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ALGEBRAIC STRUCTURE OF RECIPROCAL PYTHAGOREAN TRIPLES

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Abstract

This paper focused to study the Algebraic Structure of Set of Pythagorean triples $P = \{(x, y, z) \in Z^3 : x^2 + y^2 = z^2\}$ by introducing to define various types of Binary Operations on Set of Pythagorean triples $P_1 \cdot P_2 = (|y_1y_2 - x_1x_2|, x_1y_2 + x_2y_1, z_1z_2)$ and $P_1 \cdot P_2 = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$. Also, we know that every Pythagorean triple (x, y, z) is having Corresponding Reciprocal Pythagorean triple (xz, yz, xy). Apply this corollary, to define Binary Operations and introduce to study Algebraic Structure of Set of Reciprocal Pythagorean Triples $RP = \left\{(x, y, z) \in z^3 : \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}\right\}$.

1. Introduction

The solutions to the quadratic Diophantine equation $x^2 + y^2 = z^2$ are given by the Pythagorean Theorem. From Reference [1], It is clear that A, B, C, D are Non empty subsets of Set of Pythagorean Triples $P = \{(x, y, z) \in Z^3 : x^2 + y^2 = z^2\}$, Where

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$$A = \left\{(x,\ y,\ z) : \frac{z}{y} = 1 + \frac{2}{x^2 - 1} \text{ if } x \text{ is odd prime number or its powers}\right\}$$

$$B = \{(x,\ y,\ z) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{(2p-1)^2}\right)^2 - 1} \text{ if } x \text{ is composite or its powers,}$$

for some $P = 1, 2, 3\ldots$ with $x > (2p-1)^2\}$

$$C = \left\{(x,\ y,\ z) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{2}\right)^2 - 1} \text{ if } x \text{ grometric power of } 2\right\}$$

$$D = \{(x,\ y,\ z) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{2p^2}\right)^2 - 1}, \text{ otherwise (even composite numbers}$$

or its power); with $x > 2p^2$ and HCF of $(x, 2p^2) \neq 2\}$

Also, S_1 and S_2 are two mutually exclusive subsets of Set of Pythagorean triples as follows

$$S_1 = \left\{\left(x, \left(\frac{x}{2}\right)^2 - 1, \left(\frac{x}{2}\right)^2 + 1\right) : x \text{ is even number}\right\},$$

$$S_2 = \left\{\left(x, \frac{x^2-1}{2}, \frac{x^2+1}{2}\right) : x \text{ is odd number}\right\}$$ Every Pythagorean triple (x, y, z) is having Corresponding Reciprocal Pythagorean triple (xz, yz, xy). It follows that Corresponding subsets of Set of Reciprocal Pythagorean Triples are defined as follows.

$$A^1 = \left\{(xz,\ yz,\ xy) : \frac{z}{y} = 1 + \frac{2}{x^2 - 1} \text{ if } x \text{ is odd prime number or its powers}\right\}$$

$$B^1 = \{(xz,\ yz,\ xy) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{(2p-1)^2}\right)^2 - 1} \text{ if } x \text{ is odd composite or its}$$

powers, for some $p = 1, 2, 3,\}$

$$C^1 = \left\{(xz, yz, xy) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{2}\right)^2 - 1} \text{ if } x \text{ is geometric power of } 2\right\}$$

$D^1 = \{(xz, yz, xy) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{2p^2}\right)^2 - 1}$, otherwise even composite numbers or its power$\}$

$Rpt_0(x) = \left\{\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2}\right) : x\right.$ is an odd number greater than 1$\}$

$Rpt_e(x) = \left\{\left(x\left(\left(\frac{x}{2}\right)^2 + 1\right)\right), \left(\left(\frac{x}{2}\right)^2 - 1\right)\left(\left(\frac{x}{2}\right)^2 + 1\right), x\left(\left(\frac{x}{2}\right)^2 - 1\right) : x\right.$ is an even number greater than 2$\}$

Working Rule and Result

Now we can introduce two types of binary operations on Set of Pythagorean Triples.

Case 1. Introduce to define another binary operation of Usual multiplication '.' On Set of Pythagorean triples is $P_1 \cdot P_2 = (|\, y_1y_2 - x_1x_2\,|,$ $x_1y_2 + x_2y_1, z_1z_2)$, For some $P_1 = (x_1, y_1, z_1) \epsilon P$, $P_2 = (x_2, y_2, z_2) \epsilon P$.

