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EXACT SOLUTIONS OF WAVE AND HEAT EQUATIONS VIA ANDUALEM AND KHAN TRANSFORM

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Abstract

Recently, Andualem and Khan introduced a new integral transform method known as AK transform (AKT) to solve partial differential equations. Therefore, in this article, we use AKT to solve some partial differential equations. More exactly, waves and heat equations are solved for new exact solutions and after transformation, the results are obtained and plotted. The two counterexamples considered in this work, are quite important in terms of engineering and sciences applications.

I. Introduction

An equation that consists of derivatives of unknown function is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. Partial differential equations are mathematical formulations of problems involving two or more independent variables. Most of the problems that arise in the real world are modeled by partial differential equations. Partial differential equations arise in all fields of sciences including Physics, Chemistry and Mathematics [1-4].

In order to solve the differential equations, the integral transform was extensively used and thus there are several works on the theory and

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application of integral transform such as the Laplace, Mellin, Sumudu Transform, Elzaki Transform and [5, 6, 7, 8, 9, 11, 12, 13].

New integral transform, named as AK Transformation [10] introduce by Mulugeta Andualem and Ilyas Khan [2022], AK transform was successfully applied to fractional differential equations. In this study, we will discuss on the exact solution of some well-known partial differential equations with constant coefficient used in the fields of Engineering and Sciences with the help of AK transform. The main advantage of the method is that, it can be applied directly to various types of differential and integral equations, which are linear and nonlinear, homogeneous and non-homogeneous, with constant and with variable coefficients. The organization of this paper is as follows:

Section II describes the AK transform and fundamental property of AK transform in order to solve PDEs. In section III, some analytical examples are presented to illustrate the efficiency of the AK transform and obtained the exact solution in graphically. Finally, in section IV, we give the conclusion.

II. AK Transform

Andualem and Khan transform of the function $y(t)$ belonging to a class A, where:

$$A = \{y(t) : \exists N, \eta_1, \eta_2 > 0, |y(t)| < Ne^{\frac{|t|}{\eta i}}, if\ t \in (-1)^i \times [0\ \infty)\}$$

Andualem and Khan transform of $y(t)$ denoted by $M_i[y(t)] = \bar{y}(s, \beta)$ and is given by:

$$\bar{y}(s, \beta) = M_i\{y(t)\} = s\int_0^{\infty} y(t)e^{-\frac{s}{\beta}t}dt$$

M_i^{-1} is called inverse AK transform of and is defined as:

$$M_i^{-1}\{\bar{y}(s, \beta)\} = y(t) = \int_{\alpha-\infty}^{\alpha+\infty} \frac{s}{\beta} e^{-\frac{s}{\beta}t} \bar{y}(s, \beta)ds$$

where α is real constant.

Table 1. AK Transform of Some Basic functions.

$f(t)$	$M_i(f(t))$
C (constant)	$C\beta$
t^n	$\Gamma(n+1)\frac{\beta^{n+1}}{s^n}$
$e^{\lambda t}$	$\frac{s\beta}{s-\lambda\beta}$
$\cos t$	$\frac{s^2\beta}{\beta^2 - s^2}$
$\sin(t)$	$\frac{s\beta^2}{\beta^2 - s^2}$
$\sinh t$	$\frac{s\beta^2}{s^2 - \beta^2}$
$\cosh t$	$\frac{s^2\beta}{s^2 - \beta^2}$

The sufficient condition for the existence of AK transform.

If the function $y(t)$ is piecewise continues in every finite interval $0 \leq t \leq \alpha$ and of exponential order for $t > \beta$. Then its AK transform $\bar{y}(s, \beta)$ exists.

Proof. For any positive α, we have

$$s\int_0^\infty y(t)e^{-\frac{st}{\beta}}dt = s\int_0^\alpha y(t)e^{-\frac{st}{\beta}}dt + s\int_\alpha^\infty y(t)e^{-\frac{st}{\beta}}dt$$

Since the function $y(t)$ is piecewise continues in every finite interval $0 \leq t \leq \alpha$, then the first integral on the right hand side exists. Besides, the second integral on the right hand side exists, since the function $y(t)$ is of exponential order β for $t > \beta$.

