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ON MAXIMAL MATCHING COVER PEBBLING NUMBER

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Abstract

An edge pebbling move is defined as the removal of two pebbles from one edge and placing one on the adjacent edge. In this paper, we introduce a new graph invariant called the maximal matching cover pebbling number which is a combination of two graph invariants, 'cover pebbling' and 'maximal matching'. The maximal matching cover pebbling number, $f_{mmcp}(G)$, of a graph G is the minimum number of pebbles that must be placed on $E(G)$, such that after a sequence of pebbling moves the set of edges with pebbles forms a maximal matching regardless of the initial configuration. Some basic results on maximal matching cover pebbling number are found and the number for variants of complete graphs, some path related graphs, cycle C_n, wheel related graphs and some families of trees are determined.

1. Introduction

One of the rapidly developing areas of research in graph theory is graph pebbling, suggested by Lagarias and Sakhs and introduced into the literature by F. R. K Chung. We define a pebbling move as the removal of two pebbles from one vertex and placing one on the adjacent vertex whereas an edge pebbling move is the removal of two pebbles from one edge and placing one on the adjacent edge. The pebbling number of a graph G denoted by $f(G)$, is the least n such that however n pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves [2] and an edge pebbling number of a graph G, denoted by, $f_e(G)$, is the least n such that however n pebbles are placed on the edges of G, we can move a pebble to any

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edge by a sequence of pebbling moves [10]. For a survey of additional results, refer [6].

Also, the variants of graph pebbling can be found in [3, 5, 7, 11].

The motivation for us to introduce this topic is as follows. Let the human body be modeled into a graph where the infected cells of the body be denoted by the vertices and let two cells be joined by an edge if there is a cell junction between these infected cells. Then, by finding the maximal matching number and thereby injecting the drugs into the injured cells will help the drug to cure the maximum number of cells at a time.

In graph pebbling, we can find the minimum quantity of drugs required during the movement of drugs from one cell to another if there is a loss.

Hence, by applying the concept of pebbling in maximal matching, one can find the minimum quantity of drugs required to cure the infected cells of the human body at a time.

With this motivation, we develop the concept of maximal matching cover solution, denoted by, $f_{mmcp}(G)$, and is defined as the minimum number of pebbles that must be placed on $E(G)$, such that after a sequence of pebbling moves the set of edges with pebbles forms a maximal matching regardless of the initial configuration. But, the maximal matching set of a graph G need not be unique which makes this problem quite challenging.

For example, consider a path P_6 with six vertices. Let the vertices of the path P_6 be denoted by v_1, v_2, v_3, v_4, v_5, and v_6 and the corresponding edges which connects the vertices be denoted by $a, b, c, d,$ and e. Then, $\{a, c, e\}, \{b, d\}, \{a, d\}, \{b, e\}$ are the different possible maximal matching edge set. (See Figure 1). Here, finding the minimum number of pebbles needed to place at least one pebble on any of the maximal matching edge set under any configuration of pebbles to the edges of path P_6 makes this problem a difficult one.

In this paper, some basic results on maximal matching cover pebbling number are found and the number for variants of complete graphs, some path related graphs, cycle C_n, wheel related graphs and some families of trees are determined.

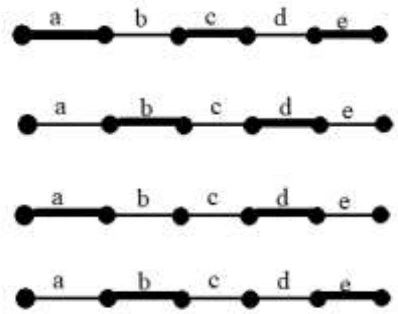

Figure 1. Possible maximal matching edge sets for path P_6.

2. Preliminaries

Definition 1 [1]**.** Given a graph $G = (V, E)$, a *matching M* in G is a set of pairwise non-adjacent edges, none of which are loops; i.e., no edges share common vertices.

Definition 2 [1]**.** A matching M of a graph G is *maximal* if every edge in G has a non-empty intersection with at least one edge in M and a *maximum matching* is a matching that contains the largest possible number of edges in M.

Definition 3 [3]**.** The *cover pebbling number* denoted by $\gamma(G)$ of a graph G is the minimum number of pebbles required to place a pebble on every vertex simultaneously under any initial configuration.

Definition 4 [7]**.** The *covering cover pebbling number* denoted by $\sigma(G)$ is the minimum number of pebbles required such that after a sequence of pebbling moves, the set of vertices with pebbles forms a covering of G.

Definition 5. The distance between the vertices in the corresponding line graph is called the *edge distance* of a graph G.

Definition 6 [1]**.** Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be a connected simple graph. Then $G_1 \cup G_2$ is the graph $G(V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ and $G_1 + G_2$ is $G_1 \cup G_2$ together with the edges joining elements of V_1 to elements of V_2.

3. Main Results

Result 1. For a connected simple graph $G(V, E)$, $f_{mmcp}(G) \geq \left\lceil \frac{|E|}{2} \right\rceil$.

Result 2. Let G be a graph with diameter 2, then $f_{mmcp}(G) = 1$.

Theorem 1. *For a connected simple graph* G, $f_{mmcp}(G) \leq \sigma(G)$, *where* $\sigma(G)$ *is the covering cover pebbling number.*

Proof. Let M be a matching and S be the vertex cover for the given graph G and since $|M| \leq |S|$, the result follows.

Theorem 2. *For a connected simple graph* $G(V, E)$ *with diameter d and edge independence number* $\alpha'(G)$, *the maximal matching cover pebbling number of graph G is,* $f_{mmcp}(G) \leq 2^{d-1}\alpha'(G)$ *and the equality holds for star graph* K_2.

Proof. For a graph G with diameter d, the maximum number of pebbles required to place a pebble on any edge of the graph is 2^{d-1} and the result follows.

