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HYBRID FIXED POINT THEOREMS FOR INTEGRAL TYPE IMPLICIT RELATIONS IN HAUSDORFF FUZZY METRIC SPACE

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Abstract

In nonlinear analysis, the fixed point theory plays a key role. In the present paper some new hybrid fixed point theorems (HFPT) for integral type implicit relations (ITIR) in Hausdorff fuzzy metric space (HFMS). Fixed point theorem (FPT) for set-valued maps was initiated by Nadler [21] and has developed more results in different directions. Our result generalized extends and modified some existing results of fuzzy metric space (FMS).

1. Introduction

For the existing fixed points in FMS, the contractive conditions and implicit function play a key role. Zadeh [31] was the first who explore fuzzy mathematics and introduced a mathematical formulation of the fuzzy set (FS) in 1965. Many researchers worked on this theory of FS and developed some interesting results to encounter the vagueness, ambiguity of daily life problems. Kramosil and Michalek [15] first to introduced the concept of FMS, which was modified by George and Veeramani [10]. The Banach contractive FPT extended by Gregori and Sapenal [11] to the fuzzy contractive mapping of complete metric space. Recently more results in FMS (see [7, 8, 9, 18, 19, 27, 29, 30]).

Investigations of the existence of fixed points of set-valued contractions in

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metric spaces were initiated by Nadler [21] and subsequently by Asad and Krik [3], Mizoguchi, and Takahashi [20], and Kim and Wardowshi [13]. To prove common FPT's of contractive maps in different spaces, Popa ([22, 23]) introduces the concept of implicit function. The integral type contraction principle, which is a version of the Banach contraction principle, was developed by Branciari [6] in 1985 and proved FPT for single-valued contractive mapping of integral type. Many researchers extended the result of Branciari in different spaces, (see [2, 24, 26] and references therein).

The Banach [5] contractive FPT extended by Gregori and Sapena [11] to the fuzzy contractive mapping of complete metric space. Lopez and Romaguera [17] introduced the concept of HFMS in 2004. After that, a number of research papers were developed by researchers to prove FPT for multivalued contraction mapping in HFMS (see [12, 13, 22, 29]). The main idea of this study is to generalize, improve and extend some new hybrid fixed point theorems for ITIR in HFMS.

2. Preliminaries

Let us memorize the terminologies and basic properties of fuzzy metric space in the sequel as follows:

Definition 2.1[30]**.** A fuzzy set $\widetilde{A} = \{x, \mu_{\widetilde{A}}(x) : x \in X\}$ is an ordered set of a non-empty set X, where $\mu_{\widetilde{A}}(x) : X \to [0, 1]$ is a function.

Definition 2.2[27]**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$, is a continuous t-norm with unit 1, if it satisfies the following conditions:

(i) $*$ is associative, commutative and continuous,

(ii) $a * b \geq ab, \forall a, b \in [0, 1]$,

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

The typical examples of continuous t-norm are $a * b = ab$ or $a * b = \min(a, b)$, and $a * b = \frac{ab}{\max\{a, b, \lambda\}}$, for $0 < \lambda < 1$.

Definition 2.3[10]**.** An ordered triple $(X, M, *)$ is called a fuzzy metric space, where X is a non-empty set, $*$ is a continuous t-norm and $M : X \times X \times (0, +\infty) \to [0, 1]$ is a function satisfying the following conditions, such that for all $x, y, z \in X$,

$(F_{GV} - 1)$ $M(x, y, t) > 0$, for all $t > 0$,

$(F_{GV} - 2)$ $M(x, y, t) = 1$, for all $t > 0$, and $M(x, y, t) = 1$, for some $t > 0 \Rightarrow x = y$,

$(F_{GV} - 3)$ $M(x, y, t) = M(y, x, t)$ for all $t > 0$,

$(F_{GV} - 4)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, for all $s, t > 0$,