Proof. Consider $(y_1y_2 - x_1x_2)^2 + (x_1y_2 + x_2y_1)^2 = (x_1x_1)^2 + (y_1y_2)^2$ $+ (x_1y_2)^2 + (x_2y_1)^2 = [x_1^2 + y_1^2][x_2^2 + y_2^2] = z_1^2z_2^2$. It follows that $(|\, y_1y_2 -$ $x_1x_2\,|, x_1y_2 + x_2y_1, z_1z_2)$ is becomes to a Pythagorean triple. Hence the binary operation of usual multiplication is well defined on Set of Pythagorean triples. Some examples are represented in the below table

Table 1. Verification by choosing some $P_1 \epsilon P$, $P_2 \epsilon P$.

P_1	P_2	$(\lvert y_1y_2 - x_1x_2 \rvert, x_1y_2 + x_2y_1, z_1z_2)$
(5, 12, 13)	(4, 3, 5)	(16, 63, 65)
(7, 24, 25)	(3, 4, 5)	(75, 100, 125)
(4, 3, 5)	(8, 15, 17)	(13, 84, 85)
(4, 3, 5)	(4, 3, 5)	(8, 24, 25)
(3, 4, 5)	(8, 15, 17)	(36, 77, 85)
(1, 0, 1)	(8, 15, 17)	(8, 15, 17)
(1, 0, 1)	(1, 0, 1)	(1, 0, 1)

Hence the binary operation of usual multiplication '.' is well defined on Set of Pythagorean triples. Apply a few calculations under this binary operation, we can verify Commutative axiom $(p_1 \cdot p_2 = p_2 \cdot p_1)$ and Associative Axioms $(p_1 \cdot p_2) \cdot p_3 = p_1 \cdot (p_2 \cdot p_3)$ are satisfies the elements of Set of Pythagorean Triples P with existence of Identity element $(1, 0, 1)$. Since $(5, 12, 13) \cdot (1, 0, 1) = (\lvert y_1y_2 - x_1x_2 \rvert, x_1y_2 + x_2y_1, z_1z_2) = (5, 12, 13)$. It proves that Set of Pythagorean triples $(P, \cdot)$ can form the commutative Monoid.

Case 2. Introduce to define another binary operation of usual multiplication '.' On Set of Pythagorean triples is $P_1 \cdot P_2 = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$.

Proof. Consider $(y_1y_2 + z_1z_2)^2 - (y_1z_2 + y_2z_1)^2 = (y_1y_2)^2 + (z_1z_2)^2 - (y_1z_2)^2 - (y_2z_1)^2 = z_2^2[z_1^2 - y_1^2] - y_2^2[z_1^2 - y_1^2] = [z_1^2 - y_1^2][z_2^2 - y_2^2] = x_1^2x_2^2$.

It follows that $(x_1x_2)^2 + (y_1z_2 + y_2z_1)^2 = (y_1y_2 + z_1z_2)^2$ implies that $(x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$ is becomes to Pythagorean triple. Some examples are represented in below table.

Table 2. verification of Lemma B, By choosing some $P_1 \epsilon P$, $P_2 \epsilon P$.

$P_1 = (x_1, y_1, z_1)$	$P_2 = (x_2, y_2, z_2)$	$(x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$
(5, 12, 13)	(4, 3, 5)	(20, 99, 101)
(7, 24, 25)	(3, 4, 5)	(21, 220, 221)
(4, 3, 5)	(8, 15, 17)	(32, 126, 130)
(4, 3, 5)	(4, 3, 5)	(16, 30, 34)
(3, 4, 5)	(8, 15, 17)	(24, 143, 145)
(1, 0, 1)	(8, 15, 17)	(8, 15, 17)
(1, 0, 1)	(1, 0, 1)	(1, 0, 1)

Hence the binary operation of usual multiplication'.' is well defined on Set of Pythagorean triples. Apply a few calculations under this binary operation, we can verify Commutative axiom $(p_1 \cdot p_2 = p_2 \cdot p_1)$ and Associative Axioms $(p_1 \cdot p_2) \cdot p_3 = p_1 \cdot (p_2 \cdot p_3)$ are satisfies the elements of Set of Pythagorean Triples P with existence of Identity element $(1, 0, 1)$. Since $(5, 12, 13) \cdot (1, 0, 1) = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2) = (5, 12, 13)$. It proves that Set of Pythagorean triples $(P, \cdot)$ can form the commutative Monoid.