Theorem 3. *Let G be a graph with* $|V|$ *vertices and diameter d, then the maximal matching cover pebbling number of graph G is,* $f_{mmcp}(G) \leq 2^{d-1}\left\lfloor \frac{|V|}{2} \right\rfloor$ *and the equality holds for* P_2.

Proof. We know that for a graph G with diameter d, the maximum number of pebbles required to place a pebble on any edge of the graph G is 2^{d-1}. Also, note that every maximum matching is maximal and since maximum matching number $\leq \left\lfloor \frac{|V|}{2} \right\rfloor$[1], the result follows.

Theorem 4. *If* $G(V, E)$ *contains a cut edge* e^*, *then* $f_{mmcp}(G) \geq |E| - (s-1)$, *where s is the number of adjacent edges of* e^*.

Proof. Suppose that $f_{mmcp}(G) < |E| - (s-1)$. Then, by placing a single pebble on all the edges except e^* and the adjacent edges of e^*, it is not possible to produce a maximal matching cover solution which is a contradiction. Hence the proof.

Theorem 5. *Let* $G(V, E), |V| = p, p \geq 2$ *and* $G'(V' E'), |V'| = p', p' \geq 2$ *be two simple connected graphs where* $p' \geq p$. *Then, the maximal matching cover pebbling number of join of two graphs G and* G' *is given by,* $f_{mmcp}(G + G') \leq 4(p-1) + 8f_{mmcp}(G'_{p'-p})$.

Proof. Let $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_p, v_{p+1}, v_{p+2}, \ldots, v_{p'}$ be the vertices of G and G' respectively. Then, $\epsilon = \{u_1v_1, u_2v_2, \ldots, u_pv_p\}$ forms an independent edge set for $G_p + G'_p$. From the construction, it is clear that a maximum of $4(p-1)$ pebbles are required to place a pebble on all the edges on the edge set ϵ and a maximum of $8f_{mmcp}(G'_{p'-p})$ pebbles are required to place at least one pebble on the remaining target edges of $G'_{p'-p}$.

Theorem 6. *Let the graph G be traceable, then*

$$f_{mmcp}(G) \leq \begin{cases} \frac{2^n - 1}{3}, & n \text{ even} \\ 2^{n-1} - 1, & n \text{ odd} \end{cases}$$

The equality holds for n^{th} *power of a graph G.*

Proof. The graph G contains a hamiltonian path since the given graph G is traceable. Assume that the number of vertices on the hamiltonian path is n. Let the vertices of the hamiltonian path be denoted by $v_1, v_2, \ldots, v_n$ and the corresponding edges which joins the vertices be denoted by $e_1, e_2, \ldots, e_{n-1}$. Then, $(e_1, e_3, e_5, \ldots, e_{n-1})$ if n is even and $(e_1, e_3, e_5, \ldots, e_{n-2})$ if n is odd will obviously form a maximal matching for the given graph G. Note that the maximum number of pebbles required to place at least one pebble on an edge which is of distance k is 2^k. Thus, the minimum number of pebbles required to place at least one pebble on the target edge set is $\frac{2^n - 1}{3}$ if n is even and $\frac{2^{n-1} - 1}{3}$ if n is odd.

4. Maximal Matching Cover Pebbling Number for Some Variants of Complete Graphs

Theorem 7. *The maximal matching cover pebbling number for complete graph* K_n, $n \geq 2$ *is*, $f_{mmcp}(K_n) = 4\left(\left\lceil \frac{n}{2} \right\rceil\right) - 3.$

Proof. Consider the distribution of pebbles to any one of the edge of the complete graph K_n, $n \geq 2$. Then, in order to produce a maximal matching cover solution, we need a minimum of $4\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 3$ pebbles. Hence, $f_{mmcp}(K_n) \geq 4\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 3.$

Note that each edge of K_n is at a maximum of distance two from any other edges of K_n and hence a maximum of four pebbles are only required to place a pebble on any edge. Since, $\left\lfloor \frac{n}{2} \right\rfloor$ edges of K_n forms a maximal matching, we need a maximum of $4\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 3$ pebbles to place a pebble on the target edge set and the result follows.

Theorem 8. *For* $l_1 \leq l_2 \leq \ldots \leq l_r$, *let* $K_{l_1, l_2, l_3, \ldots, l_r}$ *be the complete* r-*partite graph with* $l_1, l_2, \ldots l_r$ *vertices in the vertex classes* $m_1, m_2, \ldots, m_r$ *respectively*. *Then*,

$$f_{mmcp}(K_{l_1, l_2, l_3, \ldots, l_r}) = \begin{cases} 4(l_1 + l_3 + l_5 + \ldots, l_{r-1}) - 3, & m_r \text{ even} \\ 4(l_2 + l_4 + l_6 + \ldots, l_{r-1}) - 3, & m_r \text{ odd} \end{cases}$$

Proof. Consider the configuration of placing all the pebbles on a single edge of $K_{l_1, l_2, l_3, \ldots, l_r}$. Then, a minimum of $4(l_1 + l_3 + l_5 + \ldots, l_{r-1}) - 3$, if m_r is even or $4(l_2 + l_4 + l_6 + \ldots, l_{r-1}) - 3$, if m_r is odd pebbles are required in order to produce a maximal matching cover solution. Thus, we obtained the lower bound for maximal matching cover pebbling number for complete r-partite graph $K_{l_1, l_2, l_3, \ldots, l_r}$.

Note that, for a complete r-partite graph, the maximal matching set contains $l_1 + l_3 + l_5 + \ldots, l_{r-1}$ and $l_2 + l_4 + l_6 + \ldots, l_{r-1}$ edges when m_r is even and odd respectively. Here, since each edge of $K_{l_1, l_2, l_3, \ldots, l_r}$ is at a distance of two from all other edges, a maximum of four pebbles are required to place a pebble on any of the edge from an edge of $K_{l_1, l_2, l_3, \ldots, l_r}$. Hence, we obtain the upper bound for maximal matching cover pebbling number for complete r-partite graph $K_{l_1, l_2, l_3, \ldots, l_r}$.

