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A COMMON FIXED-POINT THEOREM IN LINEAR *n*-NORMED SPACE

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Abstract

In this paper, we discuss for an existence and uniqueness of common fixed-point theorem for three self-independent operators mapping in linear n-normed space.

I. Introduction

In 1963, S. Gahler [2, 3] was introduced by 2-norm and n-norm on a linear space. Raymond W. Freese and Y. J. Cho [1] gave as a survey of the latest results on the relations between linear 2-normed spaces and normed linear spaces and conclusion of linear 2-normed spaces. The idea on n-inner product spaces is also due to Misiak who had studied the same as early as 1980. Lateron A. Misiak [10] had also established the notion of an n-normin 1989.

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A logical growth of linear n-normed spaces has been widely made by Hendra Ganawan and Mashadi [5], S.S. Kim, Y. J. Cho [7], R. Malceski [8] and A. Misiak [10]. For correlated works of n-metric spaces and n-inner product spaces see [4], [5] and [10]. In recent times, many authors establish the fixed-point theorem in n-normed spaces and n-Banach spaces.

In this paper, the concept of existence and uniqueness of common fixed-point theorem for three self-independent operators mapping in linear n-normed space and illustrate the suitable corollary.

Now we will give some basic definitions and results in n-normed spaces before presenting our main results.

Definition 1.1 [1]. Let U be a genuine direct space with measurement of U is more prominent than 1 and $\|\cdot, \cdot\| : U \times U \to [0, \infty)$ be a function. Then

(i) $\| u, v \| = 0$ if and just if u and v are linearly dependent,

(ii) $\| u, v \| = \| v, u \|$,

(iii) $\| \alpha u, v \| = | \alpha | \| u, v \|$,

(iv) $\| u + v, w \| \le \| u, w \| + \| v, w \|$, where for all $u, v, w \in U$ and $\alpha \in R$.

If $\|\cdot, \cdot\|$ is Known as a 2-norm and the pair $(U, \|\cdot, \cdot\|)$ is known as a linear 2-normed space. So a 2-norm $\| u, v \|$ always fulfils [13] $\| u, v + \alpha u \| = \| u, v \|$ for all $u, v \in U$ and all scalars α.

Definition 1.2 [6]. Let n be a natural number, let X be a genuine vector space of measurement $d \ge n$ (d might be endlessness). A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n fulfilling four properties,

(i) $\| x_1, x_2, x_3, \ldots, x_n \| = 0$ if and just if $x_1, x_2, x_3, \ldots, x_n$ are linearly dependent in X,

(ii) $\| x_1, x_2, x_3, \ldots, x_n \|$ is invariant under permutation of $x_1, x_2, x_3, \ldots, x_n$,

(iii) $\| x_1, x_2, x_3, \ldots, x_{n-1}, \alpha x_n \| = | \alpha | \| x_1, x_2, x_3, \ldots, x_{n-1}, x_n \|$ for each $\alpha \in R$,

(iv) $\| x_1, x_2, x_3, \ldots, x_{n-1}, y + z \| \leq \| x_1, x_2, x_3, \ldots, x_{n-1}, y \| + \| x_1, x_2, x_3, \ldots, x_{n-1}, z \|$ for all y and z in X, is called an n-norm over X and the pair $(X \| \cdot, \ldots, \cdot \|)$ is known as a linear n-normed spaces.

Example 1.1 [5]. Let $X = R^n$ with the norm $\| \cdot, \ldots, \cdot \|$ on X by

$$\| x_1, x_2, x_3, \ldots, x_n \| | x_{ij} | = \begin{pmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{13} & \ldots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \ldots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \ldots & x_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \ldots & x_{nn} \end{vmatrix} \end{pmatrix}.$$

Where, $x_i = x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in} \in R^n$ for each $i = 1, 2, 3, \ldots, n$. Then $(X, \| \cdot, \ldots, \cdot \|)$ is a linear n-normed space.

Definition 1.3 [6]. An arrangement $\{x_n\}_{n \in N}$ in a n-normed space $(X, \| \cdot, \ldots, \cdot \|)$ is said to be a convergent to a component $x \in X$ if $\lim_{n \to \infty} \| x_n - x, a_2, a_3, \ldots, a_n \| = 0$ for all $a_2, a_3, \ldots, a_n \in X$. The point x is called the limit of the sequence.

