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AN INVESTIGATION ON THE SUBDIRECT IRREDUCIBILITY OF THE SUBGROUP LATTICES OF THE 2×2 MATRICES OVER Z_{11}

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Abstract

In this research article, we establish the subdirect irreducibility of the subgroup lattices of

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the group of 2×2 matrices over Z_{11}.

1. Introduction

If $\mathcal{G} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_p$ and $ad - bc \neq 0\}$, then $\mathcal{G}$ satisfies the conditions of a group under matrix multiplication modulo p. If $G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} : ad - bc = 1\}$, then it can be verified that G is a subgroup of $\mathcal{G}$. The order of these two groups can be computed as $o(\mathcal{G}) = p(p^2 - 1)$ $(p - 1)$ [4] and $o(\mathcal{G}) = p(p^2 - 1)$. [4]

2. Preliminaries

In this section, we provide the basic definitions that are required for the development of the research article.

Definition 2.1. An equivalence relation φon a lattice L is said to be a congruence relation on L iff $(a_0, b_0) \in \varphi$ and $(a_1, b_1) \in \varphi$ imply that $(a_0 \wedge a_1, b_0 \wedge b_1) \in \varphi$ and $(a_0 \vee a_1, b_0 \vee b_1) \in \varphi$.

Definition 2.2. The set of all congruence relations on L is denoted by Con L.

Definition 2.3. A Lattice L is said to be simple if it has only two trivial congruence relations, namely ε, the diagonal and $\theta = L \times L$. (e.g. The lattice M_3 is simple)

Definition 2.4. A lattice L is said to be subdirectly irreducible if Con L contains a unique atom. (e.g. the lattice N_5 is subdirectly irreducible)

For the case $p = 11$, the diagram of $L(G)$ is shown in Figure 2.1.

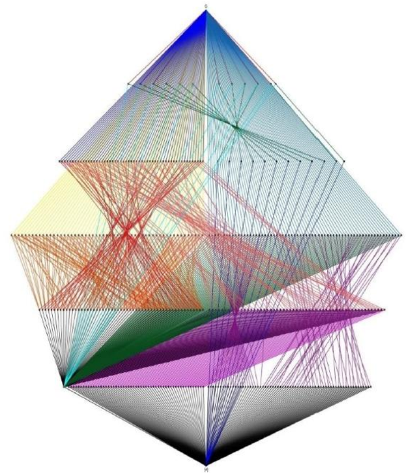

Figure 2.1. $L(G)$ for the case $p = 11$.

Line 1. (Left to right). L_1 to L_{12}

Line 2. (Left to right). J_1 to J_{55} and I_1 to I_{12}

Line 3. (Left to right). F_1 to F_{55} and H_1 to H_{66}

Line 4. (Left to right). C_1 to C_{55} and E_1 to E_{55}

Line 5. (Left to right). A_1, B_2 to B_{55} and D_1 to D_{66}.

3. Subdirect Irreducibility of $L(G)$ for the case $p = 11$

In the following two propositions and one theorem, we regard as $L(G)$ for the case $p = 11$.

Proposition 3.1. *For the case* $p = 11$, $\varphi(\{e\}, A_1)$ *is a proper congruence relation on* $L(G)$.

Proof.

For $p = 11$, $L(G)$ is as shown in Figure 2.1. The principal congruence relation generated by $\varphi(\{e\}, A_1)$ is equal to $\varepsilon U\{(\{e\}, A_1), (A_1, \{e\}), (B_1, E_1), (B_1, E_2), \ldots, (B_{55}, E_{55}), (E_1, B_1), (E_2, B_2), \ldots\}$ where ε is the diagonal relation on $L(G)$, is a proper congruence relation of $L(G)$.

Proposition 3.2. *With usual notations*, $\varphi(A, B) = L(G) \times L(G)$.