Corollary 1. *The maximal matching cover pebbling number of the complete bipartite graph* $K_{p,q}$ *is,* $f_{mmcp}(K_{p,q}) = 1 + 2^2\{\min(p, q) - 1\}$.

Proof. Consider the configuration of placing all the pebbles on any edge of complete bipartite graph $K_{p,q}$. Then, $f_{mmcp}(K_{p,q}) \geq 1 + 2^2\{\min(p, q) - 1\}$. Note that $\{\min(p, q)\}$ edges forms the maximal matching for the complete bipartite graph $K_{p,q}$. Since, each edge is at a maximum of distance two from all other edges, we require a maximum of four pebbles to place a pebble on any edge of $K_{p,q}$ and hence the result follows.

Result 3. For a star graph $K_{1,m}$, $f_{mmcp}(K_{1,m}) = 1$.

5. Maximal Matching Cover Pebbling Number for Some Path Related Graphs

Theorem 9. *For a path* P_n, $n \geq 4$, *the maximal matching cover pebbling number is given by,*

$$f_{mmcp}(P_n) = \begin{cases} \dfrac{2^n - 1}{7}, \; n \cong 0 \bmod 3 \\ \dfrac{2((2^3)^{\left\lceil \frac{n}{4} \right\rceil} - 1)}{7}, \; n \cong 1 \bmod 3 \\ \dfrac{2^{n-2} - 1}{7} + 2^{n-3}, \; n \cong 2 \bmod 3 \end{cases}$$

Proof. Let us denote the vertices of path P_n by $w_1, w_2, \ldots, w_n$ and the edges by $e_1, e_2, \ldots, e_{n-1}$. The maximal matching cover pebbling number for path P_n, $n \leq 3$ is as follows:

$$f_{mmcp}(P_n) = \begin{cases} 0, & n = 1 \\ 1, & n = 2, 3 \end{cases}$$

In order to complete the proof, let us divide the proof into following cases.

Case 1. $n \cong 0 \bmod 3$

Consider the distribution of all the pebbles on the first edge e_1. Then, a minimum of $1 + 2^3 + 2^6 + 2^9 + \ldots$ pebbles are required in order to place a pebble on all the edges of maximal matching set. Thus, $f_{mmcp}(P_n) \geq \frac{2^n - 1}{7}$, $n \cong 0 \bmod 3$, $n \geq 4$.

Let us prove $f_{mmcp}(P_n) \leq \frac{2^n - 1}{7}$ by induction on *n*. The result is obvious when $n = 3$. Assume that the result holds good for all P_{n-3}, $n \cong 0 \bmod 3$. Note that e_{n-3}, e_{n-2} and e_{n-1} are the only extra edges of P_n, $n \geq 3$ when compare to P_{n-3}, $n \geq 4$. Since, the edge e_{n-2} forms a maximal matching for the edges e_{n-3}, e_{n-2} and e_{n-1} and since the edge e_{n-2} is at a maximum distance of $n - 3$ from all other edges of P_n, we need only a maximum of 2^{n-3} pebbles to place a pebble on the edge e_{n-2}. So, we are left with $\frac{2^n - 1}{7} - 2^{n-3} = \frac{2^{n-1} - 1}{7}$ pebbles and is sufficient to place a pebble on all the edges of the maximal matching set of P_{n-3}, $n \geq 3$ and the proof follows.

Case 2. $n \cong 1 \bmod 3$

Consider the configuration of placing all the pebbles on the first edge e_1. Consequently, we need a minimum of $\frac{2((2^3)^{\left\lceil \frac{n}{4} \right\rceil} - 1)}{7}$ pebbles in order to produce a maximal matching cover solution. Thus, $f_{mmcp}(P_n) \geq \frac{2((2^3)^{\left\lceil \frac{n}{4} \right\rceil} - 1)}{7}$, $n \cong 1 \bmod 3$, $n \geq 4$.

We will prove the upper bound by induction on n. The result is obvious when $n = 4$. Assume that the hypothesis holds for all P_{n-3}, $n \cong 1 \bmod 3$, $n \geq 4$. In this case, we need only a maximum of 2^{n-3} pebbles to place a pebble on the edge e_{n-2}. Consequently, we are left with $\frac{2((2^3)^{\lceil \frac{n}{4} \rceil} - 1)}{7} - 2^{n-3} = \frac{2((2^3)^{\lceil \frac{n-1}{4} \rceil} - 1)}{7}$ pebbles and is sufficient to place a pebble on all the edges of the maximal matching set of P_{n-3}, $n \geq 4$ and the proof follows.

Case 3. $n \cong 2 \bmod 3$

e_{n-2} and e_{n-1} are the only two extra edges of P_n, $n \cong 2 \bmod 3$ when compare to Case 1. Also, a maximium of 2^{n-3} pebbles are required to place a pebble on the edge e_{n-2} and the result follows.

Definition 7. The Braid graph B_n is obtained from a pair of paths P_n and P_m by joining i^{th} vertex of path P_n with $(i+1)^{th}$ vertex of the path P_m and the i^{th} vertex of the path P_m with $(i+2)^{th}$ vertex of the path P_n for all $1 \leq i \leq n-2$ [8].

Theorem 10. *The maximal matching cover pebbling number of Braid graph B_n, $n \geq 3$ is*

$$f_{mmcp}(B_n) = \begin{cases} 5 + \dfrac{2^4(2\frac{2n}{3} - 1)}{3} & n \cong 0 \bmod 3 \\ 7 + \dfrac{2^5(2\frac{2(n-4)}{3} - 1)}{3} & n \cong 0 \bmod 3 \\ 5 + \dfrac{2^4(5^{\frac{2n}{3}-2} - 1)}{3} + 2(2^{\lfloor \frac{2n}{3} \rfloor - 1}), & n \cong 2 \bmod 3 \end{cases}$$

Proof. Let the vertices of the path P_1 and P_2 be denoted by $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ respectively. In order to prove the theorem, let us divide the proof into following cases.

