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ASYMPTOTIC NORMALITY OF THE LOCAL LINEAR ESTIMATION OF THE L_1 CONDITIONAL CUMULATIVE DISTRIBUTION FUNCTION

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Abstract

In this paper, we study the local linear estimation of the conditional distribution function of a scalar response variable Y given a functional random variable X. This estimator proposed is based on the L_1 approach. We establish it under general conditions of the asymptotic normality.

1. Introduction

Functional data analysis is dedicated to the study of statistical models involving functional data. This field of research has seen an explosion of work in recent decades. Some references on this topic are the monographs by Ferraty and Y. Romain [13], Ramsay and Silverman [20], Ferraty and Vieu [15].

Literature on the kernel method has been widely used for nonparametric functional data analysis. The first result on this subject was obtained by Ferraty and Vieu [14]. The nonparametric estimation of conditional

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distribution function has been largely studied in the last decade. Amongst these works, for example, we mention are Ferraty et al. [11], [12], M. Ezzahrioui and E. Ould-Saïd [9], Laksaci et al. [16], [18], [17], Bouadjemi [5].

In this paper we focus on the functional linear estimation of the L_1 conditional distribution function. In a fact Fan and Gibles [10] first proposed a local linear estimation of the regression in real data. Notice that in FDA the first result in this topic date backs 2009. It was provided by Baillo and Graine [3]. In this pioneer work the authors constructed a local linear estimation of the regression operator when the explanatory variable takes values in Hilbert space. The Bariento-Marin et al. [4] have proposed another version which can be used for banachic covariates. Furthermore, first results on the functional local linear fitting of conditional distribution function were obtained by Demongeot et al. [8].

Recently Bouanani et al. [6], [7], have been study the asymptotic normality of some conditional nonparametric functional parameters. On the other hand, alternative version of the local linear modeling for functional data has been investigated by a number of researchers such as (cf., Bouanani et al. [6], Ayad et al. [1], Zhou and Lin [22], Xiong et al. [21], Al-Awadhi et al. [2]).

The rest of the paper is organized as follows. The section 2 introduced the local linear estimation of conditional distribution function. Some assumptions and the main asymptotic results are established in section 3. The detailed proofs of our main results are presented in appendix.

2. The Functional Model and its Estimation

Consider n pairs of random variable (X_i, Y_i) for $i = 1, \ldots, n$ drawn from the pair (X, Y) with values in $\mathfrak{F} \times \mathbb{R}\ X$, where $\mathfrak{F}$ is a semi metric vector space, $d(\cdot, \cdot)$ denoting the semi metric. Assure there exists a regular version of the conditional probability of Y given X. For all $x \in \mathfrak{F}$, will denote the conditional cumulative distribution function of Y given $X = x$ by

$$\forall x \in \mathfrak{F} \text{ and } \forall y \in \mathbb{R} \quad F^x(y) = P(Y \leq y \mid X = x).$$

In the FDA setup, there are several ways for extending the local linear ideas proposed by Barrientos-Marin et al. [4]. For a fixed $(y, x) \in \mathbb{R} \times \mathfrak{F}$ is smoothed enough to be locally approximated by a linear function that is for all x_0 in a neighborhood of x, we have

$$F(y, x_0) = a_{yx} + b_{y,x}\beta(x, y) + o(\beta(x_0, x))$$

Where $\beta(\cdot, \cdot)$ is a known operator from $\mathfrak{F}^2$ into $\mathbb{R}$, for all $\xi \in \mathfrak{F}$, $\beta(\xi, \xi) = 0$.

The operators a_{yx} and b_{yx} are estimated by L_1-norm approach method as the minimizers of the following rule

$$\min \sum_{i=1}^{n} (1\!\!1_{Y_i < y} - a - b(X_i - x))^2 k(h^{-1}\varrho(x, X_i))$$

Where $1\!\!1_A$ denotes the indicator function on the set A, K is a kernel function, $h = h_{k,n}$ is a sequence of positive numbers and $\varrho(\cdot, \cdot)$ is an operator defined on $\mathfrak{F}^2$ such that $d(\cdot, \cdot) = |\varrho(\cdot, \cdot)|$.

