonal, tetra decagonal and octa decagonal numbers in which the product of two quantities added with distinct polynomials results a square are displayed.

1. Introduction

"A set of k positive integers $\{a_1, a_2, \ldots, a_k\}$ is termed as Diophantine k-tuple with the characteristics $\mathcal{D}(j)$, $j - \{0\} \in Z$ if $a_m \cdot a_n + j$ remains a square for all $1 \leq m < n \leq k$. Experts have assessed the formation of varies applications of Diophantine triples with $\mathcal{D}(j)$ for any arbitrary integer j [1]. For a wide examination of various article on Diophantine triples, one may refer [2-7]. In this paper, miscellany of Diophantine triples concerning $(2j, 3)$-polygonal numbers innovatively evaluated from polygonal numbers sprinkled as decagonal, dodecagonal, tetra decagonal and octadecagonal numbers with the additional condition that the product of two elements added with separate polynomials consequences a square is exhibited.

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2. Erection of Diophantine 3-Tuples

2.1. Diophantine 3-tuple with respect to $(2j, 3)$-Decagonal numbers.

The common formula for the decagonal number is well-defined by

$$d_p = 4\rho^2 - 3\rho,\ \rho \in N$$

The sequence of $(2j, 3)$-Decagonal numbers derived from the decagonal sequence is indicated by $(20j + 3), (54j + 30), (104j + 81),$ etc

The relation among the elements of $(2j, 3)$-decagonal sequence is offered by

$$\mathcal{D}_\rho = 2jd_{\rho+1} + 3d_\rho,\ \rho \geq 1 \text{ with } d_1 = 1 \text{ and } d_2 = 10$$

Let us begin with 2-tuples (τ_{11}, σ_{11}) containing $(2j, 3)$-decagonal numbers such that τ_{11}, σ_{11} added with a polynomial is a square.

For illustration, let us choose

$$\tau_{11} = \mathcal{D}_\rho = (8j + 12)\rho^2 + (10j - 9)\rho + 2j$$

$$\sigma_{11} = \mathcal{D}_{\rho+1} = (8j + 12)\rho^2 + (26j + 15)\rho + (20j + 3)$$

Note that $\tau_{11}, \sigma_{11} + J_1 = \{(8j + 12)\rho^2 + (18j + 3)\rho + (20j + 1)\}^2$

Therefore, the pair (τ_{11}, σ_{11}) be a symbol of Diophantine 2-tuples with the distinguishing property $\mathcal{D}(J_1)$ where

$$J_1 = (208j^2 + 400j + 132)\rho^2 + (468j^2 + 276j + 33)\rho + (360j^2 + 34j + 1)$$

To extend this pair into triples, suppose ξ_{11} is the third element in the above pair. Then it pleases the subsequent system of equations

$$\tau_{11}\xi_{11} + J_1 = a^2 \tag{1}$$

$$\sigma_{11}\xi_{11} + J_1 = b^2 \tag{2}$$

From (1) and (2), it is clear that

$$a^2\sigma_{11} - b^2\tau_{11} = (\sigma_{11} - \tau_{11})J_1 \tag{3}$$

Initiate the new transformations

$$a = \mathcal{J} + \tau_{11}\mathcal{K} \text{ and } b = \mathcal{J} + \sigma_{11}\mathcal{K} \tag{4}$$

Substitution of the above alterations direct (3) into the succeeding equation

$$\mathcal{J}^2 = \tau_{11}\sigma_{11}\mathcal{K}^2 + J_1 \tag{5}$$

Equation (5) is fulfilled when $\mathcal{K} = 1$ and hence

$$\mathcal{J} = (8j + 12)\rho^2 + (18j + 3)\rho + (20j + 1).$$

Exchange the value of a obtained from (4) by replacing the values of $\mathcal{J}$ and $\mathcal{K}$ in (1), the equivalent value of ξ_{11} is labeled as

$$\xi_{11} = (32j + 48)\rho^2 + (72j + 12)\rho + (62j + 5).$$

Therefore, $(\tau_{11}, \sigma_{11}, \xi_{11})$ is signified as Diophantine 3-tuples with the property $\mathcal{D}(J_1)$

Proceed with (τ_{11}, ξ_{11}) and concerning the above course of action, there exists a sequence of triples $(\tau_{11}, \xi_{11}, \xi_{12}), (\tau_{11}, \xi_{12}, \xi_{13})$ etc satisfying the same property $\mathcal{D}(J_1)$ where $\xi_{12} = (72j + 108)\rho^2 + (138j - 9)\rho + (108j + 7)$ $\xi_{13} = (128j + 192)\rho^2 + (224j + 158)\rho + (-48j + 9)$

Samples for few triples in the needed sequence for selected choices of ρ and j are listed in table 2.1.1.