2. Algebraic Structure of Set of Reciprocal Pythagorean Triples

Let RP be a Set of all Reciprocal Pythagorean triples $RP = \left\{(a, b, c) \in z^3 : \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}\right\}$.

Corollary 1. *If* $(x_1, y_1, z_1) \epsilon RP$ *then* $(x_1z_1, y_1z_1, x_1y_1) \epsilon P$ *and vice versa.*

Proof. Let $(x_1, y_1, z_1) \epsilon RP$ implies that $\frac{1}{(x_1)^2} + \frac{1}{(y_1)^2} = \frac{1}{(z_1)^2}$.

Consider $(x_1z_1)^2 + (y_1z_1)^2 = (z_1)^2[x_1^2 + y_1^2] = (x_1y_1)^2$.

It follows that if $(x_1, y_1, z_1) \epsilon RP$ then $(x_1z_1, y_1z_1, x_1y_1) \epsilon P$.

Again Consider if $(x_1z_1, y_1z_1, x_1y_1) \epsilon RP$ implies $\frac{1}{(x_1z_1)^2} + \frac{1}{(y_1z_1)^2} = \frac{1}{(x_1y_1)^2}$, which is equals to $(x_1)^2 + (y_1)^2 = (z_1)^2$. It follows that If $(x_1z_1, y_1z_1, x_1y_1) \epsilon RP$ then $(x_1, y_1, z_1) \epsilon P$. Hence the corollary is proved. That is if $(x_1, y_1, z_1) \epsilon RP$ then $(x_1z_1, y_1z_1, x_1y_1) \epsilon P$ and vice versa.

Table 3. Some examples are represented in below table.

$(x_1, y_1, z_1) \epsilon RP$	$(x_1z_1, y_1z_1, x_1y_1) \epsilon P$
(20, 15, 12)	$(240, 180, 300) = [(240)^2 + (180)^2 = (300)^2]$
(60, 80, 48)	$(2880, 3840, 4800) = [(2880)^2 + (3840)^2 = (4800)^2]$
(136, 255, 120)	$(16320, 30600, 34680) = [(16320)^2 + (30600)^2 = (34680)^2]$
(260, 624, 240)	$(62400, 149760, 162240) = [(62400)^2 + (149760)^2 = (162240)^2]$
(444, 1295, 420)	$(186480, 543900, 574980) = [(186480)^2 + (543900)^2 = (574980)^2]$

Also from Reference [1], define two mutually exclusive subsets of Set of Reciprocal Pythagorean Triples are as follows

$Rpt_0(x) = \{\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2}\right) : x$ is an odd number greater than 1$\}$

$Rpt_e(x) = \{\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right), \left(\left(\frac{x}{2}\right)^2-1\right)\left(\left(\frac{x}{2}\right)^2+1\right), x\left(\left(\frac{x}{2}\right)^2-1\right) : x$ is an even number greater than 2$\}$

Also from corollary 1, If $P_1 = (x_1z_1, y_1z_1, x_1y_1) \epsilon RP$ then $(x_1, y_1, z_1) \epsilon P$ and If $P_2 = (x_2z_2, y_2z_2, x_2y_2) \epsilon RP$ then $(x_2, y_2, z_2) \epsilon P$.

Case 3. Now we can define one of the Binary Operation '*' on RP, If $P_1 = (x_1z_1, y_1z_1, x_1y_1) \epsilon RP$ and $P_2 = (x_2z_2, y_2z_2, x_2y_2) \epsilon RP$ then

$$P_1 * P_2 = \begin{cases} \left(\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right), \left(\left(\frac{x}{2}\right)^4-1\right), x\left(\left(\frac{x}{2}\right)^2-1\right)\right) : x = x_1 \\ + x_2 \text{ is even number greater than } 2 \\ \left(\left(\frac{x(x^2+1)}{2}, \frac{(x^4-1)}{4}, \frac{x(x^2-1)}{2}\right) : x = x_1 + x_2 \\ \text{is an odd number greater than } 1) \end{cases}$$

We can verify the above binary operations from chapter 1 examples of $Rpt_0(x)$ and $Rpt_e(x)$.

