Advances and Applications in Mathematical Sciences
Volume 21, Issue 6, April 2022, Pages 3537-3551

M-POLYNOMIAL AND TOPOLOGICAL INDICES OF A SPECIAL GRAPH SG_n^3

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Abstract

Topological indices are widely studied to investigate and correlate the Physical properties and Chemical properties of hydrocarbons. M-polynomial has been introduced to compute the Topological indices for the Chemical compounds. In this paper, we have used complete graph of order 3 and constructed subdivisions of this graph to compute, its degree-based topological indices, First Zagreb index $M_1(G)$, Second Zagreb index $M_2(G)$ and the modified Zagreb index. Further, Symmetric Sum division index $SS(G)$ and Generalized Randić index $R_\alpha(G)$ are computed using M-polynomial.

Introduction

Chemical compounds have Molecular structure. In the field of Chemical graph theory Trinajstic [1] introduced representation of these molecular structure by a graph. The symbolic display of atoms and bonds in a molecule is presented by vertices and edges respectively.

2020 Mathematics Subject Classification: 05C07, 05C31, 05C90.
Keywords: Topological index, M-Polynomial, Complete graph, Edge Subdivision.
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Received November 25, 2021; Accepted December 18, 2021

The Structure and property activities of a chemical compound can be identified by using graph theory as a tool. A topological index, commonly referred as molecular descriptor, is a numerical entity which cognates with graph structure of a chemical compound. This helps to correlate the Quantitative activity-property relationship (QAPR) and Quantitative structure-property relationship (QSPR) properties of chemical compounds [2].

In this paper, we assume the graph $G = (V, E)$ simple graph. The set V contains all vertices and E includes all of the graph's of incident edges at vertex v in G. The topological indices, in general, are graph invariant. The introduction of several degree-based topological indices (DBTI) [3] have helped to understand the physio-chemical behavior of the hydrocarbons to a great extent.

In papers [5, 6, 7] the definition of M-Polynomial is employed to compute the DBTI's of graphs. Deutsch and Klavzar [4] have recently introduced the M-Polynomial in year 2015.

In graph G a M-polynomial, $M(G, x, y)$, is given by

$$M(G, x, y) = \sum_{\delta \leq i \leq j \leq \Delta} m_{ij}(G) x^i y^j$$

where, $m_{ij} = \{e = uv \in E \mid d(u) = i, d(v) = j\}, \delta = \min\{d(v); v \in V\}$ and $\Delta = \max\{d(v); v \in V\}$.

A Planar graph [9] is a graph where no edges intersect each other. A face of a planar graph is defined to be a region bounded by the edges of the graph.

The DBTI's of special graph G_n is explored by M. S Abdelgader et al. [10]. The special graph G_n is produced from a complete graph of order 3 i.e.3-cycle C_3. In this paper we have used this thought of subdivision of 3-cycle C_3 to get a sequence of special graphs $SG_1^3, SG_2^3, SG_3^3, \ldots SG_n^3$ and derived eight DBTI's for the graph SG_n^3.

First Zagreb index, denoted by $M_1(G)$, and second Zagreb index $M_2(G)$ were introduced by Gutman et al. [8] and from the paper by Garg et al. [5], we define

Table 1. Topological indices formulae and their derivations using M-Polynomial.