Table 2.1.1.

ρ	j	$\mathcal{D}(J_1)$	$(\tau_{11}, \sigma_{11}, \xi_{11})$	$(\tau_{11}, \xi_{11}, \xi_{12})$	$(\tau_{11}, \xi_{12}, \xi_{13})$
1	1	1912	(23,84,231)	(23,231,424)	(23,424,663)
	2	5730	(43,138,397)	(43,397,742)	(43,742,1173)

2	1	4909	(84,185,555)	(84,555,1093)	(84,1093,1799)
	2	13479	(138,289,889)	(138,889,1765)	(138,1765,2917)

Consider the sequence of 2-tuples $(\tau_{11}, \xi_{11}), (\tau_{11}, \xi_{12}), (\tau_{11}, \xi_{13})$ etc and manipulate the same performance as explained above, the patterns of 3-tuples $(\tau_{11}, \sigma_{11}, \xi_{12}), (\tau_{11}, \sigma_{12}, \xi_{13}), (\tau_{11}, \sigma_{13}, \xi_{14})$ etc are discovered with the characteristics $\mathcal{D}(J_1)$ where

$$\zeta_{12} = (72j + 108)\rho^2 + (186j + 6)\rho + (162j + 16)$$

$$\zeta_{13} = (128j + 192)\rho^2 + (352j + 144)\rho + (302j + 33)$$

$$\zeta_{14} = (200j + 300)\rho^2(570j + 255)\rho + (482j + 56)$$

Few statistical illustrations are clearly stated in the succeeding table 2.1.2 for appropriate choices of ρ and *j*.

Table 2.1.2.

ρ	j	$\mathcal{D}(J_2)$	$(\tau_{11}, \zeta_{11}, \zeta_{12})$	$(\tau_{11}, \zeta_{12}, \zeta_{13})$	$(\tau_{11}, \zeta_{13}, \zeta_{14})$
1	1	1912	(84,231,607)	(84,607,1151)	(84,1151,1863)
	2	5730	(138,397,1027)	(138,1027,1933)	(138,1933,3115)
2	1	4909	(185,555,1396)	(185,1396,2607)	(185,2607,4188)
	2	13479	(289,889,2218)	(289,2218,4125)	(289,4125,6610)

2.2. Diophantine 3-tuples concerning $(2j, 3)$**–Dodecagonal numbers.**

The general form of ρ^{th} dodecagonal number is dubbed as $dd_\rho = 5\rho^2 - 4\rho, \rho \in N$ and the new sequence named as $(2j, 3)$-dodecagonal number gained from this sequence is uttered as $(24j + 3), (66j + 36), (128j + 99)$ etc.

The ρ^{th} term in this sequence is provided by

$$\mathcal{DD}_\rho = 2jdd_\rho + 3dd_{\rho-1}, \rho \geq 2 \text{ with } dd_1 = 1 \text{ and } dd_2 = 12$$

Permit $\tau_{21} = \mathcal{D}\mathcal{D}_\rho = (10j + 15)\rho^2 + (12j - 12)\rho + 2j$

$\tau_{21} = \mathcal{D}\mathcal{D}_\rho = (10j + 15)\rho^2 + (32j - 18)\rho + (24j + 3)$

Together with the combination of these two options satisfying the condition that $\tau_{21}\sigma_{21} + J^2 = ((10j + 15)\rho^2 + (22j + 3)\rho + (8j - 8))^2$

Applying the identical technique referred in the previous section, it is perceived that $(\tau_{21}, \sigma_{21}, \xi_{21}), (\tau_{21}, \xi_{21}, \xi_{22}), (\tau_{21}, \xi_{22}, \xi_{23})$ etc is a sequence of 3-tuples with the characteristics $\mathcal{D}(J_2)$ where

$$\xi_{21} = (40j + 60)\rho^2 + (88j + 12)\rho + (42j - 13)$$

$$\xi_{22} = (90j + 135)\rho^2 + (168j - 18)\rho + (64j - 29)$$

$$\xi_{23} = (160j + 240)\rho^2 + (272j - 72)\rho + (90j - 45)$$

and $J^2 = (-40j - 60)\rho^2 + (88j + 12)\rho + (16j^2 - 134j + 64)$

Numerical verification for certain values of ρ and j are offered in table 2.2.1.

Table 2.2.1.