$(F_{GV} - 5)$ $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

According to Kramosil and Michalek [15], M is a fuzzy set on $X \times X \times (0, \infty)$ which satisfies $(F_{GV} - 3)$ and $(F_{GV} - 4)$ while $(F_{GV} - 1)$, $(F_{GV} - 2)$ and $(F_{GV} - 5)$ replaced by $(F_{KM} - 1)$,

$(F_{KM} - 2)$ and $(F_{KM} - 5)$, as follows:

$(F_{KM} - 1)$ $M(x, y, 0) = 0$,

$(F_{KM} - 2)$ $M(x, y, t) = 1$, for all $t > 0$, if and only if $x = y$,

$(F_{KM} - 5)$ $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

Song [29] gives two important facts that $M(\cdot, \cdot, t)$ is continuous function on $X \times X$ for $t \in (0, \infty)$ and $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$. For more examples, lemma and definition about fuzzy metric space we refer ([10, 18, 26, 27] and references therein).

For all conditions on fuzzy metric space $(X, M, *)$, it is easy to show that commuting $\Rightarrow$ weakly commuting $\Rightarrow$ compatible $\Rightarrow$ continuity and there are examples (see [26, 27]).

Example 2.1. Let $(X, M, *)$, be a fuzzy metric space, where $X = [0, 10]$, $a * b = \min\{a, b\}, \forall\ a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t + |x - y|}$, for all $x, y \in X, t > 0$ with condition $(F_{KM} - 5)$.

Define S and T by:

$$Sx = \begin{cases} 3, & \text{if } x \in (0, 2] \\ 0, & \text{if } x \in \{0\} \cup (2, 10] \end{cases}, \quad Tx = \begin{cases} 0, & \text{if } x = 0 \\ x + 8, & \text{if } x \in (0, 2] \\ x - 2, & \text{if } x \in (2, 10] \end{cases}.$$

We have $M(Sx, Tx, t) = 1 \Rightarrow Sx = Tx$ iff $x = 0,\ t > 0$, and so

$$M(ST, TS, t) = 1 \Rightarrow STx = TSx \Rightarrow ST(0) = TS(0) \Rightarrow ST = TS,\ t > 0.$$

Then, (S, T) is weakly commuting. Let $\{x_n\}$ be a sequence in X defined by:

$$x_n = 2 + \frac{1}{n},\ n \geq 1,\ Sx_n = S\left(2 + \frac{1}{n}\right) = 0,\ Tx_n = T\left(2 + \frac{1}{n}\right) = \frac{1}{n},$$

Since $\lim_{x \to \infty} M(Sx_n, Tx_n, t) = 1$, implies $\lim_{x \to \infty} M(STx_n, TSx_n, t)$ $= M\left(S\left(\frac{1}{n}\right), T(0), t\right) \Rightarrow M(3, 0, t) < 1,\ t > 0$, so that (S, T) is compatible, which implies continuity.

A collection of all non-empty closed sub sets of X denoted by $C(X)$, or generally a collection of closed bounded sub sets of X, by $CB(X)$, we define a function $M_H(A, B, t)$ on $CB(X) \times CB(X) \times (0, \infty)$ defined as follows:

$$M_H(A, B, t) = \frac{t}{t + H(A, B)} = \min\{\inf_{b \in B} M_\delta(A, b, t), \inf_{a \in A} M_\delta(a, B, t)\} \quad \text{for}$$

any $A, B \in CB(X)$ and $t > 0$, where $M(D, c, t) = M(c, D, t)$ $= \sup_{d \in D} M(c, d, t)$. For all $A, B \in CB(X)$, $M_\delta(A, B, t)$ and $M_D(A, B, t)$ be the functions defined by

$$M_\delta(A, B, t) = \frac{t}{t + \delta(A, B)} = \inf\{M_d(a, b, t) : a \in A, b \in B\}$$

and

$$M_D(A, B, t) = \frac{t}{t + D(A, B)} = \sup\{M_d(a, b, t) : a \in A, b \in B\}$$

If A is a singleton i.e. $A = \{a\}$ we write $M_\delta(A, B, t) = M_\delta(a, B, t)$