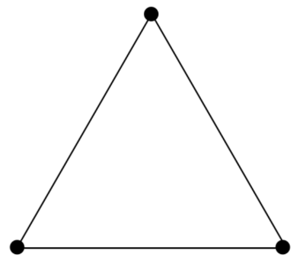
Topological index	Formula	Derivation from $M(G, x, y)$
First Zagreb index $M_1(G)$	$\sum_{v \in V(G)} d^2(v)$	$[(D_x + D_y)M(G, x, y)]_{x=y=1}$
Second Zagreb index $M_2(G)$	$\sum_{uv \in E(G)} d(u)d(v)$	$[(D_x + D_y)M(G, x, y)]_{x=y=1}$
Modified Zagreb index $mM_2(G)$	$\sum_{uv \in E(G)} \frac{1}{d(u)d(v)}$	$[(S_x + S_y)M(G, x, y)]_{x=y=1}$
Generalized Randić index $R_\alpha(G)$	$\sum_{uv \in E(G)} (d(u)d(v))^\alpha$	$[(D_x^\alpha D_y^\alpha)M(G, x, y)]_{x=y=1}$
Inverse Randić index $RR_\alpha(G)$	$\sum_{uv \in E(G)} \frac{1}{(d(u)d(v))^\alpha}$	$[(S_x^\alpha S_y^\alpha)M(G, x, y)]_{x=y=1}$
Symmetric Sum division index $SSD(G)$	$\sum_{uv \in E(G)} \frac{d^2(u)d^2(v)}{d(u)d(v)}$	$[(S_x D_y + S_y D_x)M(G, x, y)]_{x=y}$
Harmonic index $H(G)$	$\sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}$	$[S_x JM(G, x, y)]_{x=1}$
Inverse sum index $I(G)$	$\sum_{uv \in E(G)} \frac{d(u)d(v)}{d(u) + d(v)}$	$[S_x J D_x D_y M(G, x, y)]_{x=y=1}$

Where D_x, D_y, S_x, S_y and J are defined as

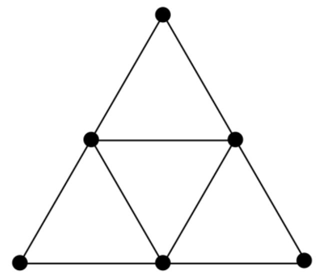
$D_x f(x, y) = x \frac{\partial f(x, y)}{\partial x}$ $D_y f(x, y) = \frac{\partial f(x, y)}{\partial y}$, $S_x f(x, y)$

$$= \int_0^x \frac{f(t, y)}{t} dt S_y f(x, y) = \int_0^x \frac{f(t, y)}{t} dt \text{ and } Jf(x, y) = f(x, y)$$

Main Results

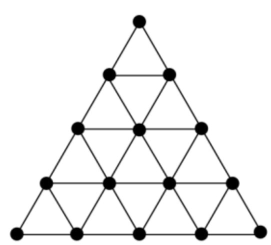


3-cycle C_3

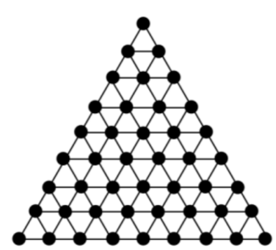


SG_1^3

We have used the subdivision of 3-cycle C_3 up to n times to obtain sequence of special graphs $SG_1^3, SG_2^3, SG_3^3, \ldots, SG_n^3$. Edge subdivision is done in C_3 to produce new vertices on every edge and construct a cycle in the inner face of this planar graph to obtain SG_1^3 Similarly, repeating the same procedure i.e. edge subdivision is used in SG_1^3 and all the new vertices are joined to form a cycle in the inner faces of SG_1^3 to obtain new special graph SG_2^3. Repeating this process $n-2$ more times, we get special graph SG_n^3.



SG_2^3



SG_3^3

3.1 Proposition: *Let* SG_n^3 *be the special graph, then* $|V(SG_1^3)| = (2^n + 1)(2^{n-1} + 1)$ *and* $|E(SG_1^3)| = 3.2^{n-1}(2^n + 1)$.

Proof. To enumerate the number of vertices we develop a counting method. We observe that there are $2^n + 1$ levels in graph SG_n^3 and every k^{th} level has exactly k vertices.

So, $|V(SG_1^3)| = 1 + 2 + 3 + \ldots + (2^n + 1)$

$|V(SG_1^3)| = (2^n + 1)(2^{n-1} + 1)$

Also, To enumerate the number of edges, we develop an approach considering edges at every level and edges between two preceding levels. We, observe that $0, 1, 2, 3, \ldots, 2^n$ edges at $1, 2, 3, \ldots, 2^n + 1$ levels and $2, 4, 6, 2.2^n$ edges between first and second level, second and third level, third and fourth level,..., 2^n and $2^n + 1$ level respectively.