Then the quantity $\hat{F}^x(y)$ is explicitly defined by the following:

$$\hat{F}^x(y) = \frac{\sum_{j=1}^{n} w_{ij} 1\!\!1_{\{Y_i \leq y\}}}{\sum_{j=1}^{n} w_i j} = \frac{\sum_{j=1}^{n} \Delta_j K_j 1\!\!1_{\{Y_i \leq y\}}}{\sum_{j=1}^{n} \Delta_j K_j}$$

with $w_i j = \beta_i(\beta_i - \beta_j)K_i K_j$, $\Delta_j = K_j^{-1}(\sum_{i=1}^{n} w_i j)$.

Notice that in nonparametric FDA the local linear smoothing method is introduced by Barrientos-Marin et al. [4]. In this contribution, we study the estimator proposed by Al-Awadhi et al. [2].

3. Assumption and Main Results

To obtain the asymptotic normality of $\hat{F}^x(y)$, for fixed point x in $\mathfrak{F}$, we need the following assumptions.

(H1) $\forall r > 0 \phi_x(r) := \phi_x(-r, r) > 0$, where $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \le \varrho(X, x) \le r_1)$, here exists a function $\varphi_x(\cdot)$ such that, for all $t \in [-1, 1]$

$$\lim \frac{\phi_x(th_k, h_k)}{\phi(h_k)} = \varphi_x(t).$$

Let Ψ be the real valued function defined as for any $l \in [0, 2] : \Psi_l(s)$ $= E[g_l(X, y) - g_l(x, y) | \beta(x, X) = s]$ with $g_l(x, y) = \frac{\partial^l F^x(y)}{\partial y^l}$

(H2) $\sup_{i \ne j} IP((X_i, X_J) \in B(x, h_k)) \le \psi_k(h_k)$. Where $\psi_x(h_k) = O(\phi_x^2(h_k))$ and $B(x, r) = \{z \in \mathfrak{F} / |\varrho(z, x)| \le r\}$.

(H3) For all $y \in \mathbb{R} \forall (x_1, x_2) \in N_x^2$,

$$| F^{x_1}(t_1) - F^{x_2}(t_2) | \le C(d(x_1, x_2)^{\beta_1} + | t_1 - t_2 |^{\beta_2}),$$

with $C > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and N_x is a fixed neighborhood of x.

(H4) the functions $\varrho(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are such that:

For all $z \in \mathfrak{F} | \varrho(x, z)| = d(x, z)$ and $C_1 | \varrho(x, z) | \le | \beta(x, z) | \le C_2 | \varrho(x, z) |$.

(H5) the kernel K is a positive and differentiable function which is supported within (–1, 1), and such that

$$\begin{pmatrix} K(1) - \int_{-1}^{1} (K'(t))\varphi(t)dt & K(1) - \int_{-1}^{1} (tK(t))' \varphi(t)dt \\ K(1) - \int_{-1}^{1} (tK(t))' \varphi(t)dt & K(1) - \int_{-1}^{1} (t^2 K(t))' \varphi(t)dt \end{pmatrix}.$$

Note that the hypotheses (H1)-(H5) are the same conditions assumed in Barrientos-Marin et al. [4] Al-Awhadi et al. [2].

Before giving the main result, we list some notations. In the sequel, we denote:

$$M_j = K^j(1) - \int_{-1}^{1} (K^j(u))\varphi(u)du, \text{ where } j = 1, 2$$

$N(a, b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))' \varphi(u) du,$ for all $a > 0$, and $b = 2, 4$.

Theorem 3.1. *Assume that* (H1)-(H5) *hold, then for any* $(x, y) \in A(x, y)$, *we have*

$$(n\phi_n(x))^{1/2}(\hat{F}^x(y) - F^x(y) - B_n(x, y)) \xrightarrow{D} N(0, V_K(x, y)) \text{ as } n \to \infty.$$

With $\xrightarrow{D}$ *denoting the convergence in distribution.*

Where

$$V_K(x, y) = \frac{M_2}{M_1^2} F^x(y)(1 - Fx(y))$$

and

$$B_n(x, y) = \mathbb{E}(\hat{F}^x(y) - F^x(y))$$

It is easy to see that the proof of Theorem 3.1 is a direct consequence of the following decomposition given by

$$\hat{F}^x(y) - F^x(y) - B_n(x) = \frac{Q_n(x) - B_n(x)(\hat{F}_D^x - \mathbb{E}(\hat{F}_D^x))}{\hat{F}_D^x}.$$

Moreover, let $\hat{F}^x(y) = \frac{\hat{F}_N^x(y)}{\hat{F}_D^x}$, where $\hat{F}_N^x(y) = \frac{1}{n\mathbb{E}_{[\Delta_1 K_1]}} \sum_{j=1}^{n} \Delta_j K_j \mathbb{1}_{\{Y_i \leq y\}}$ and $\hat{F}_D^x = \frac{1}{n\mathbb{E}_{[\Delta_1 K_1]}} \sum_{j=1}^{n} \Delta_j K_j$.