If B is also a singleton i.e. $B = \{b\}$ we write $M_\delta(A, B, t) = M_\delta(A, b, t)$

It follows immediately from the definition that

$M_\delta(A, B, t) = M_\delta(B, A, t) \leq 1$

$M_\delta(A, B, t) \geq M_\delta(A, C, t) + M_\delta(C, B, t)$

$M_\delta(A, B, t) = 1 \Leftrightarrow A = B = \{a\}$

$M_\delta(A, A, t) = 1 \Rightarrow \dim(A)$

For the collection of compact subsets $CB(X)$ the function $M_H(A, B, t)$ satisfies the conditions $(F_{GV} - 1) - (F_{GV} - 5)$. Also for all $x \in A, y \in B$, $x, y \in X, t > 0$,

$$M_H(\{x\}, \{y\}, t) = M_H(x, y, t).$$

Lemma 2.1[21]. *If* $A \subset CB(X)$, *then* $x \in A$ *if and only if* $M_\delta(x, A, t) = 1$, *for* $t > 0$.

The study of hybrid contractions, i.e., contractive conditions involving single valued and multi valued maps, initiated independently by Singh and Kulshrestha [28].

Definition 2.4. Let $F : X \to CB(X)$ and $G : X \to X$. The pair (F, G) is called a hybrid contraction, if for some $0 < \alpha < 1$ and for all $x, y, \in X$,

$$H(Fx, Fy) \leq \alpha\delta(Gx, Gy).$$

Definition 2.5. Let $(X, M, *)$ be a fuzzy metric space and $CB(X)$ is the collection set of all non-empty closed bounded subsets of X, and let $F : X \to CB(X)$ and $G : X \to X$. The pair (F, G) is called a hybrid contraction, if for some $\alpha > 1$ and for all $x, y \in X$,

$$H_M(Fx, Fy, t) \geq \alpha\delta_M(Gx, Gy, t), t > 0.$$

3. Implicit Relations

Let F^* be the set of all continuous functions $F(t_1, t_2, t_3, t_4, t_5, t_6)$ $: (R^+)^6 \to R^+$ satisfying the following conditions for all $u \geq 0, v > 0$ as

(C_a) $\int_0^{F(u, v, v, u, u+v, 1)} \phi(t)dt \geq 0,$ implies $u > v.$

(C_b) $\int_0^{F(u, v, u, v, 1, u+v)} \phi(t)dt \geq 0,$ implies $u > v.$

(C_1) $\int_0^{F(u, u, 1, 1, u, u)} \phi(t)dt \geq 0,$ for all $u \in [0, 1] \Rightarrow u = 1.$

where $\phi : R^+ \to R$ is a Lebesgue integrable mapping which is summable.

Example 3.1. Let $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4\} + b\{t_5 + t_6\},$ $b > 0$ and $\phi(t) = \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)},$ for all t in $R^+.$ Then

(C_a) $\int_0^{F(u, v, v, u, u+v, 1)} \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)} dt \geq 0,$

i.e. $\int_0^{u - \max(v, v, u) + b(u+v+1)} \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)} dt \geq 0,$

Now if $u < v \Rightarrow u < u,$ which is contradiction, so we take $u > v.$ Thus, $\sin\frac{3\pi(u - \lambda v)}{4\{1 + (u - \lambda v)\}} \geq 0,$ which implies $u \geq v,$ where $\lambda = (1-b)/(1+b).$ Similarly,

(C_b) $\int_0^{F(u, v, u, v, 1, u+v)} \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)} dt \geq 0,$

Implies $\sin\frac{3\pi(u - \lambda v)}{4\{1 + (u - \lambda v)\}} \geq 0 \Rightarrow u \geq v,$ and $\lambda = (1-b)/(1+b).$

(C_1) Further for

$\lambda = (1-b)/(1+b),$ $\int_0^{F(u, u, 1, 1, u, u)} \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)} dt \geq 0,$ so

$\sin\frac{3bu\pi}{2(1+2bu)} < 0,$ for all $u \in [0, 1] \Rightarrow u = 1.$

The following examples are also proved for the ϕ function of given above.