Definition 1.4 [11]. An arrangement $\{x_n\}_{n \in N}$ in a n-normed space $(X, \| \cdot, \ldots, \cdot \|)$ is said to be a Cauchy sequence if $\lim_{m, n \to \infty} \| x_m - x_n, a_2, a_3, \ldots, a_n \| = 0$ for all $a_2, a_3, \ldots, a_n \in X$.

Definition 1.5 [6]. A n-normed space $(X, \| \cdot, \ldots, \cdot \|)$ is said to be complete if every Cauchy sequence in X is convergent. A complete n-normed space is called an n-Banach space.

Definition 1.6 [6]. Let X be a n-Banach space and T be a self-mapping of $X \cdot T$ is said to be continuous at x if for every sequence x_n in X, $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$ in X.

Definition 1.7 [9]. Let $(X, \| \cdot, \ldots, \cdot \|)$ be a linear n-normed space. Then the mapping $T : X \to X$ is said to be a contraction if there exists $k \in [0, 1)$ such that

$$\| Tx - Ty, u_2, u_3, \ldots, u_n \| \leq k \| x - y, u_2, u_3, \ldots, u_n \|,$$

for all $x, y, u_2, u_3, \ldots, u_n \in X$.

In this paper we will utilize Picard iteration schema defined as following

Definition 1.8. Let A be any set and $T : A \to A$ a self-map. For any given $x \in A$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$, we recall $T^n(x)$ the n^{th} iterative of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n \geq 0} \subset X$ given by $x_n = Tx_{n-1} = T^n x_0$, $n = 1, 2, 3, \ldots$ is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x.

Lemma 1.1 [12]. *Let $\{x_n\}_{n \in N}$ be a sequence in a complete n-normed space $(X, \| \cdot, \ldots, \cdot \|)$ then there exists $r \in (0, 1)$ such that $\| x_n - x_{n+1}, a_2, a_3, \ldots, a_n \| \leq r \| x_{n-1} - x_n, a_2, a_3, \ldots, a_n \|$ for all non-negative integer n and every $a_2, a_3, \ldots, a_n$ in X then $\{x_n\}$ converges to a point in X.*

II. Main Result

Theorem 2.1. *If T, P and Q are three operators mapping a complete n-normed space $(X, \| \cdot, \ldots, \cdot \|)$ to itself be sequentially continuous and if for all $x, y, u_2, u_3, \ldots, u_n$ in X*

(i) $\min \{\| P^p(x) - Q^q(y), u_2, u_3, \ldots, u_n \|, \| T_x - P^p(T_x), u_2, u_3, \ldots, u_n \|,$
$\| T_y - Q^q(T_y), u_2, u_3, \ldots, u_n \|,$

$$\| P^p(T_x) - Q^q P^p(T_x), u_2, u_3, \ldots, u_n \|, \| T_y - Q^q P^p(T_x), u_2, u_3, \ldots, u_n \|\}$$

$$+ k \min \{\| T_x - Q^q(T_y), u_2, u_3, \ldots, u_n \|, \| T_y - P^p(T_x), u_2, u_3, \ldots, u_n \|,$$

$$\| T_x - P^p Q^q(T_y), u_2, u_3, \ldots, u_n \|, \| Q^q(T_y) - Q^q P^p(T_x), u_2, u_3, \ldots, u_n \|\}$$

$$\leq r \| x - y, u_2, u_3, \ldots, u_n \|$$

where $r \in (0, 1)$ *and k is a real number.*

(ii) $\| T_x - T_y, u_2, u_3, \ldots, u_n \| \leq \| x - y, u_2, u_3, \ldots, u_n \|$

(iii) $TP^p = P^pT$ *and* $TQ^q = Q^qT$ *then there exists a unique common fixed point of* T, P *and* Q *if* $k > r$.