Defined the bias terms by

$$B_n(X) = \frac{\mathbb{E}(\hat{F}_N^x(y)) - F^x \mathbb{E}(F_D^x)}{\mathbb{E}(F_D^x)}$$

and a centered variate

$$Q_n(x) = \hat{F}_N^x(y) - \mathbb{E}(\hat{F}_N^x(y)) - F^x(y)(\hat{F}_D^x - \mathbb{E}(\hat{F}_D^x)).$$

Then, in order to establish our normality asymptotic, we need the following

Lemmas. The proofs of these Lemmas are positioned to the Appendix.

Lemma 3.2 (See Bariento-Marin et al. [4]). *Under the assumptions* (H1)-(H5), *we have*

- $\mathbb{E}[K_1^a] = M_a \phi_x(h_k) + o(\phi(h_k))$ *for* $a > 0$
- $\mathbb{E}[K_1^a \beta_1] = o(h_k \phi(h_k))$, *for all* $a > 0$.
- $\mathbb{E}[K_1^a \beta_1^b] = N(a, b) h_k \phi(h_k) + o(h_k^b \phi(h_k))$. *For all* $a > 0, b > 1$.

Lemma 3.3 (See Bouanani et al. [6]). *Under the assumptions* (H1), (H5), *we have*

$$F_D^x = \mathbb{E}(F_D^x) \to 1 \text{ as } n \leftrightarrow \infty.$$

Lemma 3.4. *Under the assumptions* (H1), (H5), *we have*

- $B_n(x) = \frac{1}{2} \frac{\partial^2 F^x(y)}{\partial y^2} + h_k^2 \frac{1}{2} \Psi_0^{(2)}(0) \frac{N(0, 1)}{M_1} + o(1) + o(h_k^2)$. (*See Rachedi et al.* [19])
- $\text{var}[\hat{F}^x(y)] =: \frac{V(x, y)}{n\phi(h_k)} = \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y))$.

Lemma 3.5. *Under the assumptions* (H1), (H5), *we have*

$$\sqrt{n\phi(h_k)} Q_n(x) \to \mathbb{N}(0, V(x, y)) \text{ as } n \to \infty$$

4. Appendix

In what follows, let C be some strictly generic constant and for any $x \in \mathfrak{F}$, and for all $i = 1, \ldots, n$:

$$K_i(x) := K(h_k^{-1} d(X_i, x)) \text{ and } \beta_i := \beta(X_i, x).$$

Proof of Lemma 3.4. To prove this lemma, we use similar ideas as Ferraty and Romain et al. [13] used to deduce

$$Var[\hat{F}^x(y)] = Var(\hat{F}_N^x(y)) - 4[E(\hat{F}_N^x(y))] Cov(\hat{F}_N^x(y), \hat{F}_D^x)$$

$$+ 3[E(\hat{F}_N^x(y))]^2 Var(\hat{F}_D^x) + o\,\frac{1}{n\phi(h_k)}.$$

By some simple calculation, and using Lemma 3.3, we obtain

$$Var[\hat{F}^x(y)] = \frac{n^2 M_2}{(n-1)^2 M_1^2} F^x(y)(1 - F^x(y))$$

$$\to \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)).$$

Proof of Lemma 3.5. We have

$$\hat{F}^x(y) - F^x(y) - B_n(x) = \frac{Q_n(x) - B_n(x)(\hat{F}_D^x - \mathbb{E}(\hat{F}_D^x)}{\hat{F}_D^x}$$

then use 3.3 to imply that $\hat{F}_D^x \to 1$. Moreover, $B_n = o(1)$ as $n \to \infty$ we can obtain that

$$\hat{F}^x(y) = F^x(y) - B_n(x) = \frac{Q_n(x)}{\hat{F}_D^x}(1 + o(1)).$$

Thus, in order to gain Theorem 3.1, it suffices to show that

$$\sqrt{n\phi(h_k)}Q_n(x) \to \mathbb{N}(0, V(x, y)).$$

Write

$$\sqrt{n\phi(h_k)}Q_n(x)$$

$$= \frac{1}{n\mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^{n} \beta_1^2 K_1 \frac{\sqrt{n\phi(h_k)}\,\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} K_j(1\!\!1_{[Y_j \le y]} - F^x(y))$$