Now to this, we have following:

$$| s\int_0^\infty y(t)e^{\frac{st}{\beta}}dt | \leq s\int_0^\infty | y(t)e^{-\frac{st}{\beta}}dt |dt$$

$$\leq s\int_0^\infty e^{-\frac{st}{\beta}}dt| y(t) |dt \leq s\int_0^\infty e^{-\frac{st}{\beta}}Me^{\alpha t}dt$$

$$= Ms\int_0^\infty e^{-\frac{(s-\alpha\beta)t}{\beta}}dt$$

$$= -\frac{Ms\beta}{s-\alpha\beta}\left(\lim_{\tau\to\infty} e^{-\frac{(1-s\alpha)t}{s}} |_0^\tau\right)$$

$$= \frac{Ms\beta}{s-\alpha\beta}$$

and this improper integral is convergent for all, $s > \alpha\beta$. Thus $M_i\{y(t)\} = \bar{y}(s, \beta)$.

To obtain AK transform of partial derivatives we use integration by parts as follows:

$$M_i\left[\frac{\partial f(x, t)}{\partial t}\right] = s\int_0^\infty \frac{\partial f(x, t)}{\partial t} e^{\frac{-s}{\beta}t}dt = s \lim_{\eta\to\infty}\int_0^\eta e^{\frac{-s}{\beta}t} \frac{\partial f(x, t)}{\partial t} dt$$

$$= \lim_{\eta\to\infty}\left(s[e^{\frac{-s}{\beta}t} f(x, t]_0^\eta + \frac{s^2}{\beta}\int_0^\infty e^{\frac{-s}{\beta}t} f(x, t)dt\right)$$

$$= -sf(x, 0) + \frac{s}{\beta}\bar{f}(x, s, \beta)$$

$$= \frac{s}{\beta}\bar{f}(x, s, \beta) - sf(x, 0)$$

Next, to find $M_i\left[\frac{\partial^2 f(x, t)}{\partial t^2}\right]$, let $\frac{\partial f(x, t)}{\partial} = g(x, t)$ then by using equation (1.2) we have

$$M_i\left[\frac{\partial^2 f(x, t)}{\partial t^2}\right] = M_i\left[\frac{\partial g(x, t)}{\partial t}\right] = \frac{s}{\beta}\bar{g}(x, s, \beta) - sg(x, 0) \ (*)$$

But $\bar{g}(x, s, \beta) = M_i(g(x\ t)) = M_i\left(\frac{\partial f(x, t)}{\partial t}\right) = \frac{s}{\beta}\bar{f}(x, s, \beta) - sf(x, 0)$ (**)

Substituting (**) from (*) we obtain

$$M_i\left[\frac{\partial^2 f(x, t)}{\partial t^2}\right] = \frac{s}{\beta}\left(\frac{s}{\beta}\bar{f}(x, s, \beta) - sf(x, 0)\right) - s\frac{\partial f(x, 0)}{\partial t}$$

$$= \frac{s^2}{\beta^2}\bar{f}(x, s, \beta) - s\frac{\partial f(x, 0)}{\partial t} - \frac{s^2}{\beta}f(x, 0).$$

III. Application of AK Transform for Partial Differential Equations

In this section, some applications are given in order to show the efficiency of AK transform.

AK transform is applicable to solve many real life problems. Especially it is useful to solve initial-value problems and fractional differential equations. Here we can see some applications of AK transform for the solution of some well-known partial differential equations with constant coefficient used in the fields of Engineering and Sciences.

Example 1. Find the solution of the first-order initial value problem

$$\frac{\partial f(x, t)}{\partial x} - 2\frac{\partial f(x, t)}{\partial t} = f(x, t), \ x > 0, \ t > 0 \ f(x, 0) = e^{-3x}.$$

Solution. Applying the AK transform on both sides of Equation of (1.4) with respect to t, leads to

$$M_i\left[\frac{\partial f(x, t)}{\partial x} - 2\frac{\partial f(x, t)}{\partial t} = f(x, t)\right]$$

Using the differentiation property of AK transform we get

$$\bar{f}'(x, s, \beta) - 2\left(\frac{s}{\beta}\bar{f}(x, s, \beta) - sf(x, 0)\right) = \bar{f}(x, s, \beta)$$

where $\bar{f}(x, s, \beta)$ is the AK transform of $f(x, t)$.