Case 1. $n \cong 0 \bmod 3$

Consider the configuration of placing all the pebbles on the first edge of any path P_1 or P_2. Then, we need a minimum of $1 + 2^2 + 2(2^3 + 2^5 + 2^7 + \ldots)$ pebbles in order to produce a maximal matching cover solution. Therefore, $f_{mmcp}(B_n) \geq 5 + \frac{2^4 (2^{\frac{2n}{3}-2} - 2)}{3}$, $n \cong 0 \bmod 3$.

Let us prove the upper bound by induction on n. The result is obvious when $n = 3$. Assume that the assertion is true for all B_{n-3}, $n \cong 0 \bmod 3$. Consider the distribution of all the pebbles on the Braid graph B_n, $n \cong 0 \bmod 3$, $n \geq 3$. Note that $\{u_{n-2}, u_{n-1}, u_n\}$ and $\{v_{n-2}, v_{n-1}, v_n\}$ are the only extra vertices from the path P_1 and P_2 of B_n when compare to B_{n-3}, $n \cong 0 \bmod 3$. Since both the edges $v_{n-2}v_{n-1}$ and $u_{n-2}u_{n-1}$ are at a distance of $\frac{2n-5}{3}$ from any edge of either P_1 or P_2, we need a maximum of $2(2^{\frac{2n-3}{3}})$ pebbles to place a pebble on both the edges $u_{n-2}u_{n-1}$ and $v_{n-2}v_{n-1}$ of P_1 and P_2.

Also, $5 + 2^4 \frac{(2^{\frac{2n}{3}-2} - 1)}{3} - 2(2^{\frac{2n-3}{3}}) = 5 + 2^4 \frac{(2^{\frac{2(n-1)}{3}-2} - 1)}{3} n \cong 0 \bmod 3,$

and hence the result follows by induction.

Case 2. $n \cong 1 \bmod 3$.

Consider the distribution of all the pebbles on the first edge of Braid graph B_n, $n \cong 1 \bmod 3$. Then, a minimum of $1 + 2 + 2^2 + 2^5(1 + 2^2 + 2^4 + \ldots)$ pebbles are required in order to produce a maximum matching cover solution.

Consequently, $f_{mmcp}(B_n) \geq 7 + \frac{2^5 (2^{\frac{2(n-4)}{3}} - 1)}{3}$, $n \cong 1 \bmod 3$.

Let us prove the upper bound by induction on n. Then, a maximum of $2^{\frac{2n-5}{3}}$ pebbles are required in order to place a pebble on both the edges

$u_{n-2}u_{n-1}$ and $v_{n-2}v_{n-1}$ on the paths P_1 and P_2 by a sequence of pebbling moves.

Hence the result follows by induction as $7 + 2^5 \frac{(2^{\frac{2(n-4)}{3}} - 1)}{3} - 2^{\frac{2n-5}{3}} = 7 + 2^5 \frac{(2^{\frac{2(n-5)}{3}} - 1)}{3}$, $n \cong 1 \bmod 3$.

Case 3. $n \cong 2 \bmod 3$

$\{u_{n-1}, u_n\}$ and $\{v_{n-1}, v_n\}$ are the only extra vertices of B_n when compare to B_{n-2}, $n \cong 2 \bmod 3$. Note that, in order to place a pebble on both the target edges, we require only a maximum of $2(2^{\left\lfloor \frac{2n}{3} \right\rfloor - 1})$ and hence the result follows.

6. Maximal Matching Cover Pebbling Number for Cycle

Lemma 1. *The maximal matching cover pebbling number of cycle* C_n, $f_{mmcp}(C_n)$ *can be obtained by placing all the pebbles on a single edge*.

Proof. The proof is by contradiction. Assume that the worst case configuration is by placing the pebbles on more than one set of consecutively edges (islands). Then, the cardinality of this set be at most two. If not, one could move all the pebbles to their inner one or two edges, causes a larger number of pebbles to form a maximal matching cover solution which is a contradiction. Hence, each island is having at most two edges. Now, again consider the case of moving all the pebbles on to a single island. Then, it requires more than $f_{mmcp}(C_n)$ pebbles to form the maximal matching cover solution which is again a contradiction. Thus, we have the worst case configuration consisting of a single island with two pebbled edges. Clearly, the worst case is by placing $f_{mmcp}(G) - 1$ pebbles on any one edge, say e_1 and a pebble on the other edge, say e_2, since it need more pebbles to reach the e_2 side of the cycle. Since all the pebbles were on e_1, we need at least more pebbles to produce a maximal matching cover solution, again a contradiction and hence the result follows.

Theorem 11. *The maximal matching cover pebbling number for cycle* C_n *is given by*,

$$f_{mmcp}(C_n) \leq \begin{cases} 2f_{mmcp}(P_{\frac{n}{2}}) + 2^{\frac{n}{2}}, \; n \; even \\ 2f_{mmcp}(P_{\frac{n+1}{2}}), \; n \; odd \end{cases}$$

Proof. By the above lemma, let us assume that all the pebbles are placed on a single edge, say, e_1 of C_n. Then, we have two identical paths with $\frac{n-2}{2}$ edges on both the paths and an edge which is of distance $\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ edges on both the paths if n is odd to cover. So, we can produce a maximal matching cover solution for the paths and the edge with $2f_{mmcp}(P_{\frac{n}{2}}) + 2^{\frac{n}{2}}$ pebbles if n is even and $2f_{mmcp}(P_{\frac{n+1}{2}})$ pebbles if n is odd and hence the result follows.