ρ	j	$\mathcal{D}(J_2)$	$(\tau_{21}, \sigma_{21}, \xi_{21})$	$(\tau_{21}, \xi_{21}, \xi_{22})$	$(\tau_{21}, \xi_{22}, \xi_{23})$
1	1	−254	(27,102,229)	(27,229,410)	(27,410,645)
	2	−468	(51,168,399)	(51,399,732)	(51,732,1167)
2	1	−654	(102,227,629)	(102,629,1235)	(102,1235,2045)
	2	−1076	(168,355,1007)	(168,1007,1995)	(168,1995,3319)

Taking (τ_{21}, ξ_{21}) and working with the analogous idea specified above, there exist varieties of 3-tuples $(\sigma_{21}, \xi_{21}, \zeta_{22}), (\sigma_{21}, \zeta_{22}, \zeta_{23})$, $(\sigma_{21}, \zeta_{23}, \zeta_{24})$ etc with property $\mathcal{D}(J_2)$ where

$$\zeta_{22} = (90j + 135)\rho^2 + (168j - 18)\rho + (64j - 29)$$

$$\zeta_{23} = (160j + 240)\rho^2 + (432j + 168)\rho + (266j - 21)$$

$$\zeta_{24} = (250j + 375)\rho^2 + (700j + 300)\rho + (450j - 16)$$

The table 2.2.2 facilitate to establish the 3-tuples.

Table 2.2.2.

ρ	j	$\mathcal{D}(J_2)$	$(\sigma_{21}, \xi_{21}, \zeta_{22})$	$(\sigma_{21}, \xi_{22}, \zeta_{23})$	$(\sigma_{21}, \xi_{23}, \zeta_{24})$
1	1	−254	(102,229,635)	(102,635,1245)	(102,1245,2059)
	2	−468	(168,399,1083)	(168,1083,2103)	(168,2103,3459)
2	1	−654	(227,629,1610)	(227,1610,3045)	(227,3045,4934)
	2	−1076	(355,1007,2556)	(355,2556,4815)	(355,4815,7784)

2.3. Construction of a Diophantine 3-tuples using $(2j, 3)$ −Tetradecagonal numbers.

Note that $td_\rho = 6\rho^2 - 5\rho, \rho \in N$ is the ρ^{th} tetra decagonal number.

The $(2j, 3)$-tetra decagonal sequence received from tetra decagonal number is demarcated by $(28j + 3), (78j + 42), (152j + 117)$ etc. The ρ^{th} term in the prescribed sequence is stated by $\mathcal{TD}_\rho = 2jtd_\rho + 3td_{\rho-1}, \rho \geq 2$ with $td_1 = 1$ and $td_2 = 14$

Allow $\tau_{31} = \mathcal{TD}_\rho = (12j + 18)\rho^2 + (14j - 15)\rho + 2j$

$\sigma_{31} = \mathcal{TD}_\rho = (12j + 18)\rho^2 + (38j + 21)\rho + (28j + 3)$

with the condition that $(\tau_{31}\sigma_{31}) = \mathcal{J}^3 = \{(12j + 18)\rho^2 + (26j + 3)\rho + (7j - 9)\}^2$

Then, the triples $(\tau_{31}, \sigma_{31}, \xi_{31}), (\tau_{31}, \sigma_{31}, \xi_{32}), (\tau_{31}, \sigma_{31}, \xi_{33})$ etc are derived by the similar direction précised above with the property $\mathcal{D}(J_3)$ where

$$\xi_{31} = (48j + 72)\rho^2 + (104j + 12)\rho + (44j - 15)$$

$$\xi_{32} = (108j + 162)\rho^2 + (198j - 27)\rho + (64j - 33)$$

$$\xi_{33} = (192j + 288)\rho^2 + (320j - 96)\rho + (88j - 51)$$

$$J^3 = (48j^2 + 108j + 54)\rho^2 - (104j^2 + 90j + 9)\rho - (7j^2 + 132j - 81)$$

The resultant 3-tuples charted shown in the table 2.3.1.

Table 2.3.1.