Example 3.2. If $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c\max\{t_2, t_3, t_4, t_5 \times t_6\}$, where $(C_a), (C_b)$ and (C_1) proved as in example 3.1.

Example 3.3. If $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, 1/2(t_3 + t_4), \sqrt{t_5 + t_6}\}$, where $(C_a), (C_b)$ and (C_1) proved as in example 3.1.

Example 3.4. If $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4\, 1/2(t_5 + t_6), b\sqrt{t_5 \cdot t_6}\}$, where $(C_a), (C_b)$ proved as in example 3.1 and for

$$(C_1)\ \int_0^{F(u, u, 1, 1, u, u)} \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)}\, dt \geq 0, \text{ so } \sin\frac{3bu\pi}{2(1+2bu)} < 0,$$

$\forall\ u \in [0, 1] \Rightarrow u = 1.$

Remark 3.1. In above examples, we observe that $\phi(t) = \frac{3\pi}{4(1+t)^2} \cdot \cos\frac{3\pi t}{4(1+t)}$, is negative for $t \in (2, \infty)$, positive for $t \in [0, 2)$ and vanishes at $t = 2$.

4. Fixed Point Theorem

Let $T : X \to CB(X)$ be a set valued mapping in a fuzzy metric space $(X, M_\delta, *)$. If $x \in Tx$, then an element $x \in X$ is called a fixed point of T.

Remark 4.1. Let Y be a finite subset of a standard fuzzy metric space $(X, M_H, *)$. Then Y is compact.

Theorem 4.1. Let I, J be self mappings of a fuzzy metric space $(X, M_H, *)$ and $f, g : X \to CB(X)$ two set valued mappings satisfying

(i)° $\cup f(X) \subset J(X)$ and $\cup g(X) \subset I(X)$

(ii)° Their exists $F \in F^*$ such that

$$(C_a) \int_0^{F\{M_\delta(fx, gy, t),\ M_d(Ix, Jy, t),\ M_\delta(Ix, fx, t),\ M_\delta(Jy, gy, t),\ M_D(Ix, gy, t),\ M_D(Jy, fx, t)\}} \phi(t)dt \geq 0,$$

which implies $u > v$, $\forall\ x, y \in X$ and for which at least one of $M_d(Ix, Jy, t)$, $M_\delta(Ix, fx, t)$ and $M_\delta(Jy, gy, t)$ is positive, where $F \in F^*$ satisfies the properties $(C_a), (C_b)$ and (C_1) and $\phi : R^+ \to R$ is a Lebesgue integrable mapping which is summable.

(iii)$^\circ$ The pair $\{f, I\}$ and $\{g, J\}$ are weakly compatible,

(vi)$^\circ$ The mapping *f* and *I* are continuous.

Then there exists a unique point $u \in X$ such that $fu = gu = \{u\} = \{Iu\} = \{Ju\}$.

Proof. Let $\varepsilon = \inf\{M_d(Ix, fx, t) : x \in X\}$. Since $(X, M_H, *)$ is compact fuzzy metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that $\lim_{x\to\infty} M_d(Ix_n, fx_n, t) = \varepsilon$. Since $M_\delta(Ix_0, fx_0, t) \geq M_d(Ix_n, fx_n, t) + M_\delta(fx_n, fx_0, t)$ therefore by the continuity of *f* and *I* and $\lim_{x\to\infty} x_n = x_0$, we get

$$M_\delta(Ix_0, fx_0, t) \geq \varepsilon \Rightarrow M_\delta(Ix_0, fx_0, t) = \varepsilon.$$

