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CORDIAL LABELING IN THE PATH UNION AND CYCLE OF TORCH GRAPH

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Abstract

The *Cordial labeling* of a graph G is a function $f : V(G) \to \{0, 1\}$ such that every edge uv in G is given the label $|f(u) - f(v)|$ with the property $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1,$ where $v_f(i)$ represents the number of vertices with label i for $i = 0, 1$ and $e_f(i)$ represents the number of edges with label i for $i = 0, 1.$ The graph which accepts cordial labeling is called the cordial graph. In this paper, torch graph is proved to be cordial; path union of torch graphs as well as the cycle of torch graphs is also proved to be cordial.

Introduction

In 1967 Rosa [7] introduced graceful labeling which traced the origin of various graph labeling methods. For the last fifty years so many labeling methods have been in progress. One among such labeling methods is the cordial labeling introduced by Cahit [4] in 1987. Many graphs were proved as cordial. The helms, closed helms, flowers, gears, sunflower graphs, multiple shells were proved as cordial by Andar et al. [1], [2], [3]. Again in [1], [2], [3] their one-point union is also shown to be cordial. Cahit [5] has proved that

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every tree, all fans, the wheel W_n when $n \not\equiv 3 (\bmod 4)$, the complete graph K_n if and only if $n \leq 3$, the bipartite graph $K_{m,n}$ for all m and n, the friendship graph $C_3^{(t)}$ if and only if $t \not\equiv 2 (\bmod 4)$ are all cordial. An extensive survey on cordial labeling is available in Gallian survey [6]. In this paper, torch graph is proved as cordial, the path union of torch graphs is proved to be cordial and cycle of torch graphs is also proved as cordial.

Main Result

First we define the torch graph, the path union of graphs and cycle of graphs. Then we prove our results.

Definition 1. The *torch graph* O_n is defined by $V(O_n) = \{v_i / 1 \leq i \leq (n+4)\}$, $E(O_n) = \{v_i v_{n+1} / 2 \leq i \leq (n-2)\} \cup \{v_i v_{n+3} / 2 \leq i \leq (n-2)\} \cup \{v_1 v_i / 2 \leq i \leq (n+4)\} \cup \{v_{n-1} v_n, v_n v_{n+2}, v_n v_{n+4}, v_{n+1} v_{n+3}\}$.

Definition 2. Let $G_1, G_2, \ldots, G_n (n \geq 2)$ is finite graphs. The new graph obtained by adding an edge between a vertex of G_i and a vertex of G_{i+1}, for $i = 1, 2, \ldots, (n-1)$ is called a *path union* of $G_1, G_2, \ldots, G_n$.

Definition 3. Let $G_1, G_2, \ldots, G_n$ be given connected graphs. Then the *cycle of graphs* $C(G_1, G_2, \ldots, G_n)$ is the graph obtained by adding an edge joining G_i to G_{i+1} for $i = 1, 2, \ldots, (n-1)$ and an edge joining G_n to G_1. When the n graphs are isomorphic to G then it is denoted as $C(n \cdot G)$.

Theorem 1. *The torch graph is cordial.*

Proof of Theorem 1. Let $v_1, v_2, \ldots, v_{n+4}$ be the vertices of the Torch Graph O_n as shown in Figure 1. In general, it is denoted as v_i, for $i \leq i \leq (n+4)$. It has $p = (n+4)$ vertices and $q = (2n+3)$ edges.

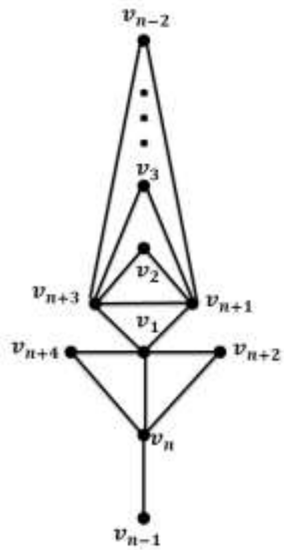

Figure 1. The torch graph O_n.