So, $|E(SG_1^3)| =$ addition of edges on and between two consecutive levels

$|E(SG_1^3)| = 3.2^{n-1}(2^n + 1)$

Next, we define the value of incidence $m_{(i, j)}$ in graph G as total number of edges in edge set E with end vertices having degree i and degrees j.

3.2 Lemma. *Let SG_n^3 be the special graph, then the values of incidence of Special graph are $m_{(2, 4)} = 3.2 = 6$, $m_{(4, 4)} = 3(2^n - 1)$, $m_{(4, 5)} = 6(2^n - 2)$ and $m_{(5, 5)} = 3(2^n - 3)(2^{n-1} - 1)$.*

Proof. From the graph, we observe that between the level 1 and level 2 there are two edges such that their end vertices have degree 2 and degree 4. Also, between the level 2^n and level $2^n + 1$ there are four edges such that end vertices have degree 2 and degree 4.

$\therefore m_{(2, 4)} = 3.2 = 6$

Next, we count the number of edges having value of incidence $m_{(4, 4)}$. At level $2^n + 1$ there is $2^n - 1$ such edges and between level 2 and level 2^n there are 2. $(2^n + 1)$ such edges.

$\therefore m_{(4,4)} = 3(2^n - 1)$

Also, from level 2 to level 2^n there are 4. $(2^n - 2)$ edges such that their end vertices have degree 4 and degree 5. Between level 2^n and level $2^n + 1$ there are 2. $(2^n - 2)$ such edges.

$\therefore m_{(4,5)} = 6(2^n - 2)$

Now, we enumerate the number of edges having value of incidence $m_{(5,5)}$. At levels 3, 4, 2^n there are 1, 2, 3, $(2^n - 3)$ such edges respectively and between third level to fourth level, fourth to fifth level, fifth to sixth level there are 2, 4, 6, …, 2. $(2^n - 3)$ such edges respectively.

$\Rightarrow m_{(5,5)} = 3(2^n - 3)(2^{n-1} - 1)$

3.3 Preposition. *Let* SG_n^3 *be the special graph, then M-polynomial of* SG_n^3 *is* $M(SG_n^3, x, y) = 6x^2y^2 + 3(2^n - 1)x^4y^4 + 6(2^n - 2)x^4y^5 + 3(2^n - 2)(2^{n-1} - 1)x^5y^5$.

Proof. Employing definition of M-polynomial of a graph we get

$$M(SG_n^3, x, y) = \sum_{0 \le i \le j 5} m_{ij}(SG_n^3)x^i y^j$$

$$\Rightarrow M(SG_n^3, x, y) = m_{(2,4)}x^2y^2 + m_{(4,4)}x^4y^4 + m_{(4,5)}x^4y^5 + m_{(5,5)}x^5y^5$$

$$\therefore M(SG_n^3, x, y) = 6x^2y^2 + 3(2^n - 1)x^4y^4 + (2^n - 2)x^4y^5$$

$$+ 3(2^n - 3)(2^n - 1)x^5y^5$$

Now, for convenience, we denote the M-polynomial of SG_n^3 by $f(x, y)$ i.e. $f(x, y) = 6x^2y^2 + 3(2^n - 1)x^4y^4 + 6(2^n - 2)x^4y^5 + 3(2^n - 3)(2^{n-1} - 1)x^5y^5$