Since $\cup f(X) \subset J(X)$ and $\cup g(X) \subset I(X)$, there exists a point $y_0 \in X$ such that $Jy_0 \in fx_0$ and $M_d(Ix_0, Jy_0, t) \geq \varepsilon$. If $\varepsilon > 1$, then, by (ii)$^\circ$ we have

$$\int_0^{F\{M_\delta(fx_0, gy_0, t), M_d(Ix_0, Jy_0, t), M_\delta(Ix_0, fx_0, t), M_\delta(Jy_0, gy_0, t), M_D(Ix_0, gy_0, t), M_D(Jy_0, fx_0, t)\}} \phi(t)dt \geq 0,$$

i.e. $$\int_0^{F\{M_\delta(fx_0, gy_0, t), \varepsilon, \varepsilon, M_\delta(Jy_0, gy_0, t), M_\delta(Ix_0, gy_0, t)+\varepsilon, 0\}} \phi(t)dt \geq 0,$$

By (C_b) it implies $M_\delta(fx_0, gy_0, t) > \varepsilon$, and hence $M_\delta(Jy_0, gy_0, t) \geq M_d(fx_0, gy_0, t) > \varepsilon$. Since $\cup g(X) \subset I(X)$, then there exists a point $z_0 \in X$ such that $Iz_0 \in gy_0$ and $M_d(Iz_0, Jy_0, t) > \varepsilon$. Now, since $M_\delta(Iz_0, Jy_0, t) \geq M_d(Iz_0, Jy_0, t) \geq \varepsilon > 1$, then we have,

$$\int_0^{F\{M_\delta(fz_0, gy_0, t), M_d(Iz_0, Jy_0, t), M_\delta(Iz_0, fz_0, t), M_\delta(Jy_0, gy_0, t), M_D(Iz_0, gy_0, t), M_D(Jy_0, fz_0, t)\}} \phi(t)dt \geq 0,$$

i.e.

$$\int_0^{F\{M_\delta(fz_0, gy_0, t), M_d(Jy_0, gy_0, t), M_\delta(fz_0, gy_0, t), M_\delta(Jy_0, gy_0, t), 1, M_\delta(fz_0, gy_0, t), M_\delta(Jy_0, fz_0, t)\}} \phi(t)dt \geq 0,$$

which by (C_b), yields $M_\delta(fz_0, gy_0, t) > M_\delta(Jy_0, gy_0, t)$, but then, $\varepsilon \leq M_\delta(Iz_0, fz_0, t) \geq M_\delta(fz_0, gy_0, t) > M_\delta(Jy_0, gy_0, t) > \varepsilon$, which is a contradiction. Thus $\varepsilon = 1$. Then we get $\{Ix_0\} = \{Jy_0\} = fx_0$. If $M_\delta(Jy_0, gy_0, t) < 1$, then by $(\text{ii})^\circ$, we have

$$\int_0^{F\{M_\delta(fx_0, gy_0, t), M_d(Ix_0, Jy_0, t), M_\delta(Ix_0, fx_0, t), M_\delta(Jy_0, gy_0, t), M_D(Ix_0, gy_0, t), M_D(Jy_0, fx_0, t)\}} \phi(t)dt \geq 0,$$

i.e. $\int_0^{F\{M_\delta(Jy_0, gy_0, t), 1, 1, M_\delta(Jy_0, gy_0, t), M_\delta(Jy_0, gy_0, t), 0\}} \phi(t)dt \geq 0,$

which, by (C_1), implies that $M_\delta(Jy_0, gy_0, t) > 1$, a contradiction. Thus $M_\delta(Jy_0, gy_0, t) = 1$, and so $gy_0 = \{Jy_0\}$. Therefore

$$\{Ix_0\} = fx_0 = \{Jy_0\} = \{gy_0\} = \{p\}. \tag{4}$$

Then, by weak compatibility of the pair $\{f, I\}$, we have

$$fp = f\{Ix_0\} = \{Ifx_0\} = \{Ip\}. \tag{5}$$

If $Ip \neq p = Jy_0$, then by an application of (ii), we have

$$\int_0^{F\{M_\delta(fp_0, gy_0, t), M_d(Ip_0, Jy_0, t), M_\delta(Ip_0, fp_0, t), M_\delta(Jy_0, gy_0, t), M_D(Ip_0, gy_0, t), M_D(Jy_0, fp_0, t)\}} \phi(t)dt \geq 0, \tag{6}$$