The vertex labels for the torch graph O_n are as follows

Case 1. When $n \equiv 1 (\mathrm{mod}\ 2)$

$$f(v_j) = \begin{cases} 1, & \text{for } j \equiv 0 (\mathrm{mod}\ 2), \quad 2 \le j \le n-2 \\ 0, & \text{for } j \equiv 1 (\mathrm{mod}\ 2), \quad 2 \le j \le n-2 \end{cases}$$

$$f(v_j) = \begin{cases} 0, & \text{for } j = 1, n-1, n+1, n+3 \\ 1, & \text{for } j = n, n+2, n+4 \end{cases}$$

Case 2. When $n \equiv 0 (\mathrm{mod}\ 2)$

For $1 \le j \le (n+4)$

$$f(v_j) = \begin{cases} 1, & \text{for } j \equiv 0 (\mathrm{mod}\ 2) \\ 0, & \text{for } j \equiv 1 (\mathrm{mod}\ 2) \end{cases}$$

From the above labelings we get

$$v_f(0) = \left\lceil \frac{p}{2} \right\rceil, v_f(1) = \left\lfloor \frac{p}{2} \right\rfloor \text{ and } e_f(0) = \left\lceil \frac{q}{2} \right\rceil, e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor \text{ for case 1.}$$

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \left\lceil \frac{q}{2} \right\rceil,\ e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor \text{ for case 2.}$$

Thus in both the cases $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$ which proves that the torch graph is cordial. An illustration is given in Figure 2.

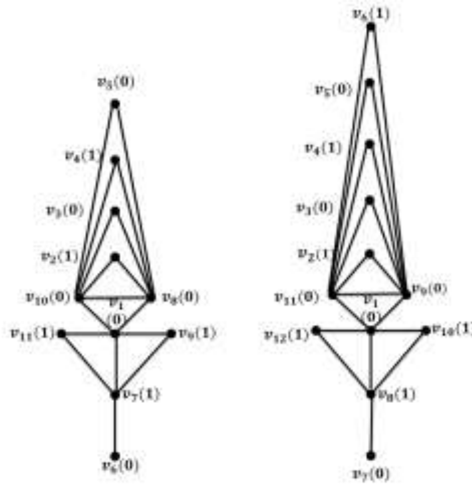

Figure 2. The torch graph O_7 and O_8.

Theorem 2. *Path union of torch graphs is cordial.*

Proof of Theorem 2. Let $G_1, G_2, \ldots, G_m$ be m copies of the torch graph as described in Theorem 1. Each copy of the graph is connected by an edge and thus forms the path union of $G_1, G_2, \ldots, G_n$ which is shown in Figure 3. The path union of torch graphs is denoted by PO_n. The vertices of G_1 are represented as $v_1^1, v_2^1, \ldots, v_{n+4}^1$. Let $v_1^2, v_2^2, \ldots, v_{n+4}^2$ denote the vertices of G_2. The vertices of G_m, (that is, m^{th} copy), are denoted as $v_1^m, v_2^m, \ldots, v_{n+4}^m$. Thus the vertices in the i^{th} copy are denoted as v_j^i, for $1 \leq i \leq m$, $1 \leq j \leq (n+4)$.

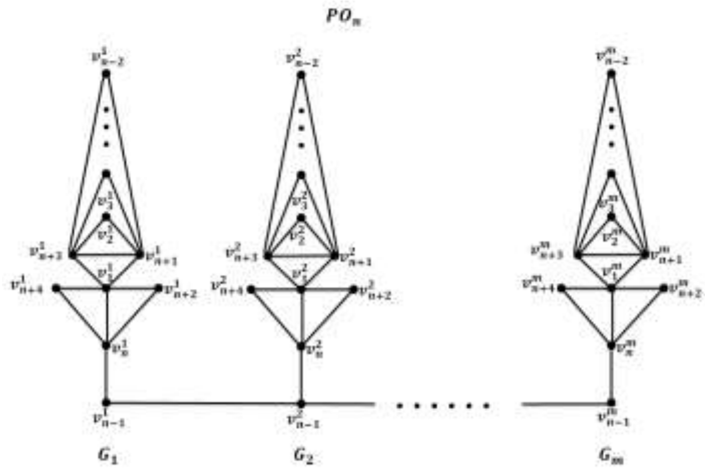

Figure 3. Path union of torch graph PO_n.

Let the number of vertices in the path union of the torch graphs be denoted as p and the number of edges be denoted as q. Note that $p = m(n+4)$ and $q = m(2n+3) + (m-1) = 2m(n+2) - 1$.