Next, we compute the operators needed for estimation of the DBTI's. $D_x f(x, y) = x\dfrac{\partial f(x, y)}{\partial x}$

$$= x\frac{\partial}{\partial x}(6x^2y^2 + 3(2^n - 1)x^4y^4 + 6(2^n - 2)x^4y^5 + 3(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$= 12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 2)x^4y^5 + 15(2^n - 3)(2^{n-1} - 1)x^5y^5 \quad (1)$$

$$D_y f(x, y) = 12x^2y^2 + 12(2^n - 1)x^4y^4 + 30(2^n - 2)x^4y^5$$

$$+ 15(2^n - 3)(2^{n-1} - 1)x^5y^5 \quad (2)$$

$$S_x f(x, y) = \int_0^x \frac{f(t, y)}{t}dt$$

$$= \int_0^x \frac{1}{t}(6t^2y^2 + (2^n - 1)t^4y^4 + 6(2^n - 2)t^4y^5 + (2^n - 3)(2^{n-1} - 1)t^5y^5)dt$$

$$= \int_0^x \frac{1}{t}(6ty^2 + 3(2^n - 1)t^3y^4 + 6(2^n - 2)t^3y^5 + 3(2^n - 3)(2^{n-1} - 1)t^4y^5)dt$$

$$\therefore S_x f(x, y) = 3x^2y^2 + \frac{3}{4}(2^n - 1)x^4y^4 + \frac{3}{2}(2^n - 1)x^4y^5$$

$$+ \frac{3}{5}(2^n - 3)(2^{n-1} - 1)x^5y^5 \quad (3)$$

$$S_y f(x, y) = \int_0^y \frac{f(t, y)}{t}dt$$

$$= \int_0^y \frac{1}{t}(6x^2t^2 + 3(2^n - 1)x^4t^4 + 6(2^n - 2)x^4y^5 + 3(2^n - 3)(2^{n-1} - 1)x^5t^5)dt$$

$$= \int_0^y (6x^2t + 3(2^n - 1)x^4y^3 + 6(2^n - 2)x^4t^4 + 3(2^n - 3)(2^{n-1} - 1)x^5t^4)dt$$

$$\therefore S_y f(x, y) = 3x^2y^2 + \frac{3}{4}(2^n - 1)x^4y^4 + \frac{6}{5}(2^n - 2)x^4y^5$$

$$+ \frac{3}{5}(2^n - 3)(2^{n-1} - 1)x^5y^5 \quad (4)$$

$$Jf(x, y) = f(x, x)$$

$$= 6x^4 + 3(2^n - 1)x^8 + 6(2^n - 2)x^9 + 3(2^n - 3)(2^{n-1} - 1)x^{10} \quad (5)$$

3.4 Theorem. *Let* SG_n^3 *be the special graph, then it's first Zagreb index of* SG_n^3 *is* $M_1(SG_n^3) = 30.2^{2n-1} + 48.2^n - 90.2^{n-1} - 18$

The second Zagreb index is

$$M_2(SG_n^3) = 75.2^{2n-1} + 93.2^n - 225.2^{n-1} - 39$$

and Modified Zagreb index is

$$mM_2(SG_n^3) = \frac{3}{25} \cdot 2^{2n-1} + \frac{147}{400} \cdot 2^n - \frac{9}{25} \cdot 2^{n-1} + \frac{429}{400}$$

Proof. Initially, computation of first Zagreb index of SG_n^3 involves addition of (1) and (2).

Adding (1) and (2), we have

$$(D_x + D_y)f(x, y) = (12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 2)x^4y^5 + 15(2^n - 3)$$
$$(2^{n-1} - 1)x^5y^5) + (12x^2y^2 + 12(2^n - 1)x^4y^4 + 30(2^n - 2)x^4y^4 + 15(2^n - 3)$$
$$(2^{n-1} - 1)x^5y^5)$$

$$= 24x^2y^2 + 24 + (2^n - 1)x^4y^4 + 54(2^n - 2)x^4y^5 + 30(2^n - 3)(2^{n-1} - 1)x^5y^5$$