Now using, (4), (5) and (6), we get

$$\int_0^{F\{M_\delta(fp, p, t), M_\delta(fp, p, t), 1, 1, M_d(fp, p, t), M_d(fp, p, t)\}} \phi(t)dt \geq 0,$$

which, by (C_1) is a contradiction. Therefore $M_d(fb, p, t) = 1$, and hence

$$fp = \{p\} \text{ and so } fp = \{Ip\} = \{Jp\}. \tag{7}$$

Now, since J and g are weakly compatible $\{Jp\} = \{Jgy_0\} = gJy_0 = gp.$ Suppose $Ip \neq Jp$, then $M_d(Ip, Jp, t) < 1$ and so

$$\int_0^{F\{M_\delta(fp, gp, t), M_d(Ip, Jy, t), M_\delta(Ip, fp, t), M_\delta(Jp, gp, t), M_D(Ip, gp, t), M_D(Jp, fp, t)\}} \phi(t)dt \geq 0,$$

i.e. $\int_0^{F\{M_d(Ip, Jp, t), M_d(Ip, Jp, t), 1, 1, M_d(Ip, Jp, t), M_d(Ip, Jp, t)\}} \phi(t)dt \geq 0,$

which by (C_1) is a contradiction. Thus $Ip = Jp$ and hence $fp = gp = \{Jp\} = \{p\}$. Suppose, q be a point such that, i.e. $fq = gq = \{Iq\} = \{Jq\} = q$, then, by (ii)$^\circ$ we have,

$$\int_0^{F\{M_\delta(fp, gq, t), M_d(Ip, Jq, t), M_\delta(Ip, fp, t), M_\delta(Jq, gq, t), M_D(Ip, gq, t), M_D(Jq, fp, t)\}} \phi(t)dt \geq 0,$$

i.e. $\int_0^{F\{M_d(p, q, t), M_d(p, q, t), 1, 1, M_d(p, q, t), M_d(p, q, t)\}} \phi(t)dt \geq 0,$

which by (C_1), yields $M_d(p, q, t) = 1$, and so $p = q$.

Theorem 4.2. *Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying the following conditions*

(i)$^\circ$ $S(X) \subset B(X)$ *and* $T(X) \subset A(X)$

(ii)$^\circ$ *Their exists* $F \in F^*$ *such that*

$$(C_a)^\circ \int_0^{F\{M(Sx, Ty, t), M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(Sx, By, t)\}} \phi(t)dt \geq 0,$$

for all $x, y \in X$ for which at least one of $M(Ax, By, t), M(Ax, Sx, t)$ and $M(By, Ty, t)$ is positive, where $F \in F^*$ satisfies the properties $(C_a)^\circ, (C_b)$ and (C_1) and $\phi : R^+ \to R$ is a Lebesgue integrable mapping which is summable.

(iii)$^\circ$ *The pair $\{S, A\}$ and $\{T, B\}$ are weakly compatible,*

(iv)$^\circ$ *The mapping A and B are continuous.*

Then there exists a unique common fixed point $u \in X$. Further u is a unique common fixed point of A, B, S and T.

Proof. Let $\varsigma = \inf\{M(Ax, Sx, t) : x \in X\}$. Since X is compact space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that $\lim_{x\to\infty} M(Ax_n, Sx_n, t) = \varsigma$. Since $M(Ax_0, Sx_0, t) \le M(Ax_0, Ax_n, t) + M(Ax_n, Sx_n, t) + M(Sx_n, Sx_0, t)$ therefore by the continuity of S and A and $\lim_{x\to\infty} x_n = x_0$, we get $M(Ax_0, Sx_0, t) \ge \varsigma$ and that

$$M(Ax_0, Sx_0, t) = \varsigma.$$

Since $S(X) \subset B(X)$, there exists a point $y_0 \in X$ such that $By_0 \in Sx_0$ and