Substituting the given initial condition, we get

$$\bar{f}'(x, s, \beta) - 2\left(\frac{s}{\beta}\bar{f}(x, s, \beta) - se^{-3x}\right) = \bar{f}(x, s, \beta)$$

$$\Rightarrow \bar{f}'(x, s, \beta) - \left(2\frac{s}{\beta} + 1\right)\bar{f}(x, s, \beta) = -2se^{-3x}$$

If you consider equation (1.5) has the form $y' + g(x)y = r(x)$ and solved the method of integrating factor.

$$\mu(x) = e^{\int g(x)dx} = e^{-\int\left(2\frac{s}{\beta}+1\right)dx} = e^{-\left(2\frac{s}{\beta}+1\right)x}$$

where $\mu(x)$ is integrating factor.

Now, multiplying equation (1.5) by integrating factor we obtain

$$e^{-\left(2\frac{s}{\beta}+1\right)x}\bar{f}'(x, s, \beta) - \left(2\frac{s}{\beta} + 1\right)e^{-\left(2\frac{s}{\beta}+1\right)dx}\bar{f}(x, s, \beta) = -2se^{-3x}e^{-\left(2\frac{s}{\beta}+1\right)x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-\left(2\frac{s}{\beta}+1\right)x}\bar{f}(x, s, \beta)\right) = -e^{-\left(2\frac{s}{\beta}+1\right)x} \cdot 2se^{3x}$$

$$\Rightarrow \int\left(e^{-\left(2\frac{s}{\beta}+1\right)x}\bar{f}(x, s, \beta)\right)' = \int\left(-e^{-\left(2\frac{s}{\beta}+1\right)x} \cdot 2se^{-3x}\right)dx$$

After simple calculation we get

$$\bar{f}(x, s, \beta) = \frac{s\beta}{s + 2\beta}e^{-3x}$$

Now applying the inverse AK transform of the above result

$$f(x, t) = M_i^{-1}\left(\frac{s\beta}{s + 2\beta}e^{-3x}\right) = e^{-3x}M_i^{-1}\left(\frac{s\beta}{s + 2\beta}\right)$$

$$f(x, t) = e^{-3x} \cdot e^{-2t} = e^{-3x-2t}$$

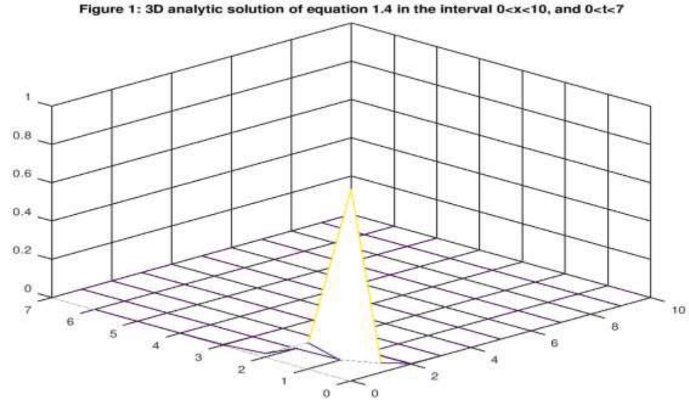
One may readily check that this is indeed the solution to the initial value problem.

$$\Rightarrow f(x, 0) = e^{-3x-0} = e^{-3x}$$

Therefore, the solution of equation (1.4) is

$$f(x, t) = e^{-3x-2t}, \; x > 0, \; t > 0$$

Figure 1: 3D analytic solution of equation 1.4 in the interval 0<x<10, and 0<t<7



Example 2. Consider initial/boundary value problem of heat equation

$$u_t(x, t) = u_{xx}(x, t), \; 0 < x < 2, \; t > 2 \tag{1.6}$$

$$u(0, t) = 0 = u(2, t)$$

Solution. Applying the AK transform on both sides of Equation of (1.6) with respect to t, we get

$$\frac{s}{\beta}\bar{u}(x, s, \beta) - su(x, 0) = \bar{u}''(x, s, \beta)$$

Using the given initial condition

$$\frac{s}{\beta}\bar{u}(x, s, \beta) - 3s\sin(2\pi x) = \bar{u}''(x, s, \beta)$$