Now,

$$M_1(SG_n^3) =$$

$$[24x^2y^2 + 24(2^n - 1)x^4y^4 + 54(2^n - 2)x^4y^5 + 30(2^n - 3)(2^{n-1} - 1)x^5y^5]_{x=y=1}$$

$$= 24 + 24(2^n - 1) + 54(2^n - 2) + 30(2^n - 3)(2^{n-1} - 1)$$

$$= 24.2^n + 54.2^n - 108 + 30.2^{2n-1} - 30.2^n - 90.2^{n-1} + 90$$

$$= 48.2^n - 18 + 30.2^{2n-1} - 90.2^{n-1}$$

$$= 30.2^{2n-1} + 48.2^n - 90.2^{n-1} - 18 \tag{6}$$

We evaluate the second Zagreb index of SG_n^3

$$D_yD_xf(x, y) = D_y(12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 2)x^4y^5$$

$$+ 15(2^n - 3)(2^{n-1} - 1)x^5y^5)$$

$$= y\frac{\partial}{\partial y}(12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 1)x^4y^5 + 15(2^n - 3)(2^{n-1} - 1)x^5y^5)$$

$$= 24x^2y^2 + 48(2^n - 1)x^4y^4 + 120(2^n - 2)x^4y^5 + 75(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$M_2(SG_n^3) = [24x^2y^2 + 48(2^n - 1)x^4y^4 + 120(2^n - 2)x^4y^5$$

$$+ 75(2^n - 3)(2^{n-1} - 1)x^5y^5]_{x=y=1}$$

$$= 24 + 48(2^n - 1) + 120(2^n - 2) + 75(2^n - 3)(2^{n-1} - 1)$$

$$= 24 + 48.2^n - 48 + 120.2^n - 240 + 75 + 75.2^{2n-1} - 75.2^n - 225.2^{n-1} + 225$$

$$= -39 + 93.2^n + 75.2^{n-1} - 225.2^{n-1}$$

$$= 75.2^{2n-1} + 93.2^n - 225.2^{n-1} - 39 \tag{7}$$

The Modified Zagreb index is

$$S_xS_yf(x, y) = \int_0^x \frac{1}{t}(3t^2y^2 + \frac{3}{4}(2^n - 1)t^4y^4 + \frac{6}{5}(2^n - 2)t^4y^5$$

$$+ \frac{3}{5}(2^n - 3)(2^{n-1} - 1)t^5y^5)dt$$

$$= \frac{3}{2}x^2y^2 + \frac{3}{16}(2^n - 1)x^4y^4 + \frac{6}{20}(2^n - 2)x^4y^5$$

$$+ \frac{3}{25}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$mM_2(SG_n^3) = [\frac{3}{2}x^2y^2 + \frac{3}{16}(2^n - 1)x^4y^4 + \frac{6}{20}(2^n - 1)x^4y^5$$

$$+ \frac{3}{25}(2^n - 3)(2^{n-1} - 1)x^5y^5]_{x=y=1}$$

$$= \frac{429}{400} + \frac{147}{400}\cdot 2^n + \frac{3}{25}\cdot 2^{2n-1} - \frac{9}{25}\cdot 2^{n-1}$$

$$= \frac{3}{25}\cdot 2^{2n-1} + \frac{147}{400}\cdot 2^n - \frac{9}{25}\cdot 2^{n-1} + \frac{429}{400} \tag{8}$$

3.5 Theorem. *Let SG_n^3 be the special graph, then the generalized Randić index of SG_n^3 is* $R_\alpha(SG_n^3) = 6.2^{2\alpha} + 3.4^{2\alpha}(2^n - 1) + 6.4^\alpha.5^\alpha(2^n - 2)$ $+ 3.5^{2\alpha}(2^n - 3)(2^{n-1} - 1)$ *and the inverse randić index is*

$$RR_\alpha(SG_n^3) = \frac{6}{2^{2\alpha}} + \frac{3}{4^{2\alpha}}(2^n - 1) + \frac{6}{4^\alpha.5^\alpha}(2^n - 2) + \frac{3}{5^{2\alpha}}(2^n - 3)(2^{n-1} - 1)$$