Case 4. Also we know that from Reference [1], subsets of Set of Reciprocal Pythagorean Triples are defined as follows.

$$A^1 = \left\{(xz, yz, xy) : \frac{z}{y} = 1 + \frac{2}{x^2-1} \text{ if } x \text{ is odd prime number or its powers}\right\}$$

$$B^1 = \{(xz, yz, xy) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{(2p-1)^2}\right)^2 - 1} \text{ if } x \text{ is odd composite or its}$$

powers, for some $p = 1, 2, 3,$}

$$C^1 = \left\{(xz, yz, xy) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{2}\right)^2 - 1} \text{ if } x \text{ is geometric power of } 2\right\}$$

$$D^1 = \{(xz, yz, xy) : \frac{z}{y} = 1 + \frac{2}{\left(\frac{x}{2p^2}\right)^2 - 1}, \text{ otherwise even composite numbers}$$

or its power}. If $P_1 = (x_1z_1, y_1z_1, x_1y_1) \epsilon RP$ with $(x_1, y_1, z_1) \epsilon P$ and $P_2 = (x_2z_2, y_2z_2, x_2y_2) \epsilon RP$ with $(x_2, y_2, z_2) \epsilon P$ then Binary Operation 'o' on Set of Reciprocal Pythagorean Triples is defined as follows.

$$P_1 * P_2 = \begin{cases} (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{x^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is odd prime number and its powers} \\ (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{\left(\dfrac{x}{(2p-1)^2}\right)^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is odd composite and is powers, for some } p = 1, 2, 3.. \\ (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{\left(\dfrac{x}{2}\right)^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is only power of 2 i.e. geometric series of } \{2^x\} \\ (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{\left(\dfrac{x}{2p^2}\right)^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is even composite not only power of 2, for some } p = 1, 2, 3. \end{cases}$$

Apply a few calculations under this binary operation, we can verify Commutative axiom $(p_1 \cdot p_2 = p_2 \cdot p_1)$ and Associative Axioms $(p_1 \cdot p_2) \cdot p_3 = p_1 \cdot (p_2 \cdot p_3)$ are satisfies the elements of set of Reciprocal Pythagorean Triples RP. *so we can say the Algebraic Structure* $(RP, .)$ can *become to semi group*.

Case 5. For some $P_1^1 = (x_1, y_1, z_1) \epsilon RP$, $P_2^1 = (x_2, y_2, z_2) \epsilon RP$, define corresponding Binary Operation on RP_T with using of CASE 3 and COROLLARY 1 is

$$P_1^1 \cdot P_2^1 = \begin{pmatrix} |\ y_1y_2z_1z_2 - x_1x_2z_1z_2\ | (x_1x_2y_1y_2), \\ (x_1y_2z_1z_2 + y_1x_2z_1z_2)(x_1x_2y_1y_2), \\ (y_1y_2z_1z_2 - x_1x_2z_1z_2)(x_1y_2z_1z_2 + y_1x_2z_1z_2) \end{pmatrix}.$$

Proof. Let $P_1^1 = (x_1, y_1, z_1) \epsilon RP$, $P_2^1 = (x_2, y_2, z_2) \epsilon RP$. From corollary 1, it is having corresponding Pythagorean triple $(x_1z_1, y_1z_1, x_1y_1) \epsilon P$, $(x_2z_2, y_2z_2, x_2y_2) \epsilon P$.

From Case 3, $(x_1z_1, y_1z_1, x_1y_1) \cdot (x_2z_2, y_2z_2, x_2y_2)$

$= \begin{pmatrix} | y_1 y_2 z_1 z_2 - x_1 x_2 z_1 z_2 |, \\ (y_1 y_2 z_1 z_2 + x_1 x_2 z_1 z_2) \\ (x_1 x_2 y_1 y_2) \end{pmatrix} \epsilon P$ Also, from Corollary 1, we can obtain

Corresponding Reciprocal Pythagorean triple is

$$P_1^1 \cdot P_2^1 = \begin{pmatrix} | y_1 y_2 z_1 z_2 - x_1 x_2 z_1 z_2 | (x_1 x_2 y_1 y_2), \\ (x_1 y_2 z_1 z_2 + y_1 x_2 z_1 z_2)(x_1 x_2 y_1 y_2), \\ (y_1 y_2 z_1 z_2 - x_1 x_2 z_1 z_2)(x_1 y_2 z_1 z_2 + y_1 x_2 z_1 z_2) \end{pmatrix}$$

Apply a few calculations under this binary operation, we can verify Commutative axiom $(P_1^1 \cdot P_2^1 = P_2^1 \cdot P_1^1)$ and Associative Axioms $(P_1^1 \cdot P_2^1) \cdot P_3^1 = P_1^1 \cdot (P_2^1 \cdot P_3^1)$ are satisfies the elements of Set of Reciprocal Pythagorean Triples RP. so we can say the Algebraic Structure $(RP, .)$ can become to semi group.