Equation (1.6) can be also rewritten as non-homogeneous, second order linear constant coefficient equation

$$\bar{u}''(x, s, \beta) - \frac{s}{\beta}\bar{u}(x, s, \beta) = -3\frac{s}{\beta}\sin(2\pi x)$$

The general solution of the non-homogeneous equation is given by the sum of the homogeneous solution and the particular solution. Now, first let us find the general solution of the homogeneous part.

$$\overline{u}''(x, s, \beta) - \frac{s}{\beta}\overline{u}(x, s, \beta) = 0$$

Suppose the solution is $\overline{u}_h(x, s, \beta) = e^{rx} \Rightarrow \overline{u}_h''(x, s, \beta) = r^2 e^{rx}$

If we substitute this result to the homogeneous equation, we have

$$r^2 e^{rx} - \frac{s}{\beta} e^{rx} = 0 \Rightarrow e^{rx}\left(r^2 - \frac{s}{\beta}\right) = 0$$

$$\Rightarrow \left(r^2 - \frac{s}{\beta}\right) = 0 \Rightarrow r = \pm\sqrt{\frac{s}{\beta}}$$

Therefore, the general solution of the homogeneous problem is

$$\overline{u}_h(x, s, \beta) = c_1 e^{x\sqrt{\frac{s}{\beta}}} + c_2 e^{-x\sqrt{\frac{s}{\beta}}}$$

with the associated conditions

$$\overline{u}(0, s, \beta) = 0 = \overline{u}(2, s, \beta)$$

In order to find the constants and we use the boundary conditions

$$0 = \overline{u}_h(x, s, \beta) = c_1 + c_2 \Rightarrow c_1 - c_2 \ (*)$$

$$\overline{u}_h(2, s, \beta) = 0 = c_1 e^{2\sqrt{\frac{s}{\beta}}} + c_2 e^{-2\sqrt{\frac{s}{\beta}}} \ (**)$$

From and we do have the following

$$0 = -c_2 e^{2\sqrt{\frac{s}{\beta}}} + c_2 e^{-2\sqrt{\frac{s}{\beta}}}$$

$$\Rightarrow c_2\left[e^{2\sqrt{\frac{s}{\beta}}} - e^{2\sqrt{\frac{s}{\beta}}}\right] = 0$$

$$\Rightarrow e^{-2\sqrt{\frac{s}{\beta}}} - e^{2\sqrt{\frac{s}{\beta}}} = 0 \text{ or } c_2 = 0.$$

Therefore the value of $c_2 = 0$, consequently the value of $c_1 = 0$.

Suppose the particular solution of non-homogeneous problem is given by

$$\bar{u}_p(x, s, \beta) = A\cos(2\pi x) + B\sin(2\pi x)$$

In order to find the constant A, and B, we use the method of undetermined coefficients

$$\Rightarrow \frac{d}{dx}\bar{u}_p(x, s, \beta) = -2\pi A\sin(2\pi x) + 2\pi B\cos(2\pi x)$$

$$\frac{d}{dx^2}\bar{u}_p(x, s, \beta) = -4\pi^2 A\cos(2\pi x) - 4\pi^2 B\sin(2\pi x)$$

Therefore

$$\bar{u}_p''(x, s, \beta) - \frac{s}{\beta}\bar{u}_p(x, s, \beta)$$

$$= -4\pi^2 A\cos(2\pi x) - 4\pi^2 B\sin(2\pi x) - \frac{s}{\beta}[A\cos(2\pi x) + B\sin(2\pi x)]$$

$$= \left(-4\pi^2 - \frac{s}{\beta}\right)[A\cos(2\pi x) + B\sin(2\pi x)] = -3\frac{s}{\beta}\sin(2\pi x)$$