7. Maximal Matching Cover Pebbling Number for Some Wheel Related Graphs

Definition 8. A wheel graph denoted by W_n on $n+1$ vertices is the graph obtained from $K_1 + C_n$, where C_n is a cycle with n vertices [5].

Theorem 12. *The maximal matching cover pebbling number of wheel graph* $K_1 + C_n$, *is*

$$f_{mmcp}(W_n) = \begin{cases} 13 + 8(\frac{n-9}{3}), \; n \cong 0 \bmod 3 \\ 9 + 8(\frac{n-7}{3}), \; n \cong 1 \bmod 3 \\ 5 + 8(\frac{n-5}{3}), \; n \cong 2 \bmod 3 \end{cases}$$

Proof. Let the vertices of the cycle C_n be denoted by $u_1, u_2, \ldots, u_n$ and the apex vertex as u. Let the edges which connects the vertices of the cycle C_n be denoted by $e_1, e_2, \ldots, e_n$ and let $e'_i, i = 1, 2, \ldots, n$ be the edge which connects the vertex v_i and the apex vertex u.

Let us divide the proof into following cases based on the distribution of pebbles to the edges of wheel graph W_n.

Case 1. $n \cong 0 \bmod 3$

Consider the distribution of all the pebbles on an edge of cycle C_n. Since, there exists $\frac{n}{3} + 1$ independent edges, we need at least $13 + 8(\frac{n-9}{3})$, $n \cong 0 \bmod 3$ pebbles to produce a maximal matching cover solution (See Figure 2). Therefore, $f_{mmcp}(W_n) \geq 13 + 8(\frac{n-9}{3})$, $n \cong 0 \bmod 3$.

Now, let us divide the case into following subcases based on the distribution of pebbles to the edges e_i, $i = 1, 2, \ldots, n$ and e'_i, $i = 1, 2, \ldots, n$.

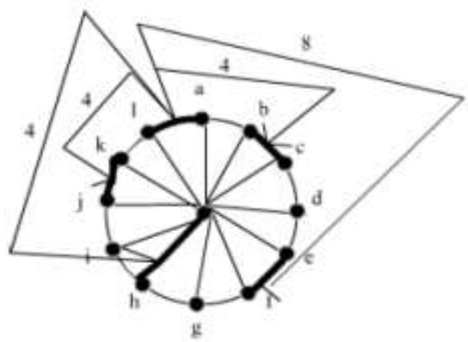

Figure 2. Distribution of pebbles for the wheel graph W_{12}.

Case 1.1. Any one of the e'_i, $i = 1, 2, \ldots, n$ has $t \geq 4(\frac{n}{3})$ pebbles.

Let us denote the pebbled edge by e^*. Place a pebble on the adjacent edge of e^* on the cycle C_n by a pebbling move. Name the pebbled edge on the cycle C_n as e^{**}. Now, place a pebble on any e'_i, $i = 1, 2, \ldots, n$ where the edge should not be adjacent to e^{**} by a pebbling move. So, here, a total of four pebbles are used. Now, in order to produce a maximal matching cover solution, we need to place a pebble on the remaining $(\frac{n}{3} - 1)$ independent edges of cycle C_n and we require a maximum of $4(\frac{n}{3} - 1)$ pebbles for it by a sequence of pebbling moves. Hence the result follows.

Case 1.2. Let none of the e'_i, $i = 1, 2, 3, \ldots, n$ has $t < \frac{4n}{3}$ pebbles, $(t \geq 1)$.

Then, there are at least $13 + 8(\frac{n-9}{3}) - (\frac{4n}{3})$ pebbles on the edges of the cycle C_n and the edges e'_i, $i = 1, 2, 3, \ldots, n$. Choose any one maximal matching edge set with more number of pebbled edges. If any of the two maximal matching edge sets are having equal number of pebbled edges, then choose the set which has more number of edges with odd number of pebbles. If not, choose any one such set. Without loss of generality, let us assume that there are k number of edges which receives pebbles in the set $S(say)$. Thus, clearly $k \geq 1$. Note that there are $\frac{n}{3}$ independent edges for cycle C_n. So, our aim is to place a pebble $\frac{n}{3} - k$ edges of S.

If any one of the $\frac{n}{3} - k$ edges of S is adjacent to an edge having at least two pebbles, then, pebble those edges of S by a pebbling move. Let us assume that k' number of edges receives pebbles in this way. i.e., we have used $2k'$ number of pebbles to pebble these k' number of edges of S. Also, if anyone of the $\frac{n}{3} - k$ edges of S is adjacent to an edge having at least four pebbles then, pebble those edges of S by a pebbling move. Let us assume that k'' number of edges receives pebble in this way, i.e., we have used $4k''$ number of pebbles to place a pebble on these k'' number of edges of S. Also, if anyone of the $\frac{n}{3} - k$ edges of S is adjacent to an edge e'_i, $i = 1, 2, 3, \ldots, n$ having at least two pebbles, then pebble those edges of S by a pebbling move. Let us assume that k''' number of edges of S receives pebble in this way, i.e., we have used $2k'''$ number of pebbles to place a pebble on these k''' number of edges of S.

Now, place at most one pebble on every edge of $C_n - S$ and move the remaining to any one of the corresponding edge e'_i, $i = 1, 2, 3, \ldots, n$ which connects the hub vertex and the cycle C_n by a pebbling move. Also, place at most two pebbles on the edges of S and transfer the remaining to any one of the corresponding edge e_i', $i = 1, 2, 3, \ldots, n$ by a pebbling move. Hence, we have at least $t - 2k''' + \left\lfloor \frac{13 + 8(\frac{n-9}{3}) - t - 2k - 2k' - 4k''}{2} \right\rfloor$ pebbles on the edges e_i', $i = 1, 2, 3, \ldots, n$.