ρ	j	$\mathcal{D}(J_3)$	$(\tau_{31}, \sigma_{31}, \xi_{31})$	$(\tau_{31}, \sigma_{31}, \xi_{32})$	$(\tau_{31}, \sigma_{32}, \xi_{33})$
1	1	−471	(31,120,265)	(31,265,472)	(31,472,741)
	2	−1278	(59,198,461)	(59,461,842)	(59,842,1341)
2	1	−1304	(120,269,741)	(120,741,1453)	(120,1453,2405)
	2	−3269	(198,421,1185)	(198,1185,2345)	(198,2345,3901)

Initiative the identical process to each pair (σ_{31}, ξ_{31}), $(\sigma_{31}, \zeta_{32})$, $(\sigma_{31}, \zeta_{33})$ etc leads to the 3-tuples $(\sigma_{31}, \xi_{31}, \zeta_{32})$, $(\sigma_{31}, \zeta_{32}, \zeta_{33})$, $(\sigma_{31}, \zeta_{33}, \zeta_{34})$ etc with an equivalent property $\mathcal{D}(J_3)$ where

$$\zeta_{32} = (108j + 162)\rho^2 + (270j + 81)\rho + (142j - 24)$$

$$\zeta_{33} = (192j + 288)\rho^2 + (512j + 192)\rho + (296j - 27)$$

$$\zeta_{34} = (300j + 450)\rho^2(830j + 345)\rho + (506j - 24)$$

Numerical computations for few values of ρ and j are classified in table 2.3.2.

Table 2.3.2.

ρ	j	$\mathcal{D}(J_3)$	$(\sigma_{31}, \xi_{31}, \zeta_{32})$	$(\sigma_{31}, \zeta_{32}, \zeta_{33})$	$(\sigma_{31}, \zeta_{33}, \zeta_{34})$
1	1	−471	(120,265,739)	(120,739,1453)	(120,1453,2407)
	2	−1278	(198,461,1259)	(198,1259,2453)	(198,2453,4043)
2	1	−1304	(269,741,1900)	(269,1900,3597)	(269,3597,5832)
	2	−3269	(421,1185,3014)	(421,3014,5685)	(421,5685,9198)

2.4. Diophantine 3-tuples pertaining to $(2j, 3)$-Octadecagonal numbers.

The octadecagonal number is of the form $\mathcal{O}d_{\rho} = 8\rho^2 - 7\rho, \rho \in N$ and $(36j+3), (102j+54), (200j+153)$ etc epitomize the $(2j, 3)$-octadecagonal sequence evaluated by the relation

$$\mathcal{OD}_{\rho} = 2j\mathcal{O}d_{\rho} + 3j\mathcal{O}d_{\rho-1}, \rho \geq 2 \text{ with } \mathcal{O}d_1 = 1 \text{ and } \mathcal{O}d_2 = 18.$$

Select $\tau_{41} = OD_{\rho} = (16j+24)\rho^2 + (18j-21)\rho + 2j$

$$\sigma_{41} = \mathcal{OD}_{\rho+1} = (16j+24)\rho^2 + (50j+27)\rho + (36j+3)$$

with the statement that the multiplication of τ_{41} and σ_{41} enlarged by J^4 grades a square number.

Following the method specified in the previous section provides the non-zero integer sequence of 3-uples $(\tau_{41}, \sigma_{41}, \xi_{41}), (\tau_{41}, \xi_{41}, \xi_{42}), (\tau_{41}, \xi_{42}, \xi_{43})$ etc conforming the feature $\mathcal{D}(J_4)$ where

$$\xi_{41} = (64j+96)\rho^2(136j+12)\rho + (58j-27)$$

$$\xi_{42} = (144j+216)\rho^2(258j-45)\rho + (84j-57)$$

$$\xi_{43} = (256j+384)\rho^2(416j-144)\rho + (114j-87)$$

$$J^4 = (32j^2 + 192j + 216)\rho^2 - (68j^2 + 312j + 27)\rho + (28j^2 - 306j + 225)$$

The subsequent table 2.4.1 shows numerical values for some ρ and j.

Table 2.4.1.

ρ	j	$\mathcal{D}(J_4)$	$(\sigma_{41}, \sigma_{41}, \xi_{41})$	$(\sigma_{41}, \xi_{41}, \xi_{42})$	$(\sigma_{41}, \sigma_{42}, \xi_{43})$
1	1	−900	(39,156,339)	(39,339,600)	(39,600,939)
	2	−1926	(75,258,597)	(75,597,1086)	(75,1086,1725)
2	1	−2627	(156,353,967)	(156,967,1893)	(156,1893,3131)
	2	−5033	(258,553,1553)	(258,1553,3069)	(258,3069,5101)

Usage of similar approach promising 3-tuples $(\sigma_{41}, \sigma_{41}, \xi_{42})$, $(\sigma_{41}, \sigma_{42}, \xi_{43})$, $(\sigma_{41}, \sigma_{43}, \xi_{44})$ etc extending from 2-tuple (σ_{41}, ξ_{41}), (σ_{41}, ξ_{42}), (σ_{41}, ξ_{43}), etc with property $\mathcal{D}(J_4)$ where

$$\zeta_{42} = (144j + 216)\rho^2(354j + 99)\rho + (186j - 48)$$

$$\zeta_{43} = (256j + 384)\rho^2(672j + 240)\rho + (386j - 63)$$

$$\zeta_{44} = (400j + 600)\rho^2(1090j + 435)\rho + (658j - 72)$$

Some numerical figures acknowledged in the following table 2.4.2.