The vertices of PO_n are labeled as follows

Case 1. When $n \equiv 1 (\text{mod } 2)$

For $1 \le i \le m$

Case (i). When $i \equiv 1 (\text{mod } 2)$

$$f(v_j^i) = \begin{cases} 1, & \text{for } 2 \le j \le n-2, \quad j \equiv 0 (\text{mod } 2) \\ 0, & \text{for } 2 \le j \le n-2, \quad j \equiv 1 (\text{mod } 2) \end{cases}$$

$$f(v_j^i) = \begin{cases} 0, & \text{for } j = 1, n-1, n+1, n+3 \\ 1, & \text{for } j = n, n+2, n+4 \end{cases}$$

Case (ii). When $i \equiv 0 (\text{mod } 2)$

$$f(v_j^i) = \begin{cases} 0, & \text{for } 2 \le j \le n-2, \quad j \equiv 0 (\text{mod } 2) \\ 1, & \text{for } 2 \le j \le n-2, \quad j \equiv 1 (\text{mod } 2) \end{cases}$$

$$f(v_j^i) = \begin{cases} 1, & \text{for } j = 1, n-1, n+1, n+3 \\ 0, & \text{for } j = n, n+2, n+4 \end{cases}$$

Case (a). When $m \equiv 0 (\bmod 2)$

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \left\lceil \frac{q}{2} \right\rceil,\ e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor$$

$$|\, v_f(0) - v_f(1)\,| = \left| \frac{p}{2} - \frac{p}{2} \right| \text{ and } |\, e_f(0) - e_f(1)\,| = \left| \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor \right| = 1$$

Therefore $|\, v_f(0) - v_f(1)\,| \le 1$ and $|\, e_f(0) - e_f(1)\,| \le 1$.

Case (b). When $m \equiv 1 (\bmod 2)$

$$v_f(0) = \left\lceil \frac{p}{2} \right\rceil,\ v_f(1) = \left\lfloor \frac{p}{2} \right\rfloor \text{ and } e_f(0) = \left\lceil \frac{q}{2} \right\rceil,\ e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor$$

$$|\, v_f(0) - v_f(1)\,| = \left| \left\lceil \frac{p}{2} \right\rceil - \left\lfloor \frac{p}{2} \right\rfloor \right| = 1 \text{ and } |\, e_f(0) - e_f(1)\,| = \left| \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor \right| = 1$$

Therefore $|\, v_f(0) - v_f(1)\,| \le 1$ and $|\, e_f(0) - e_f(1)\,| \le 1$.

Thus the path union of torch graphs is shown to be cordial for Case 1. This is illustrated in Figure 4.

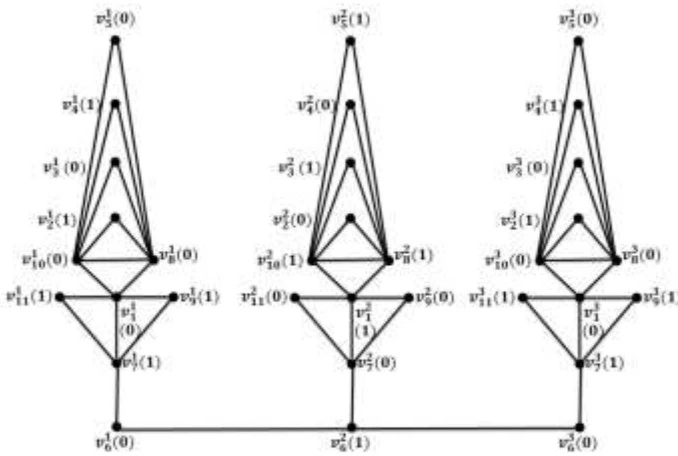

Figure 4. Path union of PO_7 admits cordial.

Case 2. When $n \equiv 0 (\bmod 2)$

For $1 \le i \le m, 1 \le j \le n + 4$.

$$f(v_j^i) = \begin{cases} 1, & \text{for } j \equiv 0 (\text{mod } 2) \\ 0, & \text{for } j \equiv 1 (\text{mod } 2) \end{cases}$$

From the above labelings we get

$$v_f(0) = \frac{p}{2}, v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \left\lfloor \frac{q}{2} \right\rfloor, e_f(1) = \left\lceil \frac{q}{2} \right\rceil$$

$$| v_f(0) - v_f(1) | = \left| \frac{p}{2} - \frac{p}{2} \right| \text{ and } | e_f(0) - e_f(1) | = \left| \left\lfloor \frac{q}{2} \right\rfloor - \left\lceil \frac{q}{2} \right\rceil \right| = 1$$

From the above definition it is clear that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

Thus the path union of torch graphs is proved to be cordial for Case 2.