Also, to define this binary operation in another methodology, which is defined as follows, For some $P_1^1 = (x_1 z_1, y_1 z_1, x_1 y_1) \epsilon RP$, $P_2^1 = (x_2 z_2, y_2 z_2, x_2 y_2) \epsilon RP$ with $(x_1, y_1, z_1) \epsilon P, (x_2, y_2, z_2) \epsilon P$ then $P_1^1 \cdot P_2^1 = (x_1 z_1, y_1 z_1, x_1 y_1) \cdot (x_2 z_2, y_2 z_2, x_2 y_2) = (x_1, y_1, z_1) \cdot (x_2, y_2, z_2)$ [from corollary 1].

Now we can apply Case 3, and corollary 1 successively, we obtain that

$P_1^1 \cdot P_2^1 = (| y_1 y_2 - x_1 x_2 | z_1 z_2, (x_1 y_2 + x_2 y_1) z_1 z_2, | y_1 y_2 - x_1 x_2 | (x_1 y_2 + x_2 y_1))$ be comes to well defined binary operation on Set of Reciprocal Pythagorean triple.

Table 4. Some examples are represented in below table.

$P_1^1 = (x_1 z_1, y_1 z_1, x_1 y_1)$	$P_1 = (x_1, y_1, z_1)$	$P_2^1 = (x_2 z_2, y_2 z_2, x_2 y_2)$	$P_2 = (x_2, y_2, z_2)$	Pythagorean triple $\begin{pmatrix} \| y_1 y_2 - x_1 x_2 \|, \\ (y_1 y_2 + x_1 x_2) \\ (z_1 z_2) \end{pmatrix}$	Reciprocal Pythagorean triple $\begin{pmatrix} \| y_1 y_2 - x_1 x_2 \| z_1 z_2, \\ (x_1 y_2 + x_2 y_1) z_1 z_2, \\ \| y_1 y_2 - x_1 x_2 \| (x_1 y_2 + x_2 y_1) \end{pmatrix}$
(65,156,60)	(5,12,13)	(20,15,12)	(4,3,5)	(16,63,65)	(1040,4095,1008)

(175,600,168)	(7,24,25)	(15,20,12)	(3,4,5)	(75,100,125)	(9375,12500,7500)
(20,15,12)	(4,3,5)	(136,255,120)	(8,15,17)	(13,84,85)	(1105,7140,1092)
(20,15,12)	(4,3,5)	(20,15,12)	(4,3,5)	(8,24,25)	(200,600,192)
(15,20,12)	(3,4,5)	(136,255,120)	(8,15,17)	(36,77,85)	(3060,6545,2772)

Case 6. For some $P_1^1 = (x_1, y_1, z_1) \epsilon RP_T$, $P_2^1 = (x_2, y_2, z_2) \epsilon RP_T$, we can go to define corresponding Binary Operation on RP is

$$P_1^1 \cdot P_2^1 = \begin{pmatrix} (x_1z_1x_2z_2)(y_1z_1y_2z_2 + x_1y_1x_2y_2), \\ (y_1z_1x_2y_2 + y_2z_2x_1y_1)(x_1z_1y_2z_2 + x_1y_1x_2y_2), \\ (x_1z_1x_2z_2)(y_1z_1x_2y_2 + y_2z_2x_1y_1) \end{pmatrix}.$$

Proof. Let $P_1^1 = (x_1, y_1, z_1) \epsilon RP$, $P_2^1 = (x_2, y_2, z_2) \epsilon RP$. From corollary 1, it is having corresponding Pythagorean triple $(x_1z_1, y_1z_1, x_1y_1) \epsilon P$, $(x_2z_2, y_2z_2, x_2y_2) \epsilon P$.