$$M(Ax_0, By_0, t) \ge \varsigma.$$

If $\varsigma > 0$, then, by $(\mathrm{ii})^\circ$ we have

$$(C_a)^\circ \int_0^{F\{M(Sx_0, Ty_0, t), M(Ax_0, By_0, t), M(Ax_0, Sx_0, t), M(By_0, Ty_0, t), M(Ax_0, Ty_0, t), M(Sx_0, By_0, t)\}} \phi(t)dt \ge 0,$$

i.e. $$\int_0^{F\{M(Sx_0, Ty_0, t), \varsigma, \varsigma, M(Sx_0, Ty_0, t), M(Sx_0, Ty_0, t)+\varsigma, 1\}} \phi(t)dt \ge 0,$$

By (C_b) it implies $M(Sx_0, Ty_0, t) > \varsigma$ and hence $M(Bx_0, Ty_0, t) \ge M(Sx_0, Ty_0, t) > \varsigma$. Since $T(X) \subset A(X)$, then there exists a point $z_0 \in X$ such that $Az_0 \in Ty_0$ and $M(Az_0, By_0, t) > \varsigma$. Now, since $M(Az_0, By_0, t) \ge \varsigma > 1$. Then, we have,

$$\int_0^{F\{M(Sz_0, Ty_0, t), M(Az_0, By_0, t), M(Az_0, Sz_0, t), M(By_0, Ty_0, t), M(Az_0, Ty_0, t), M(Sz_0, By_0, t)\}} \phi(t)dt \ge 0,$$

i.e. $$\int_0^{F\{M(Sz_0, Ty_0, t), M(By_0, Ty_0, t), M(Sz_0, Ty_0, t), M(By_0, Ty_0, t), 1, M(Sz_0, Ty_0, t)+M(By_0, Ty_0, t)\}} \phi(t)dt \ge 0,$$

which by (C_b) yields, $M(Sz_0, Ty_0, t) > M(By_0, Ty_0, t)$, but then, $\varsigma \ge M(Az_0, Sz_0, t) > M(Sz_0, Ty_0, t) > M(By_0, Ty_0, t) > \varsigma$, a contradiction.

Thus $\varsigma = 1$, then we get $Ax_0 \in By_0 = Sx_0$. If $M(By_0, Ty_0, t) < 1$, then by $(\mathrm{ii})^\circ$ we have

$$\int_0^{F\{M(Sx_0, Ty_0, t), M(Ax_0, By_0, t), M(Ax_0, Sx_0, t), M(By_0, Ty_0, t), M(Ax_0, Ty_0, t), M(Sx_0, By_0, t)\}} \phi(t)dt \geq 0,$$

i.e. $\int_0^{F\{M(By_0, Ty_0, t), 1, 1, M(By_0, Ty_0, t), M(By_0, Ty_0, t)\}} \phi(t)dt \geq 0,$

which, by (C_1), implies that $M(By_0, Ty_0, t) > 1$, a contradiction. Thus $M(By_0, Ty_0, t) = 1$ and so $By_0 = Ty_0$. Therefore $Ax_0 = Sx_0 = By_0 = Ty_0 = l.$

(8)

Then, by weak compatibility of the pair $\{A, S\}$ we have

$$Al = SAl_0 = ASl_0 = Al. \tag{9}$$

If $Sl_0 \neq l = By_0$, then by an application of $(\text{ii})^\circ$, we have,

$$\int_0^{F\{M(Sl, Ty_0, t), M(Al, By_0, t), M(Al, Sl, t), M(By_0, Ty_0, t), M(Al, Ty_0, t), M(Sl, By_0, t)\}} \phi(t)dt \geq 0,$$