As we need to place a pebble on $(\frac{n}{3} - k - k' - k'' - k''')$ edges of the cycle C_n and an independent edge e_i', $i = 1, 2, 3, \ldots, n$, we need at most $1 + 4(\frac{n}{3} - k - k' - k'' - k''')$ pebbles on the edges e_i', $i = 1, 2, 3, \ldots, n$.

But, $t - 2k''' + \left\lfloor \frac{13 + 8(\frac{n-9}{3}) - t - 2k - 2k' - 4k'}{2} \right\rfloor - 1 - 4(\frac{n}{3} - k - k' - k'' - k''')$

$= 2k''' + 3k + 3k' + 2k'' - 5 \geq 0$ as $k \geq 1, t \geq 1$ and k', k'', k''' cannot be equal to zero simultaneously.

Case 2. $n \cong 1 \bmod 3$

It is not possible to place a pebble on the maximal matching set if we place $8(\frac{n-7}{3})$ pebbles on any edge of the cycle C_n of W_n, $n \cong 1 \bmod 3$. Therefore, $f_{mmcp}(W_n) \geq 9 + 8(\frac{n-7}{3})$, $n \geq 12$.

Case 2.1. Any one of the e_i', $i = 1, 2, 3, \ldots, n$ has $4(\frac{n+2}{3})$ pebbles on it.

Similarly, by distributing the pebbles as we discussed in Case 1.1, we need a maximum of $4(\frac{n+2}{3})$ pebbles to produce a maximal matching cover solution.

Case 2.2. Any one of the e_i', $i = 1, 2, 3, \ldots, n$ has $t < 4(\frac{n+2}{3})$ pebbles on it.

Then, the cycle C_n of W_n has at least $9 + 8(\frac{n+7}{3}) - 4(\frac{n+2}{3})$ pebbles on it. Then, by proceeding as in Case 1.2, we can obtain a maximal matching cover solution for W_n, $n \cong 1 \bmod 3$.

Case 3. $n \cong 2 \bmod 3$

Consider the configuration of placing all the pebbles on any one of the edge of cycle C_n of W_n, $n \cong 2 \bmod 3$. Then, we need a maximum of $5 + 8(\frac{n+5}{3})$, $n \cong 2 \bmod 3$ pebbles to produce a maximal matching cover solution. Hence, $f_{mmcp}(W_n) \geq 5 + 8(\frac{n+5}{3})$, $n \cong 2 \bmod 3$.

Case 3.1. Any one of the e_i', $i = 1, 2, 3, \ldots, n$ has $t \geq 1 + 4(\frac{n+2}{3})$ pebbles on it.

In a similar manner, by distributing the pebbles as we discussed in Case 1.1, a maximum of $1 + 4(\frac{n+2}{3})$ pebbles are required to produce a maximal matching cover solution.

Case 3.2. None of the e_i', $i = 1, 2, \ldots, n$ has $t < 1 + 4(\frac{n+2}{3})$ pebbles on it $(t \geq 1)$.

In this case, a minimum of $5 + 8(\frac{n-5}{3}) - 1 - 4(\frac{n-2}{3})$ pebbles are distributed to the edges of cycle C_n of W_n, $n \cong 2 \bmod 3$. Then, by proceeding as in Case 1.2, the result follows.

Definition 9. A Friendship graph denoted by F_n can be constructed by joining n cycles C_n with a common vertex. It has $2n+1$ vertices and $3n$ edges [4].

Theorem 13. *The maximal matching cover pebbling number for friendship graph* F_n *is given by,* $f_{mmcp}(F_n) = 2^3(n-1) + 2$, $n > 1$.

Proof. Let us denote the edges which are adjacent to the apex vertex be

$e_1, e_2, \ldots, e_{2n}$ and the edges between the two edge $e_i, i = 1, 2, \ldots, 2n$ be $w_i, i = 1, 2, \ldots, n.$ In order to prove the theorem, let us divide the proof into following cases based on the number of pebbles distributed to the edges $e_i, i = 1, 2, \ldots, 2n$ and $w_i, i = 1, 2, \ldots, n.$

Case 1. Let any one of the $e_i, i = 1, 2, \ldots, n$ has $t \geq 4(n-1)+1$ pebbles.

Without loss of generality, let the pebbled edge of F_n be denoted by e_1. Retain one pebble on the edge e_1 itself. Consequently, all the edges $e_i, i = 2, 3, \ldots, 2n$ are adjacent to a pebbled edge. Now, in order to produce a maximal matching cover solution, we need to place at least one pebble on $(n-1)$ independent edges $w_i', i = 2, 3, \ldots, n.$ Since, these independent $w_i, i = 2, \ldots, n$ edges are of distance two from e_1, we need a maximum of $2^2(n-1)$ pebbles and the proof follows.

Case 2. Let none of the $e_i, i = 1, 2, \ldots, n$ has $t < 4(n-1)+1$ pebbles on it.

Then, there are at least $2^3(n-1)+2-(4(n-1)+1) = 4n-3$ pebbles on the edges $w_i, i = 1, 2, \ldots, n$ and $e_i, i = 1, 2, \ldots, n.$ Let us assume that m number of $w_i, i = 1, 2, \ldots, n$ edges have pebbles on it. Now, if there exists at least two pebbles on any $e_i, i = 1, 2, \ldots, 2n.$ then place a pebble on the corresponding non-pebbled edge $w_i, i = 2, \ldots, n$ by a pebbling move. Let us assume that m' edges get pebbles in this way and hence we have used $2m'$ number of pebbles to pebble m' edges by a pebbling move.

Keep a maximum of two pebbles on the edges $w_i, i = 1, 2, \ldots, n$ and transfer the remaining pebbles from the edge $w_i, i = 1, 2, \ldots, n$ to any one of the corresponding edge $e_i, i = 1, 2, 3, \ldots, 2n$ by a pebbling move. Consequently, the edges $e_i, i = 1, 2, \ldots, 2n$ of the friendship graph F_n has a minimum of $\left\lfloor \frac{2^3(n-1)+2-(2m+2m'+t)}{2} \right\rfloor$ pebbles on it.