$$- \frac{1}{n\mathbb{E}(\beta_1 K_1)} \sum_{i=1}^{n} \beta_1 K_1 \frac{\sqrt{n\phi(h_k)}\,\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} \beta_j K_j(1\!\!1_{[Y_j \le y]} - F^x(y))$$

$$-\mathbb{E}\left(= \frac{1}{n\mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^{n} \beta_1^2 K_1 \frac{\sqrt{n\phi(h_k)}\,\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} K_j(1\!\!1_{[Y_j \le y]} - F^x(y))\right)$$

$$+ \mathbb{E}\left(= \frac{1}{n\mathbb{E}(\beta_1 K_1)} \sum_{i=1}^{n} \beta_1 K_1 \frac{\sqrt{n\phi(h_k)}\, \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} \beta_j K_j (1\!\mathrm{I}_{[Y_j \leq y]} - F^x(y)) \right)$$

$$=: [A_n \cdot B_n - E[A_n \cdot B_n]] - [C_n \cdot D_n - E[C_n \cdot D_n]].$$

To prove Lemma 3.5, we only need to show the following two claims:

Claim 1. $A_n \cdot B_n - E[A_n \cdot B_n] \to \mathbb{N}(0, V(x, y))$

Claim 2. $C_n \cdot D_n - E[C_n \cdot D_n] \to 0.$

As for Claim 1, rewrite $A_n \cdot B_n - E[A_n \cdot B_n] = [B_n - E[B_n]] + [(A_n - 1) B_n - E(A_n - 1)B_n]$. By the Cauchy-Schwarz inequality and some simple calculation, which implies that $(A_n - 1)B_n - E[(A_n - 1)]B_n = o(1)$. Therefore, to prove Claim 1, we just need to prove that

$$B_n - E[B_n] \to \mathbb{N}(0, V(x, y)). \tag{1}$$

Denote

$$B_n - E[B_n] = \frac{\sqrt{n\phi(h_k)}\, \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} [K_j (1\!\mathrm{I}_{[Y_j \leq y]} - F^x(y))] - E[K_j (1\!\mathrm{I}_{[Y_j \leq y]} - F^x(y))]$$

$$=: \sum_{j=1}^{n} \epsilon_{nj},$$

where

$$\epsilon_{nj} = \frac{\sqrt{n\phi(h_k)}\, \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} [K_j (1\!\mathrm{I}_{[Y_j \leq y]} - F^x(y))] - E[K_j (1\!\mathrm{I}_{[Y_j \leq y]} - F^x(y))]$$

are i.i.d. random variables with mean 0.

The variance of $\sum_{j=1}^{n} \epsilon_{nj}$ goes to

$$E\left(\sum_{j=1}^{n} \epsilon_{nj} \right)^2 \to \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)).$$

By using the central limit theorem, the proof of equation 1 is completed if the Lindeberg condition is verified. While, the Lindeberg condition holds since for any $\eta > 0$

$$\sum_{j=1}^{n} E[\epsilon_{nj}^2 \mathbb{1}_{(|\epsilon_{nj}|>\eta)}] - nE[\epsilon_{n1}^2 \mathbb{1}_{(|\epsilon_{n1}|>\eta)}] = E[(\sqrt{n}\epsilon_{nj})^2 \mathbb{1}_{(|\sqrt{n}\epsilon_{nj}|>\sqrt{n}\eta)}] \to 0$$

as

$$E[(\sqrt{n}\epsilon_{n1})^2] = nE[\epsilon_{n1}^2] \to \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)).$$

Next, we present the proof of Claim 2. Rewrite

$$C_n \cdot D_n - E[C_n \cdot D_n] = [(C_n - 1)D_n - E[(C_n - 1)C_n] + [D_n - E[D_n]].$$

Similar to the proof of Claim 1, we have

$$E|(C_n - 1)D_n - E[(C_n - 1)C_n]| \leq 2\sqrt{E[(C_n - 1)C_n]^2} \cdot \sqrt{E(D^2)}$$

which implies that $(C_n - 1)D_n - E(C_n - 1)C_n = o(1)$. Therefore, to show Claim 2, it suffices to show

$$D_n - E(D_n) = o(1).$$

By some simple calculation we can obtain

$$E[D_n - E(D_n)]^2$$

which completes the proof of Claim 2. Then, Theorem 3.1 is proved.

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