From the above result we conclude that

$$\left(-4\pi^2 - \frac{s}{\beta}\right)A = 0 \Rightarrow A = 0 \text{ and } \left(-4\pi^2 - \frac{s}{\beta}\right)B = -3\frac{s}{\beta}$$

$$\Rightarrow B = 3\frac{s}{\beta}\left(\frac{\beta}{(4\beta\pi^2 + s)}\right) \Rightarrow B = \frac{3s}{s + 4\pi^2\beta}$$

Hence the particular solution is

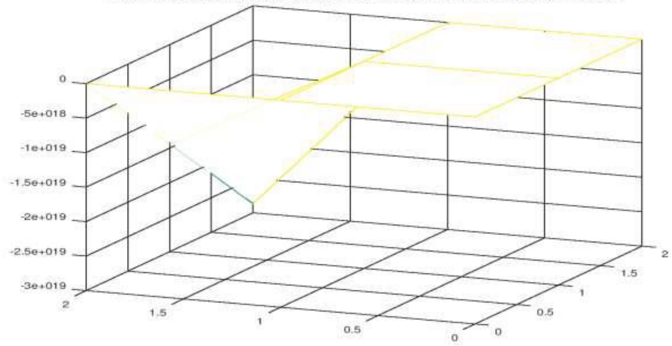
$$\bar{u}_p(x, s, \beta) = \frac{3s}{s + 4\pi^2\beta}\sin(2\pi x)$$

Taking the inverse of AK transform of the above result to obtain the solution

$$M_i^{-1}[\bar{u}_p(x, s, \beta)] = M_i^{-1}\left(\frac{3s}{s + 4\pi^2\beta}\sin(2\pi x)\right)$$

$$u(x\ t) = \sin(2\pi x)3M_i^{-1}\left(\frac{s}{s - (-4\pi^2\beta))}\right) = \sin(2\pi x)\cdot 3e^{-4\pi^2 t}$$

Figure 2: 3D analytical solution of equation 1.6 in the interval 0<x<2, and 0<t<2



Example 3. Find the solution of initial/boundary value problem

$$u_{tt}(x,\ t) = u_{xx}(x,\ t),\ 0 < x < \pi,\ t > 0 \tag{1.9}$$

with boundary conditions

$$u(x,\ 0) = \sin x,\ u_t(x,\ 0) = 0$$

and initial conditions

$$u(x,\ 0) = \sin x,\ u_t(x,\ 0) = 0$$

Solution. First applying the AK transform on both sides of Equation of (1.9), we get

$$\frac{s^2}{\beta^2}\,\overline{u}(x,\ s,\ \beta) - s\,\frac{\partial u(x,\ 0)}{\partial t} - \frac{s^2}{\beta}\,u(x,\ 0) = \overline{u}''(x,\ s,\ \beta)$$

Using initial conditions, we get

$$\frac{s^2}{\beta^2}\,\overline{u}(x,\ s,\ \beta) - \frac{s^2}{\beta}\sin x = \overline{u}''(x,\ s,\ \beta)$$

$$\Rightarrow \overline{u}''(x,\ s,\ \beta) - \frac{s^2}{\beta^2}\,\overline{u}(x,\ s,\ \beta) = -\frac{s^2}{\beta}\sin x \tag{2.0}$$

If we look equation (2.0), which is second order linear non-homogeneous differential equation. A general solution of the non-homogeneous equation is given by in the form

$$\overline{u}(x,\ s,\ \beta)\,\overline{u}_h(x,\ s,\ \beta) + \overline{u}_p(x,\ s,\ \beta)$$

Now, first let us find the general solution of the homogeneous part

$$\overline{u}''(x, s, \beta) - \frac{s^2}{\beta^2} u(x, s, \beta) = 0$$

Suppose the solution is $\overline{u}_h(x, s, \beta) = e^{rx} \Rightarrow \overline{u}_h''(x, s, \beta) = r^2 e^{rx}$

If we substitute the above result to the homogeneous equation, we have

$$r^2 e^{rx} - \frac{s^2}{\beta^2} e^{rx} = 0 \Rightarrow \left(r - \frac{s}{\beta}\right)\left(r + \frac{s}{\beta}\right) = 0 \Rightarrow r = -\frac{s}{\beta},\ r = \frac{s}{\beta}$$

Therefore, the general solution of the homogeneous problem is

$$\overline{u}_h(x, s, \beta) = c_1 e^{\frac{s}{\beta}x} + c_2 e^{-\frac{s}{\beta}x}$$

In order to find the constants c_1 and c_2 we use the boundary conditions.