Proof. We first compute the randić index of SG_n

$$D_y f(x, y) = 12x^2y^2 + 12(2^n - 1)x^4y^4 + 30(2^n - 2)x^4y^5$$

$$+ 15(2^n - 3)(2^n - 1)x^5y^5$$

$$= 6.2x^2y^2 + 3.4(2^n - 1)x^4y^4 + 6.5(2^n - 2)x^4y^5$$

$$+ 3.5(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$D_y^\alpha f(x, y) = 6.2^\alpha x^2y^2 + 3.4^\alpha(2^n - 1)x^4y^4 + 6.5^\alpha(2^n - 2)x^4y^5$$

$$+ 3.5^\alpha(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$D_x D_y^\alpha f(x, y) = x\frac{\partial}{\partial x}[6.2^\alpha x^2y^2 + 3.4^\alpha(2^n - 1)x^4y^4 + 6.5(2^n - 2)x^4y^5$$

$$+ 3.5^\alpha(2^n - 3)(2^{n-1} - 1)x^5y^5]$$

$$D_x D_y^\alpha f(x, y) = 12.2^\alpha x^2y^2 + 12.4(2^n - 1)x^4y^4 + 24.5^\alpha(2^n - 2)x^4y^5$$

$$+ 15.5^\alpha(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$D_x^\alpha D_y^\alpha f(x, y) = 6.2^\alpha.2^\alpha x^2y^2 + 3.4^\alpha 4^\alpha(2^n - 1)x^4y^4 + 6.4^\alpha.5^\alpha(2^n - 2)x^4y^5$$

$$+ 3.5^\alpha.5^\alpha(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$= 6.2^{2\alpha}x^2y^2 + 3.4^{2\alpha}(2^n - 1)x^4y^4 + 6.4^\alpha \cdot 5^\alpha(2^n - 2)x^4y^5$$

$$+ 3.5^{2\alpha}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

Thus, $R_\alpha(SG_n^3) = [6.2^{2\alpha}x^2y^2 + 3.42^\alpha(2^n - 1)x^4y^4 + 6.4^\alpha 5^\alpha(2^n - 2)x^4y^5$

$$+ 3.5^{2\alpha}(2^n - 1)x^5 y^5]_{x=y=1}$$

$$R_\alpha(SG_n^3) = 6.2^{2\alpha} + 3.4^{2\alpha}(2^n - 1) + 6.4^\alpha.5^\alpha(2^n - 2)$$

$$+ 3.5^{2\alpha}(2^n - 3)(2^{n-1} - 1) \tag{9}$$

Next, we compute inverse randić index of special graph SG_n

$$S_y f(x, y) = 3x^2y^2 + \frac{3}{4}(2^n - 1)x^4y^4 + \frac{6}{5}(2^n - 2)x^4y^5$$

$$+ \frac{3}{5}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$= \frac{6}{2}x^2y^2 + \frac{3}{4}(2^n - 1)x^4y^4 + \frac{6}{5}(2^n - 2)x^4y^5 + \frac{3}{5}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$S_y^\alpha f(x, y) = \frac{6}{2^\alpha}x^2y^2 + \frac{3}{4^\alpha}(2^n - 1)x^4y^4 + \frac{6}{5^\alpha}(2^n - 2)x^4y^5$$

$$+ \frac{3}{5^\alpha}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$S_x S_y^\alpha f(x, y) = \int_0^x \frac{1}{t}[\frac{6}{2^\alpha}t^2y^2 + \frac{3}{4^\alpha}(2^n - 1)t^4y^4 + \frac{6}{5^\alpha}(2^n - 2)t^4y^5$$

$$+ \frac{3}{5^\alpha}(2^n - 3)(2^{n-1} - 1)t^5y^5]dt$$

$$= \frac{6}{2.2^\alpha}x^2y^2 + \frac{3}{4.4^\alpha}(2^n - 1)x^4y^4 + \frac{6}{4.5^\alpha}(2^n - 2)x^4y^5$$