Now using, (8), (9) and (10), we get

$$\int_0^{F\{M(Sl, l, t), (Sl, l, t), 1, 1, (Sl, l, t), (Sl, l, t)\}} \phi(t)dt \geq 0,$$

which, by (C_1), is a contradiction. Therefore $M(Sl, l, t) = 1$, and hence

$$Sl = \{l\} \text{ and so } Sl = Al = l. \tag{11}$$

Now, since B and T are weakly compatible $Bl = BTy_0 = TBy_0 = Tl.$ Suppose $Al \neq Bl$, then $M(Al, Bl, t) < 1$, and so

$$\int_0^{F\{M(Sl, Tl, t), M(Al, Bl, t), M(Al, Sl, t), M(Bl, Tl, t), M(Al, Tl\, t), M(Sl, Bl, t)\}} \phi(t)dt \geq 0,$$

i.e. $\int_0^{F\{M(Al, Bl, t), M(Al, Bl, t), 1, 1, M(Al, Bl, t), M(Al, Bl, t)\}} \phi(t)dt \geq 0,$

which by (C_1), is a contradiction. Thus $Al = Bl$, and hence $Sl = Tl = Bl = l.$ Suppose, m be a point such that, i.e. $Sm = Tm = Am = Bm = m.$ Then, by $(\text{ii})^\circ$ we have, if

$$\int_0^{F\{M(Sl, Tm, t), M(Al, Bm, t), M(Al, Sl, t), M(Bm, Tm, t), M(Al, Tm\, t), M(Sl, Bm, t)\}} \phi(t)dt \geq 0,$$

i.e. $\int_0^{F\{M(l, m, t), M(l, m, t), 1, 1, M(l, m, t), M(l, m, t)\}} \phi(t)dt \geq 0,$

which by (C_1), yields $M(l, m, t) = 1 \Rightarrow l = m.$ Hence the theorem.

For $\phi(t) = 1$ in theorem 4.1, we obtain the result of Saini et al. [25] as follows:

Corollary 4.1. *Let* I, J *be self mappings of a compact fuzzy metric space* $(X, M_H, *)$ *and* $f, g : X \to CB(X)$ *two set valued mappings satisfying* $(\text{i})^\circ, (\text{iii})^\circ,$ *and* $(\text{iv})^\circ$ *and* $(\text{ii})^\circ F\{M_\delta(fx, gy, t), M_d(Ix, Jy, t), M_\delta(Ix, fx, t), M_\delta(Jy, gy, t), M_D(Ix, gy, t), M_D(Jy, gy, t)\} > 1$ *for all* $x, y \in X$ *for which at least one of* $M_d(Ix, Jy, t), M_\delta(Ix, fx, t)$ *and* $M_\delta(Jy, gy, t)$ *is less than unity, and* $F \in F^*$. *Then there exists a unique point* $u \in X$ *such that* $fu = gu = \{u\} = \{Iu\} = \{Ju\}$.

Corollary 4.2. *Let* I, J *be self mappings of a compact fuzzy metric space* $(X, M_H, *)$ *and* $f, g : X \to CB(X)$ *two set valued mappings satisfying* $(\text{i})^\circ, (\text{iii})^\circ,$ *and* $(\text{iv})^\circ$ *and*

$$(\text{ii})^{\circ\circ\circ} \int_0^{F\{M_\delta(fx, gy, t), M_d(Ix, Jy, t), M_\delta(Ix, fx, t), M_\delta(Jy, gy, t)} \phi(t)dt \geq 0,$$

and $F\{M_\delta(fx, gy, t)\} > F\{M_d(Ix, Jy, t), M_\delta(Ix, fx, t), M_\delta(Jy, gy, t)\}$ *for all* $x, y \in X$ *for which the right hand side of the inequality* $(\text{ii})^\circ$ *is less than unity, where* $F \in F^*$, *satisfies the properties* $(C_a), (C_b)$ *and* (C_1) *and* $\phi(t) : R^+ \to R^+$ *is Lebesgue integrable mapping which is summable. Then* f, g, I *and* J *have a unique common fixed point.*

Proof. Follows from theorem 4.1, we get proof easily, which is extension of Fisher theorem [7] to integral type of implicit relations in fuzzy metric space.

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