So, we get

$$\overline{u}(0, s, \beta) = 0 \Rightarrow c_1 + c_2 = 0 \tag{2.1}$$

And also

$$0 = \overline{u}(\pi, s, \beta) \Rightarrow c_1 e^{\frac{s}{\beta}\pi} + c_2 e^{-\frac{s}{\beta}\pi} = 0 \tag{2.2}$$

From the two results (2.1 and 2.2), we have $c_1 = c_2 = 0$

Assume the particular solution is given by:

$$\overline{u}_p(x, s, \beta) = A\cos(x) + B\sin(x)$$

In order to find the constant A and B we use the method of undetermined coefficients

$$\frac{d}{dx}\overline{u}_p(x, s, \beta) = -A\sin(x) + \cos(x) \Rightarrow \frac{d^2}{dx^2}\overline{u}_p(x, s, \beta)$$

$$= -A\cos(x) - B\sin(x)$$

$$\Rightarrow \bar{u}_p''(x, s, \beta) - \frac{s^2}{\beta^2}\bar{u}_p(x, s, \beta) = -A\cos(x) - B\sin(x)$$

$$- \frac{s^2}{\beta^2}(A\cos(x) + \sin(x))$$

$$= \left[-1 - \frac{s^2}{\beta^2}\right](A\cos(x) + B\sin(x)) = -\frac{s^2}{\beta}\sin x$$

From the above result we conclude that

$$\left[-1 - \frac{s^2}{\beta^2}\right]A = 0 \Rightarrow A = 0 \text{ and } \left[-1 - \frac{s^2}{\beta^2}\right]B = -\frac{s^2}{\beta} \Rightarrow B = \frac{s^2\beta}{s^2 + \beta^2}$$

Therefore, the general solution $\bar{u}(x, s, \beta) = \bar{u}_h(x, s, \beta) + \bar{u}_p(x, s, \beta)$ is equal to

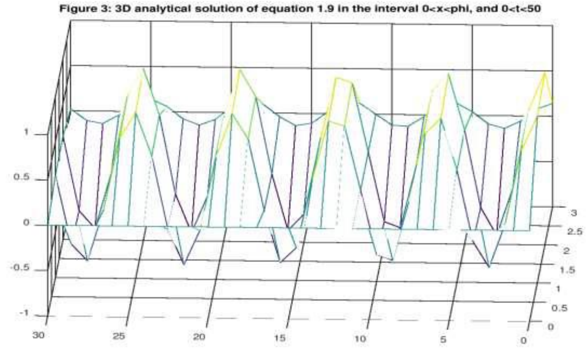
$$\Rightarrow \bar{u}(x, s, \beta) = \frac{s^2\beta}{s^2 + \beta^2}\sin(x) \tag{2.3}$$

Applying the inverse AK transform on both sides of equation (2.3)

$$\Rightarrow M_i^{-1}[\bar{u}(x, s, \beta)] = M_i^{-1}\left(\frac{s^2\beta}{s^2 + \beta^2}\sin(x)\right)$$

$$u(x, t) = \sin(x)M_i^{-1}\left(\frac{s^2\beta}{s^2 + \beta^2}\right) = \sin(x)\cos(t)$$

Figure 3: 3D analytical solution of equation 1.9 in the interval 0<x<phi, and 0<t<50



IV. Conclusion

Partial differential equations are fundamental and importance in engineering and mathematics because any physical laws and relations appear mathematically in the form of such equations. In this paper, authors successfully discussed the use of AK Transform for the solution of partial differential equations. First, some fundamental properties of AK transform are provided and then used to solve initial value problems of PDEs. We successfully found an exact solution in all the examples and the result gives a guaranty for AK transform, which plays a great role in finding exact solution of initial and boundary value problems of partial differential equations. In this paper we may be concluded that AK transform is very powerful and efficient in finding analytical solution, due to the properties of AK transform simplify its computation. AK transform can also be applied to similar nonlinear problems that appear in different branches of engineering and research in future.

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