Now, our aim is to place at least one pebble on the remaining

$(n-1-m-m')$ number of w_i, $i = 2, \ldots, n$ edges and any one of the edge e_i, $i = 1, 2, \ldots, 2n$, where the edge e_i, $i = 1, 2, \ldots, 2n$ is not adjacent to a pebbled edge w_i, $i = 1, 2, \ldots, n$. So, in order to produce a maximal matching cover solution, we need a maximum of $4(n-1-m-m')+2$ pebbles more.

But, $t + \left\lfloor \frac{2^3(n-1)+2-(2m+2m'+t)}{2} \right\rfloor - (4(n-1-m-m')+2) = 4n-1$ $+ 3m + 3m' + t - \left\lfloor \frac{t}{2} \right\rfloor \geq 0$ Hence the proof.

8. Maximal Matching Cover Pebbling Number for Some Families of Trees

Theorem 14. The maximal matching cover pebbling number of comb graph $P_n \odot K_1$ is, $f_{mmcp}(P_n \odot K_1) = \frac{2(4^{\left\lceil \frac{n}{2} \right\rceil} - 1)}{3}$, $n \geq 2$.

Proof. Let the vertices of the path P_n be denoted by $v_1, v_2, \ldots, v_n$ and the corresponding pendant edges from the vertices of the path P_n be $u_1, u_2, \ldots, u_n$.

Consider the configuration of placing all the pebbles on the edge v_1u_1. Then, a minimum of $2 + 2^3 + 2^5 + \ldots$ pebbles are required in order to produce a maximal matching cover solution. Therefore, $f_{mmcp}(P_n \odot K_1)$ $\geq \frac{2(4^{\left\lceil \frac{n}{2} \right\rceil} - 1)}{3}$, $n \geq 2$.

Let us prove the upper bound by induction on n. Since $P_n \odot K_1 \cong P_4$, the result follows from Theorem 9 when $n = 2$. Assume that the assertion is true for all comb graphs $P_{n-1} \odot K_1$. Consider the configuration of all the pebbles on $P_n \odot K_1 \cdot v_{n-1}v_n$ and u_nu_n are the only extra edges of $P_n \odot K_1$ when compare to $P_{n-1} \odot K_1$. But, we need only a maximum of 2^{n-1} or 2^n pebbles to place a pebble on the edge $v_{n-1}v_n$ or v_nu_n when n is even or odd respectively. Also, the remaining pebbles are sufficient to produce a maximal matching cover solution for $P_{n-1} \odot K_1$ and hence the result follows by induction.

Definition 10. An olive tree T_n is a rooted tree consisting of n branches, where i^{th} branch is a path of length i [9].

Theorem 15. *The maximal matching cover pebbling number of Olive Tree* T_n, $n \geq 3$ *is,*

$f_{mmcp}(T_n) =$

$$\begin{cases} \dfrac{2^{n+3}-1}{7} + 2^{n+1}(\dfrac{n-3}{3}) + \sum_{j=1}^{\frac{n-3}{3}} \sum_{i=1}^{n-3j} 2^{n+i}, \ n \cong 0 \bmod 3 \\ \dfrac{2(2^{n+3}-1)}{7} + 2^{n+1}(\dfrac{2n-5}{3}) + \sum_{j=1}^{\frac{n-4}{3}} \sum_{i=1}^{n-1-3j} 2^{n+1+i}, \ n \cong 1 \bmod 3 \\ \dfrac{2^{n+1}-1}{7} 7(2^n) + 2^{n+3}(\dfrac{n-5}{3}) \sum_{j=1}^{\frac{n-5}{3}} \sum_{i=1}^{n-2-3j} 2^{n+2+i}, \ n \cong 2 \bmod 3 \end{cases}$$

Proof. Let us denote the root vertex by u. Let u_i^j be the remaining vertices of the Olive tree T_n where i, j indicates the branch and the vertices of T_n respectively.

Let us divide the proof into following cases based on the number of pebbles distributed to the edges of Olive tree T_n.

Case 1. $n \cong 0 \bmod 3$

Consider the configuration of all the pebbles on the edge $u_n^n u_{n-1}^n$. Then, a minimum of $\dfrac{2^{n+3}-1}{7} + 2^{n+1}(\dfrac{n-3}{3}) + \sum_{j=1}^{\frac{n-3}{3}} \sum_{i=1}^{n-3j} 2^{n+i}$, $n \cong 0 \bmod 3$ pebbles are required to produce a maximal matching cover solution.

Now, let us prove the upper bound by induction on n. The assertion is true when $n = 3$. Assume that the result is true for all Olive tree T_{n-3}, $n \cong 0 \bmod 3$. Consider the distribution of all the pebbles on the Olive tree T_n, $n \geq 3$, $n \cong 0 \bmod 3$. Note that by adjoining P_{n-2}, P_{n-1}, and P_n to the root vertex of T_{n-3}, we obtain T_n. Also, the maximum distance between the edges of P_{n-1}, P_n, and P_{n-1} to the other edges of T_{n-3} is $n-3$. So, a maximum of $2^{n-3}(f_{mmcp}(P_{n-2}) + f_{mmcp}(P_{n-1}) + f_{mmcp}(P_n))$ pebbles are required to place at least one pebble on any edge of the maximal matching of P_{n-2}, P_{n-1} and P_n.

Also, $\frac{2^{n+3}-1}{7} + 2^{n+1}(\frac{n-3}{3}) + \sum_{j=1}^{\frac{n-3}{3}} \sum_{i=1}^{n-3j} 2^{n+i} - 2^{n-3}\ (f_{mmcp}(P_{n-2})$ $+f_{mmcp}(P_{n-1}) + f_{mmcp}(P_n)) \geq f_{mmcp}(T_{n-3})$ and hence the result follows by induction.