$$+ \frac{3}{5.5^\alpha}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$S_x^\alpha S_y^\alpha f(x, y) = \frac{6}{2^{2\alpha}}x^2y^2 + \frac{3}{4^{2\alpha}}(2^n - 1)x^4y^4 + \frac{6}{4^\alpha \cdot 5^\alpha}(2^n - 2)x^4y^4$$

$$+ \frac{3}{5^{2^\alpha}}(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$RR_\alpha(SG_n^3) = [S_x^\alpha S_y^\alpha f(x, y)]_{x=y=1}$$

$$\therefore\ RR_{\alpha}(SG_n^3) = \frac{6}{2^{2^{\alpha}}} + \frac{3}{4^{2\alpha}}(2^n - 1) + \frac{6}{4^{\alpha} \cdot 5^{\alpha}}(2^n - 2)$$

$$+ \frac{3}{5^{2\alpha}}(2^n - 3)(2^{n-1} - 1) \tag{10}$$

3.6 Theorem. *Let SG_n^3 be the special graph, then the Harmonic index of SG_n^3 is $H(SG_n^3) = \frac{3}{5} \cdot 2^{2n-1} + \frac{89}{60} \cdot 2^n - \frac{9}{5} \cdot 2^{n-1} + \frac{83}{60}$ and the inverse sum index is $I(SG_n^3) = \frac{15}{2} \cdot 2^{2n-1} + \frac{71}{6} \cdot 2^n - \frac{45}{2} \cdot 2^{n-1} - \frac{-25}{6}$*

Proof. We first compute the Harmonic index of SG_n^3

$$Jf(x, y) = 6x^4 + 3(2^n - 1)x^8 + 6(2^n - 2)x^9 + 3(2^n - 3)(2^{n-1} - 1)x^{10}$$

$$S_x Jf(x, y) = \int_0^x \frac{1}{2}[6t^4 + 3(2^n - 1)t^8 + 6(2^n - 2)t^9 + 3(2^n - 3)(2^{n-1} - 1)t^{10}]dt$$

$$= \frac{3}{2}x^4 + \frac{3}{8}(2^n - 1)x^8 + \frac{2}{3}(2^n - 2)x^9 + \frac{3}{10}(2^n - 3)(2^{n-1} - 1)x^{10}$$

$$H(SG_n^3) = 2[S_x Jf(x, y)]_{x=1}$$

$$\Rightarrow H(SG_n^3) = 3 + \frac{3}{4}(2^n - 1) + \frac{4}{3}(2^n - 2) + \frac{3}{5}(2^n - 3)(2^{n-1} - 1)$$

$$\therefore\ H(SG_n^3) = \frac{3}{5} \cdot 2^{2n-1} + \frac{89}{60} \cdot 2^n - \frac{9}{5} \cdot 2^{n-1} + \frac{83}{60} \tag{11}$$

Next, the inverse sum index is determined by considering

$$D_x f(x, y) = 12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 2)x^4y^5$$

$$+ 15(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$D_y D_x f(x, y) = y\frac{\partial}{\partial y}[12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 2)x^4y^5$$

$$+ 15(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$JD_yD_xf(x,\ y) = 24x^4 + 48(2^n - 1)x^8 + 120(2^n - 2)x^9$$

$$+ 75(2^n - 3)(2^{n-1} - 1)x^{10}$$

$$S_xJD_yD_xf(x,\ y)$$

$$= \int_0^x \frac{1}{t}[24t^4 + 48(2^n - 1)t^8 + 120(2^n - 2)t^9 + 75(2^n - 3)(2^{n-1} - 1)t^{10}]dt$$

$$= 6x^4 + 6(2^n - 1)x^8 + \frac{40}{3}(2^n - 2)x^9 + \frac{15}{2}(2^n - 3)(2^{n-1} - 1)x^{10}$$