From Case 4, $(x_1z_1, y_1z_1, x_1y_1) \cdot (x_2z_2, y_2z_2, x_2y_2)$ $= \begin{pmatrix} (x_1z_1x_2z_2), \\ (y_1z_1x_2y_2 + y_2z_2x_1y_1), \\ (y_1z_1y_2z_2 + x_1y_1x_2y_2) \end{pmatrix} \epsilon P$ Also, from Corollary 1, we can obtain Corresponding Reciprocal Pythagorean triple is

$$P_1^1 \cdot P_2^1 = \begin{pmatrix} (x_1z_1x_2z_2)(y_1z_1y_2z_2 + x_1y_1x_2y_2), \\ (y_1z_1x_2y_2 + y_2z_2x_1y_1)(y_1z_1y_2z_2 + x_1y_1x_2y_2), \\ (x_1z_1x_2z_2)(y_1z_1x_2y_2 + y_2z_2x_1y_1) \end{pmatrix}$$

Apply a few calculations under this binary operation, we can verify Commutative axiom $(p_1 \cdot p_2 = p_2 \cdot p_1)$ and Associative Axioms $(p_1 \cdot p_2) \cdot p_3 = p_1 \cdot (p_2 \cdot p_3)$ are satisfies the elements of Set of Reciprocal Pythagorean Triples RP. *so we can say the Algebraic Structure* $(RP, .)$ *can become to semi group*.

Also, to define this binary operation in another methodology, which is defined as follows, For some $P_1^1 = (x_1z_1, y_1z_1, x_1y_1) \epsilon RP$, $P_2^1 = (x_2z_2, y_2z_2, x_2y_2) \epsilon RP$ with $(x_1, y_1, z_1) \epsilon P$, $(x_2, y_2, z_2) \epsilon P$ then

$P_1^1 \cdot P_2^1 = (x_1z_1,\ y_1z_1,\ x_1y_1) \cdot (x_2z_2,\ y_2z_2,\ x_2y_2) = (x_1,\ y_1,\ z_1) \cdot (x_2,\ y_2,\ z_2)$ [from corollary 1].

Now we can apply Case 4, and corollary 1 successively, we obtain that $P_1^1 \cdot P_2^1 = (x_1x_2(y_1y_2 + z_1z_2),\ y_1z_2 + y_2z_1(y_1y_2 + z_1z_2),\ x_1x_2(y_1z_2 + y_1z_1))$ becomes to well defined binary operation on Set of Reciprocal Pythagorean triple.

Table 5. Some examples are represented in below table.

$P_1^1 = (x_1z_1,$ $y_1z_1,\ x_1y_1)$	$P_1 = (x_1,$ $y_1,\ z_1)$	$P_2^1 = (x_2z_2,$ $y_2z_2,\ x_2y_2$	$P_2 = (x_2,$ $y_2,\ z_2)$	Pythagorean triple $\begin{pmatrix} x_1x_2, \\ (y_1z_2 + y_1z_2) \\ y_1y_2 + z_1z_2 \end{pmatrix}$	Reciprocal Pythagorean triple $\left(\begin{pmatrix} x_1x_2(y_1y_2 + z_1z_2), \\ y_1z_2 + y_2z_1(y_1y_2 \\ + z_1z_2), \\ x_1x_2(y_1z_2 + y_1z_1 \end{pmatrix}\right)$
(65,156,60)	(5,12,13)	(20,15,12)	(4,3,5)	(16,63,65)	(1040,4095,1008)
(175,600,168)	(7,24,25)	(15,20,12)	(3,4,5)	(75,100,125)	(9375,12500,7500)
(20,15,12)	(4,3,5)	(136,255,120)	(8,15,17)	(13,84,85)	(1105,7140,1092)
(20,15,12)	(4,3,5)	(20,15,12)	(4,3,5)	(8,24,25)	(200,600,192)
(15,20,12)	(3,4,5)	(136,255,120)	(8,15,17)	(36,77,85)	(3060,6545,2772)

Conclusion and Remarks

Introduced to define another binary operation of Usual multiplication '.' On Set of Pythagorean triples is $P_1 \cdot P_2 = (|\ y_1y_2 - x_1x_2\ |,\ x_1y_2 + x_2y_1,\ z_1z_2)$ and $P_1 \cdot P_2 = (x_1x_2,\ y_1z_2 + y_2z_1 + y_1y_2 + z_1z_2)$. Also verified under this binary operations Algebraic Structure of set of Pythagorean triple can form as Commutative Semi group with identity element $(1, 0, 1)$. Also, we know that every Pythagorean triple (x, y, z) is having Corresponding Reciprocal Pythagorean triple (xz, yz, xy). Apply this corollary, to define Binary Operations to study Algebraic Structure of Set of Reciprocal Pythagorean Triples $RP = \left\{(a, b, c) \in z^3 : \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}\right\}$. If $P_1 = \{x_1z_1,\ y_1z_1,\ x_1y_1\} \epsilon RP$ with $(x_1,\ y_1z_1) \in P$ and $P_2 = \{x_2z_2,\ y_2z_2,\ x_2y_2\} \epsilon RP$ with $(x_2,\ y_2z_2) \in P$

then Binary Operation 'o' on Set of Reciprocal Pythagorean Triples is defined as follows.