Case 2. $n \cong 1 \bmod 3$

Consider the configuration of all the pebbles on a single edge $u_n^n u_{n-1}^n$. Then, in order to produce a maximal matching cover solution, a minimum of $\frac{2(2^{n+2}-1)}{7} + 2^{n+1}(\frac{2n-5}{3}) + \sum_{j=1}^{\frac{n-4}{3}} \sum_{i=1}^{n-1-3j} 2^{n+1+i}$, $n \cong 1 \bmod 3$ pebbles are required to place a pebble on all the edges of maximal matching set.

Let us prove the upper bound by induction on n. The result is obvious when $n = 4$. Assume that the assertion is true for all Olive Tree T_{n-3}, $n \cong 1 \bmod 3$. Consider the distribution of all the pebbles on the Olive tree T_{n-3}, $n \geq 3$, $n \cong 1 \bmod 3$. Then, as we discussed in Case 1, a maximum of $2^{n-3}(f_{mmcp}(P_{n-2}) + f_{mmcp}(P_{n-1}) + f_{mmcp}(P_n))$ pebbles are required to place at least one pebble on any edge of the maximal matching set of P_{n-2}, P_{n-1} and P_n.

Also, $\frac{2(2^{n+2}-1)}{7} + 2^{n+1}(\frac{2n-5}{3}) + \sum_{j=1}^{\frac{n-4}{3}} \sum_{i=1}^{n-1-3j} 2^{n+1+i} - 2^{n-3}$ $2^{n-3}(f_{mmcp}(P_{n-2}) + f_{mmcp}(P_{n-1}) + f_{mmcp}(P_n)) \geq f_{mmcp}(T_{n-3})$ and the result follows by induction.

Case 3. $n \cong 2 \bmod 3$

If we distribute all the pebbles on the edge $u_n^n u_{n-1}^n$, then a minimum of

$\frac{2^{n+1}-1}{7}+7(2^n)+2^{n+3}(\frac{n-5}{3})+\sum_{j=1}^{\frac{n-5}{3}}\sum_{i=1}^{n-2-3j}2^{n+2+i}$ pebbles are required to produce a maximal matching cover solution.

The upper bound can be proved by induction on n. The result is true when $n = 5$. Assume that the result is true for all Olive tree T_{n-3}, $n \geq 3$, $n \cong 2 \bmod 3$. Then, as we discussed in Case 1, a maximum of $2^{n-3}(f_{mmcp}(P_{n-2})+f_{mmcp}(P_{n-1})+f_{mmcp}(P_n))$ pebbles are required to place at least one pebble on all the edges of maximal marching of P_{n-2}, P_{n-1} and P_n. However, $\frac{2^{n+1}-1}{7}+7(2^n)+2^{n+3}(\frac{n-5}{3})+\sum_{j=1}^{\frac{n-5}{3}}\sum_{i=1}^{n-2-3j}2^{n+2+i}-2^{n-3}$ $2^{n-3}(f_{mmcp}(P_{n-2})+f_{mmcp}(P_{n-1})+f_{mmcp}(P_n)) \geq f_{mmcp}(T_{n-3})$ and hence the result follows by induction.

9. Conclusion

Graph pebbling and matching are the two main areas of research in graph theory which have tremendous applications. By combining the two graph invariants, namely, graph pebbling and matching, one can find the solution for many real world problems. So, in this paper, we combined the graph invariants cover pebbling and maximal matching and obtained a new graph invariant called 'maximal matching cover pebbling'. Some basic results on maximal matching cover pebbling number are found, a bound for join of two graphs G and G' and the number for variants of complete graphs, some path related graphs, cycle C_n, wheel related graphs and some families of trees are determined.

Given below are some interesting open problems.

- Finding the maximal matching cover pebbling number for networks and for directed graphs.
- Finding the exact maximal matching cover pebbling number for a graph with diameter d.
- Finding $f_{mmcp}(G) \leq k$?

References

[1] J. A. Bondy and U. S. Murty, Graph theory with applications, London: Macmillan, 1976.

[2] F. R. K. Chung, Pebbling in hypercubes, SIAM Journal on Discrete Mathematics 2(4) (1989), 467-472.

[3] B. Crull, T. Cundiff, P. Feltman, G. H. Hurlbert, L. Pudwell, Z. Szaniszlo and Z. Tuza, The cover pebbling number of graphs, Discrete Mathematics 296(1) (2005), 15-23.

[4] H. Fernau, J. F. Ryan, K. A. Sugeng, A sum labelling for the generalised friendship graph, Discrete Mathematics (2008), 734-740.

[5] J. Gardner, A. P. Godbole, A. M. Teguia, A. Z. Vuong, N. Watson and C. R. Yerger, Domination cover pebbling: graph families, JCMCC 64 (2008), 255-271.

[6] G. H. Hurlbert, A survey of graph pebbling, In Congr. Numer. 139 (1999), 41-64.

[7] A. Lourdusamy and A. Punitha Tharani, Covering cover pebbling number, Utilitas Mathematica (2009), 41-54.

[8] A. Lourdusamy and S. Saratha Nellainayaki, Pebbling on some braid graphs, J. Math. Comput. Sci. 11(1) (2020), 625-634.

[9] P. T. Marykutty and K. A. Germina, Open distance pattern edge coloring of a graph, Annals of Pure and Applied Mathematics 6(2) (2014), 191-198.

[10] Priscilla Paul, A, On Edge Pebbling Number and Cover Edge Pebbling Number of Some Graphs, Journal of Information and Computational Science 10(6) (2020), 337-344.

[11] S. Sarah and Lian Mathew, Secure domination cover pebbling number for variants of complete graphs, Advances ad Applications in Discrete Mathematics 27(1) (2021), 105-122.