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BANACH ALGEBRAS (THE GELFAND-MAZUR THEOREM)

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Abstract

We are concerned with the development of the more general real case of the classical theorem of Gelfant on representation of a complex commutative unital Banach Algebra. We obtain two representative theorems for unital real Banach Algebras.

Introduction

A Banach algebra, is an associative algebra A over the real or complex numbers that at the time also a Banach space, i.e. a normed space and complete in the metric induced by the norm. The norm is required to satisfy

$$\forall,\ y \in A : xy \leq \| x \| \| y \|.$$

This ensures that the multiplication operation is continuous.

A Banach algebra is called unital if it has an identity element for the multiplication whose norm is 1. In particular we shall give a proof of the Gelfand-Mazur theorem.

Theorem. (The Gelfand-Mazur Theorem) *If a unital Banach algebra X is a division algebra, then X is isometrically isomorphic to $\mathbb{C}$. In other words, the only normed field is $\mathbb{C}$ within an isometric isomorphism.*

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Proof. Let $\sigma(x) \neq \emptyset \forall x \in X$. Hence, $\lambda \in \sigma(x)$ for some $\lambda \in \mathbb{C}$.

So, $(x - \lambda e)$ is not invertible. But is a division algebra.

Therefore, $x - \lambda e = 0$. In other words, $x = \lambda e$.

We assume that this representation is unique.

If $x = \mu e$, then $(\lambda - \mu) e = 0$,

The zero element of *X*, so $e = 0$, a contradiction.

This proves our assumption.

Define. $X \to \mathbb{C}$ by $f(x) = \lambda$, where $x = \lambda e$.

Let $x = \lambda e$, $y = \mu e$. Then $x + y = (\lambda + \mu)e$, so that

$$f(x + y) = \lambda + \mu = f(x) + f(y). \tag{1}$$

Also, for every complex scalar α,

$$\alpha x = \alpha(\lambda e) = (\alpha\lambda)e$$

and so

$$f(a\alpha) = \alpha\lambda = \alpha f(x). \tag{2}$$

Thus form (1) and (2) we infer that *f* is linear.

Furthermore $y = (\lambda\mu)e$ and so

$$f(xy) = \lambda\mu = f(x)\ f(y).$$

Therefore, *f* is multiplicative. Since $\| e \| = 1$, we obtain

$$\begin{aligned} \| x \| &= \| \lambda e \| \\ &= | \lambda | \, \| e \| \\ &= | \lambda | \\ &= | (\lambda) |. \end{aligned}$$

This shows that *f* is an isometry.

Finally, given $\lambda \in \mathbb{C}$, choose $x = \lambda e$ in X.

Hence f is surjective. Thus is an isometric isomorphism of X onto $\mathbb{C}$.

Hence proved.

Theorem. *If X is a Banach algebra and is a proper closed ideal in X, then $\frac{X}{M}$ is a Banach algebra. Also, if e is the identity in X, then $e + M$ is the identity in $\frac{X}{M}$.*

Proof. We have already seen that $\frac{X}{M}$ is a Banach space.

Define multiplication in $\frac{X}{M}$.

$$(x + M)(y + M) = xy + M \ \forall x, \ y \in X.$$

Then X is an algebra. Also commutativity of X implies commutativity of $\frac{X}{M}$.

Furthermore

$$(x + m_1)(y + m_2) = xy + xm_2 + m_1 y + m_1 m_2 \in xy + M.$$

Hence $\| (x + M)(y + M) \| = (x + y + M)$

$$\leq \| (x + m_1)(y + m_2) \|, m_1, m_2 \in M$$

$$\leq \| x + m_1 \| \| y + m_2 \|$$

Therefore

$$\| (x + M)(y + M) \| \leq \inf \{\| x + m_1 \| \| y + m_2 \|\}$$

Where $m_1, m_2 \in M$

$$= \| x + M \| \| y + M \|.$$

Now

$$\| e + M \| = \| (e + m)^2 \| \leq \| e + M \|^2 \Rightarrow \| e + M \| \geq 1.$$

Also $\| e + M \| \leq \| e \| \leq 1$.

Hence $\| e + M \| = 1$.

Hence proved.

Theorem. A Banach algebra A without a unit can be embedded into A unital Banach algebra A_1 as an ideal of codimension one .

Proof. Let $A_1 = A \oplus \mathbb{C}$ as a linear space, and define a Multiplication in A_1 by

$$(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu).$$

It is easily checked that this is associative and distributive.

Moreover, the element (0, 1) is a unit for this multiplication:

$$(x, y)(0, 1) = (x0 + x + \lambda 0, \lambda 1) = (x, \lambda) = (0, 1)(x, \lambda).$$

Put $\| (x, \lambda) \| = \| x \| + \| \lambda \|$

Then A_1 is a Banach space when equipped with this norm. Furthermore,

$$\begin{aligned} \| (x, \lambda)(y, \mu) \| &= \| (xy + \mu x + \lambda y, \lambda\mu) \| \\ &= \| xy + \mu x + \lambda y \| + | \lambda\mu | \\ &\leq \| x \| \| y \| + | \mu | \| x \| + | \lambda | + | \lambda | | \mu | \\ &= (\| x \| + | \lambda |)(\| y \| + | \mu |) \\ &= \| (x, \lambda) \| \| (y, \mu) \|. \end{aligned}$$

Hence A_1 is a Banach algebra with unity. We may identify A with the ideal $\{(x, 0) : x \in A\}$ in A_1 via the isometric isomorphism $x \mapsto (x, 0)$.

Hence proved.

Theorem. Every maximal ideal in a unital Banach is closed.

Proof. Let J be a maximal ideal in the unital Banach algebra.

Then J cannot contain any invertible elements, otherwise we would have $J = A$.

Hence $J \subseteq \setminus(A)$.

Now, (A) is open and so $A \setminus \mathcal{G}(A)$ is closed,

Hence $J \subseteq J \subseteq A \setminus \mathcal{G}(A)$

In particular, $\overline{J} \neq A$.

But $\overline{J}$ is an ideal containing J, and so $\overline{J} = J$ since J is a maximal ideal. That is, J is closed.

Hence proved.

Conclusion

In this project work, we discussed about the theorems that are based on Banach algebra. In particular, we seen special theorem in the Gelfand-Mazur theorem.

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