$$P_1 \circ P_2 = \begin{cases} (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{x^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is odd prime number and its powers} \\ (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{\left(\dfrac{x}{(2p-1)^2}\right)^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is odd composite and is powers, for some } p = 1, 2, 3 \ldots \\ (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{\left(\dfrac{x}{2}\right)^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is only power of 2 i.e. geometric series of } \{2^x\} \\ (xz,\ yz,\ xy) : \dfrac{z}{y} = 1 + \dfrac{2}{\left(\dfrac{x}{2p^2}\right)^2 - 1} \text{ if } x = x_1 + x_2 \\ \text{is even composite not only power of 2, for some } p = 1, 2, 3. \end{cases}$$

Also, we can define one of the Binary Operation '*' on RP, If $P_1 = (x_1z_1,\ y_1z_1,\ x_1y_1) \epsilon\, RP$ and $P_2 = (x_2z_2,\ y_2z_2,\ x_2y_2) \epsilon\, RP$ then

$$P_1 * P_2 = \begin{cases} \left(\left(x\left(\left(\dfrac{x}{2}\right)^2 + 1\right)\right), \left(\left(\dfrac{x}{2}\right)^4 - 1\right) x\left(\left(\dfrac{x}{2}\right)^2 - 1\right)\right) : x = x_1 + x_2 \\ \text{is even number greater than 2} \\ \left(\left(\dfrac{x(x^2+1)}{2}, \dfrac{(x^4-1)}{4}, \dfrac{x(x^2-1)}{2}\right) : x = x_1 + x_2 \right. \\ \text{is an odd number greater than 1)} \end{cases}$$

Also, define this binary operation in another methodology, which is defined as follows, For some $P_1^1 = (x_1z_1,\ y_1z_1,\ x_1y_1) \epsilon\, RP$, $P_2^1 = (x_2z_2,\ y_2z_2,\ x_2y_2) \epsilon\, RP$ with $(x_1,\ y_1z_1) \epsilon\, P$, $(x_2,\ y_2z_2) \in P$ then

$$P_1^1 \cdot P_2^1 = \begin{pmatrix} |\ y_1y_2 - x_1x_2\ | z_1z_2, \\ (x_1y_2 + x_2y_1) z_1z_2, \\ |\ y_1y_2 - x_1x_2\ | (x_1y_2 + x_2y_1) \end{pmatrix} \text{ and } \left(\begin{pmatrix} x_1x_2(y_1y_2 + z_1z_2), \\ y_1z_2 + y_2z_1(y_1y_2 + z_1z_2), \\ x_1x_2(y_1z_2 + y_1z_1 \end{pmatrix}\right)$$

Also, verified under this binary operations Algebraic Structure of set of Reciprocal Pythagorean triples can form as semi group.

References

[1] K. Sridevi and Thiruchinapalli Srinivas, A new approach to define two types of binary operations on set of Pythagorean triples to form as at most commutative cyclic semi group Journal of Critical Reviews, (2020).

[2] K. Sridevi and T. Srinivas, A new approach to define cryptographic coding on binary operations on set of Pythagorean triples, Materials Today Proceedings, Elsevier (In press), (2021).

[3] K. Sridevi and T. Srinivas, Transcendental Representation of Diophantine Equations and Some of Its Inherent Properties, Materials Today Proceedings, Elsevier (In Press), (2021).

[4] K. Sridevi and T. Srinivas, Existence of inner addition and inner multiplication of triangular numbers and some of its inherent properties, Materials Today Proceedings, Elsevier (In Press), (2021).

[5] K. Sridevi and T. Srinivas, Proof of Fermat's last theorem by choosing two unknowns in the integer solution are prime exponents, pacific international Journal, ISSN 2616-4825 [Online] 2663-8991 [Print] 03 (2020).