Table 2.4.2.

ρ	j	$\mathcal{D}(J_4)$	$(\sigma_{41}, \xi_{41}, \zeta_{42})$	$(\sigma_{41}, \zeta_{42}, \zeta_{43})$	$(\sigma_{41}, \zeta_{43}, \zeta_{44})$
1	1	−900	(156,339,951)	(156,951,1875)	(156,1875,3111)
	2	−1926	(258,597,1635)	(258,1635,3189)	(258,3189,5259)
2	1	−2627	(353,967,2484)	(353,2484,4707)	(353,4707,7636)
	2	−5033	(553,1553,3954)	(553,3954,7461)	(553,7461,12074)

3. Python Program

Python Program for arithmetical authentications for all additional adoptions of the parameters significant our proposals are demonstrated below: import math

```
while True:
c = input("Enter choice(1/2/3/4):")
if c in ('1','2','3','4'):
p=int(input('Enter p value: '))
T=1
Def triple(a,b,D):
z= math.sqrt(a*b+D)
r=z+a*T
```

```
c1=(r∗∗2−D)/a
z= math.sqrt(a∗c1+D)
r=z+a∗T
c2=(r∗∗2−D)/a
z= math.sqrt(a∗c2+D)
r=z+a∗T
c3=(r∗∗2−D)/a
print(′(a, b, c1)′, a, b, c1)
print(′(a, c1, c2)′, a, c1, c2)
print(′(a, c2, c3)′, a, c2, c3)
z= math.sqrt(a∗b+D)
r=z+b∗T
c1=(r∗∗2−D)/a
z= math.sqrt(b∗c1+D)
r=z+b∗T
c2=(r∗∗2−D)/a
z= math.sqrt(b∗c2+D)
r=z+b∗T
c3=(r∗∗2−D)/a
z= math.sqrt(b∗c3+D)
r=z+b∗T
c4=(r∗∗2−D)/b
print(′(b, c1, c2)′, b, c1, c2)
print(′(b, c2, c3)′, b, c2, c3)
print(′(b, c3, c4)′, b, c3, c4) if (c==’1’):
for q in range(1,3):
```

```
print("q":, q)
a = (8*q+12)*(p**2)+(10*q−9)*p+(2*q)
b = (8*q+12)*(p**2)+(26*q+15)*p+((20*q)+3)
D=(((208*q**2)+(400*q)+132)*(p**2)+(468*q**2+276*q+33)*p+(360*q**2+
34*q+1))
triple(a,b,D)
elif (c=='2'):
for q in range(1,3):
print("q":, q)
a = (10*q+15)*(p**2)+(12*q−12)*p+(2*q)
b = (10*q+15)*(p**2)+(32*q+18)*p+((24*q)+3)
D = (((−40*q)−60)*(p**2))−((88*q +12)*p)+(16*q **2−134*q +64)
triple(a,b,D)
elif(c=='3'):
for q in range(1,3):
print("q":, q)
a = (12*q+18)*(p**2)+(14*q−15)*p+(2*q)
b = (12*q+18)*(p**2)+(38*q+21)*p+((28*q)+3)
D=((−(48*q**2+108*q+54)*(p**2))−(104*q**2+90*q+9)*p)−(7*q**2+132*q−
81)
triple(a,b,D)
elif(c=='3'):
for q in range(1,3):
print("q":, q)
a = (16*q+24)*(p**2)+(18*q−21)*p+(2*q)
b = (16*q+24)*(p**2)+(50*q+27)*p+((36*q)+3)
```

```
D=(−(32*q**2+192*q+216)*(p**2))−(68*q**2+312*q+27)*p)+(28*q**2−306
*q+225)

triple(a,b,D) else:

print('Invalid Input')

break
```

4. Conclusion

In this paper, a sequence of Diophantine triples consisting $(2j, 3)$-polygonal numbers obtained from polygonal numbers are calculated with right properties. To conclude, one may look for Diophantine triples, quadruples and so on for further selections of number patterns with an appropriate property.

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