Theorem 3. *Let G be the cycle C_m and H be the torch graph. Then the cycle of graphs $C(m \circ H)$ is cordial.*

Proof of Theorem 3. Let v_i, for $1 \leq i \leq (n+4)$ denote the vertices of a torch graph H as shown in Figure 1. Let $H_1, H_2, \ldots, H_m$ be m copies of the torch graphs. Let the cycle of graphs $C(m \circ H)$ which was obtained by considering a cycle C_m whose vertices are denoted as $u_1, u_2, \ldots, u_m$ in the anticlockwise direction and replacing each vertex of C_m by the graph H as shown in Figure 5. In other words, each vertex u_i, for $1 \leq i \leq m$ of the cycle C_m is identified with the vertex v_{n-1}^i of H. Let $v_1^1, v_2^1, \ldots, v_{n+4}^1$ denote the vertices of H_1. The vertices of H_2 are represented as $v_1^2, v_2^2, \ldots, v_{n+4}^2$. The vertices in the m^{th} copy, that is H_m are denoted as $v_1^m, v_2^m, \ldots, v_{n+4}^m$. Thus, v_j^i for $1 \leq i \leq m, 1 \leq j \leq (n+4)$ denotes the vertices in the i^{th} copy, that is H_i. We denote the cycle of torch graph as CO_n as shown in Figure 5.

Here $p = m(n+4)$ indicates the number of vertices in CO_n and $q = 2m(n+2)$ indicates the number of edges in CO_n.

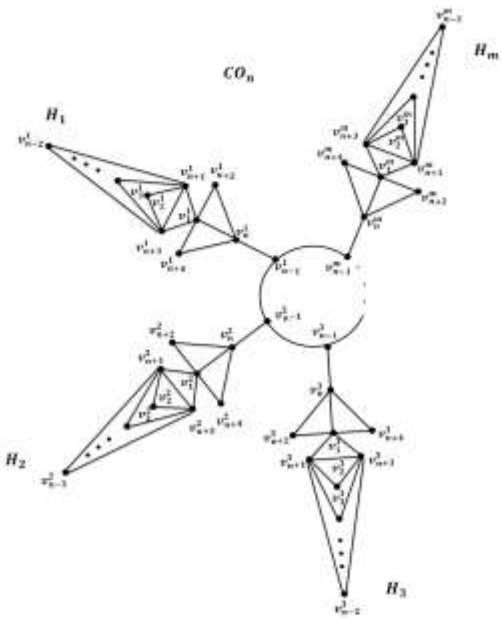

Figure 5. Cycle of torch graph CO_n.

The vertices of CO_n are labeled as follows

Case 1. When $n \equiv 1 (\bmod 2)$

For $1 \leq i \leq m$ where $m \equiv 0 (\bmod 2)$ and $2 \leq j \leq n-2$.

Case (a). When $i \equiv 1 (\bmod 2)$

$$f(v_j^i) = \begin{cases} 1, & \text{for } j \equiv 0 (\bmod 2) \\ 0, & \text{for } j \equiv 1 (\bmod 2) \end{cases}$$

$$f(v_j^i) = \begin{cases} 0, & \text{for } j = 1,\, n-1,\, n+1,\, n+3 \\ 1, & \text{for } j = n,\, n+2,\, n+4 \end{cases}$$

Case (b). When $i \equiv 0 (\bmod 2)$

$$f(v_j^i) = \begin{cases} 0, & \text{for } j \equiv 0 (\bmod 2) \\ 1, & \text{for } j \equiv 1 (\bmod 2) \end{cases}$$

$$f(v_j^i) = \begin{cases} 1, & \text{for } j = 1,\, n-1,\, n+1,\, n+3 \\ 0, & \text{for } j = n,\, n+2,\, n+4 \end{cases}$$

Case 2. When $n \equiv 0(\bmod 2)$

For $1 \leq i \leq m, 1 \leq j \leq n+4$.

$$f(v_j^i) = \begin{cases} 0, & \text{for } j \equiv 1(\bmod 2) \\ 1, & \text{for } j \equiv 0(\bmod 2) \end{cases}$$

From the above labelings we get

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \frac{q}{2},\ e_f(1) = \frac{q}{2}$$

$$|\, v_f(0) - v_f(1)\,| = \left|\frac{p}{2} - \frac{p}{2}\right| = 0 \text{ and } |\, e_f(0) - e_f(1)\,| = \left|\frac{q}{2} - \frac{q}{2}\right| = 0$$

Thus it is clear that $|\, v_f(0) - v_f(1)\,| \leq 1$ and $|\, e_f(0) - e_f(1)\,| \leq 1.$ Hence the cycle of torch graphs CO_n is cordial.

Conclusion

The torch graph, the path union of torch graphs and the cycle of torch graphs is proved to be cordial.

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