Proof.

$$\varphi(\{e\}, B_1) = \varepsilon U\{(\{e\}, B_1), (B_1, \{e\}), (B_2, G), (B_3, G) \ldots\}$$

$$= L(G) \times L(G)$$

Likewise,

$$\varphi(\{e\}, B_2\} = L(G) \times L(G)$$

$$\varphi(\{e\}, B_3\} = L(G) \times L(G)$$

$$\varphi(\{e\}, B_{55}\} = L(G) \times L(G)$$

$$\varphi(\{e\}, E_1) = \varepsilon U\{(\{e\}, E_1), (E_1, \{e\}), (E_2, G), (E_3, G) \ldots\}$$

$$= L(G) \times L(G)$$

In the same way,

$$\varphi(\{e\}, E_2\} = L(G) \times L(G)$$

$$\varphi(\{e\}, E_3\} = L(G) \times L(G)$$

$$\varphi(\{e\}, E_{55}\} = L(G) \times L(G)$$

$$\varphi(\{e\}, F_1) = \varepsilon U\{(\{e\}, F_1), (F_1, \{e\}), (F_2, G), (F_3, G) \ldots\}$$

$$= L(G) \times L(G)$$

Correspondingly,

$$\varphi(\{e\}, F_2\} = L(G) \times L(G)$$

$$\varphi(\{e\}, F_3\} = L(G) \times L(G)$$

$$\varphi(\{e\}, F_{55}\} = L(G) \times L(G)$$

$$\varphi(\{e\}, C_1) = \varepsilon U\{(\{e\}, C_1), (C_1, \{e\}), (C_2, G), (C_3, G) \ldots\}$$

$$= L(G) \times L(G)$$

Likewise,

$\varphi(\{e\}, C_2\} = L(G) \times L(G)$

$\varphi(\{e\}, C_3\} = L(G) \times L(G)$

$\varphi(\{e\}, C_{55}\} = L(G) \times L(G)$

$$\varphi(A_1, G) = \varepsilon U\{(A_1, G), (\{e\}, B_2), (\{e\}, \mathrm{B}_3)\ldots\}$$
$$= L(G) \times L(G)$$

$$\varphi(A_1, E_1) = \varepsilon U\{(A_1, E_1), (E_1, A_1), (C_1, G), (C_2, G)\ldots\}$$
$$= L(G) \times L(G)$$

$$\varphi(A_1, E_1) = \varepsilon U\{(A_1, E_1), (C_1, G), (C_2, G)\ldots\}$$
$$= L(G) \times L(G)$$

In the same way,

$\varphi(A_1, E_2\} = L(G) \times L(G)$

$\varphi(A_1, E_3\} = L(G) \times L(G)$

$\varphi(A_1, E_{55}\} = L(G) \times L(G)$

$$\varphi(\{e\}, I_1) = \varepsilon U\{(\{e\}, I_1), (I_1, \{e\}), (I_2, G), (I_3, G)\ldots\}$$
$$= L(G) \times L(G)$$

Correspondingly,

$\varphi(\{e\}, I_2\} = L(G) \times L(G)$

$\varphi(\{e\}, I_3\} = L(G) \times L(G)$

$\varphi(\{e\}, I_{12}\} = L(G) \times L(G)$

Therefore, $\varphi(A, B)$ is an improper congruence for all pairs A and B of elements in $L(G)$.

Theorem 3.3. *For the case* $p = 11$, $Con(L(G))$ *is a* 3-*element chain.*

Equivalently, $L(G)$ is subdirectly irreducible for the case $p = 11$.

Proof. The proof of the theorem is based on the propositions 3.1 and 3.2. The Hasse diagram of $Con(L(G))$ is as shown in Figure 3.1.

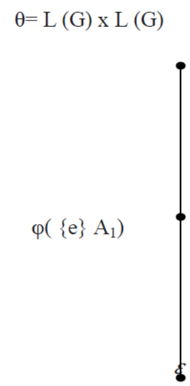

Figure 3.1. The Hasse diagram of $Con(L(G))$.

4. Conclusion

In this research article, we have established that the subgroup lattices of the group of 2×2 matrices over Z_{11} is subdirectly irreducible.

References

[1] G. Gratzer, General Lattice theory, Birkhauser Veslag, Basel, (1998).

[2] I. N. Herstien, Topics in Algebra, John Wiley and sons, New York, (1975).

[3] R. Seethalakshmi, V. Durai Murugan and R. Murugesan, On the lattice of subgroup of 2×2 Matrices over Z_{11}, Malaya Journal of Matmatik S(1) (2020), 451-456.

[4] A. Vethamanickam and Jebaraj Thiraviam, On the lattices of subgroups, Int. Journal of Mathematical Archiv-6(9), (2015), 1-11.