Now, $I(SG_n^3) = [S_xJD_xD_xf(x,\ y)]_{x=y=1}$

$$\Rightarrow I(SG_n^3) = 6 + 6(2^n - 1) + \frac{40}{3}(2^n - 2) + \frac{15}{2}(2^n - 3)(2^{n-1} - 1)$$

$$\therefore I(SG_n^3) = \frac{15}{2}\cdot 2^{2n-1} + \frac{71}{6}\cdot 2^n - \frac{45}{2}\cdot 2^{n-1} - \frac{25}{6} \tag{12}$$

3.7 Theorem. For special graph SG_n^3, symmetric sum division index of SG_n^3 is $SSD(SG_n^3) = 6.2^{2n-1} + \frac{123}{10}\cdot 2^n - 18.2^{n-1} - \frac{3}{5}$.

Proof. In order to compute the symmetric sum division index of SG_n we first consider $D_xf(x,\ y) = 12x^2y^2 + 12(2^n - 1)x^4y^4 + 24(2^n - 2)x^4y^5$ $+ 15(2^n - 3)(2^{n-1} - 1)x^5y^5$

$$S_yD_xf(x,\ y) = \int_0^y \frac{1}{t}[12x^2t^2 + 12(2^n - 1)x^4t^4 + 24(2^n - 2)x^4t^5$$

$$+ (2^n - 3)(2^{n-1} - 1)x^5t^5]dt$$

$$= 6x^2y^2 + 3(2^n - 1)x^4y^4 + \frac{24}{5}(2^n - 2)x^4y^5 + 3(2^n - 3)(2^{n-1} - 1)x^5t^5$$

Now,

$$D_yf(x,\ y) = 12x^2y^2 + 12(2^n - 1)x^4y^4 + 30(2^n - 2)x^4y^5$$

$$+ 15(2^n - 3)(2^{n-1} - 1)x^5y^5$$

$$\Rightarrow S_x D_y f(x, y) = \int_0^x \frac{1}{t}[12t^2y^2 + 12(2^n - 1)t^4y^4 + 30(2^n - 2)t^4y^4$$

$$+ 15(2^n - 3)(2^{n-1} - 1)t^5y^5]dt$$

$$= 6x^2y^2 + 3(2^n - 1)x^4y^4 + \frac{15}{2}(2^n - 2)x^4y^5 + 3(2^n - 3)(2^{n-1} - 1)x^5y^5$$

From, the definition of symmetric sum division index, we have

$$SSD(SG_n^3) = [(S_xD_y + S_yD_x)M(G, x, y)]_{x=y=1}$$

$$\Rightarrow SSD(SG_n^3) = [(6x^2y^2 + 3(2^n - 1)x^4y^4 + \frac{24}{5}(2^n - 2)x^4y^5$$

$$+ (x^2y^2 + 3(2^n - 1)x^4y^4 + \frac{15}{2}(2^n - 2)x^4y^5 + 3(2^n - 3)(2^{n-1} - 1)x^5y^5)]_{x=y=1}$$

$$= [12x^2y^2 + 6(2^n - 1)x^4y^4 + \frac{123}{10}(2^n - 2)x^4y^5$$

$$+ 6(2^n - 3)(2^{n-1} - 1)x^5y^5]_{x=y=1}$$

$$= 12 + (2^n - 1) + \frac{123}{10}(2^n - 2) + 6(2^n - 3)(2^{n-1} - 1)$$

$$= 6.2^{2n-1} + \frac{123}{10}\cdot 2^n - 18.2^{n-1} - \frac{3}{5} \tag{13}$$

Conclusion

The formulas of the topological indices provide a direct method to compute these parameters. But, in this paper we have used an alternate way to derive eight DBTI's of SG_n^3 exercising the M-polynomials. The edge subdivision leads to formation of new graphs. This course of study can be pursued and extended by considering subdivision of edges and formation of cycles on outer face of 3-cycle and constructing a sequence of special graphs.

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