Proof. Utilizing condition (ii) and (iii), condition (i) becomes,

$$\min\{\| P^p(x) - Q^q(y), u_2, u_3, \ldots, u_n \|, \| x - P^p(y), u_2, u_3, \ldots, u_n \|,$$

$$\| y - Q^q(y), u_2, u_3, \ldots, u_n \|,$$

$$\| P^p(T_x) - Q^qP^p(x), u_2, u_3, \ldots, u_n \|, \| y - Q^qP^p(x), u_2, u_3, \ldots, u_n \|\}$$

$$+ k\min\{\| x - Q^q(y), u_2, u_3, \ldots, u_n \|, \| y - P^p(x), u_2, u_3, \ldots, u_n \|,$$

$$\| x - P^pQ^q(y), u_2, u_3, \ldots, u_n \|, \| Q^q(y) - Q^qP^p(x), u_2, u_3, \ldots, u_n \|\}$$

$$\leq r\| x - y, u_2, u_3, \ldots, u_n \|.$$

Presently for given x_0 in X, we consider a sequence $\{x_n\}_{n \in N}$ as

$$x_0, x_1 = P^p(x_0), x_2 = Q^q(x_1), \ldots, x_{2n} = Q^q(x_{2n-1}), x_{2n+1} = P^p(x_{2n}).$$

If for some m, $x_m = x_{m+1}$, then P^p and Q^q have a common fixed point x_n in X. Thus, we suppose that $x_m \neq x_{m+1}$, $\forall m = 1, 2, 3, \ldots$. From the condition for $x = x_{2n}$ and $y = x_{2n+1}$, we have,

$$\min\{\| P^p(x_{2n}) - Q^q(x_{2n+1}), u_2, u_3, \ldots, u_n \|, \| x_{2n} - P^p(x_{2n+1}), u_2, u_3, \ldots, u_n \|,$$

$$\| x_{2n+1} - Q^q(x_{2n+1}), u_2, u_3, \ldots, u_n \|,$$

$$\| P^p(x_{2n}) - Q^qP^p(x_{2n}), u_2, u_3, \ldots, u_n \|, \| x_{2n+1} - Q^qP^p(x_{2n}), u_2, u_3, \ldots, u_n \|\}$$

$$+ k\min\{\| x_{2n} - Q^q(x_{2n+1}), u_2, u_3, \ldots, u_n \|, \| x_{2n+1} - P^p(x_{2n}), u_2, u_3, \ldots, u_n \|,$$

$$\| x_{2n} - P^pQ^q(x_{2n+1}), u_2, u_3, \ldots, u_n \|, \| Q^q(x_{2n+1}) - Q^qP^p(x_{2n}), u_2, u_3, \ldots, u_n \|\}$$

$$\leq r\| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \|$$

for every non-negative integer n, or,

$$\min\{\| x_{2n+1} - x_{2n+2}, u_2, u_3, \ldots, u_n \|, \| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \|\}$$

$$\leq r \| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \|$$

for every non-negative integer n.

Since, $(X, \| \cdot, \ldots, \cdot \|)$ is an-normed space, $\| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \| \neq 0$ for some $u_2, u_3, \ldots, u_n$ in X.

Hence if $\| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \| < \| x_{2n} - x_{2n+2}, u_2, u_3, \ldots, u_n \|$.

Then we have $\| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \| \leq r \| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \|$ $\forall r \in (0, 1)$ which is impossible and so, we have, $\| x_{2n+1} - x_{2n+2}, u_2, u_3, \ldots, u_n \| \leq r \| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \|$.

Similarly, we have $\| x_{2n} - x_{2n+1}, u_2, u_3, \ldots, u_n \| \leq r \| x_{2n+1} - x_{2n}, u_2, u_3, \ldots, u_n \|$.

Therefore, $\| x_m - x_{m+1}, u_2, u_3, \ldots, u_n \| \leq r \| x_{m-1} - x_m, u_2, u_3, \ldots, u_n \|$ for every non-negative integer m and by Lemma (1.1).

The sequence $\{x_n\}$ converges to some point x_0 in X, i.e., $\lim_{n \to \infty} x_n = x_0$.

Now,

$$\| x_0 - P^p(x_0), u_2, u_3, \ldots, u_n \| \leq \| x_0 - P^p(x_0), x_{2n} \| + \| x_0 - x_{2n}, u_2, u_3, \ldots, u_n \|$$

$$+ \| x_{2n+1} - P^p(x_0), u_2, u_3, \ldots, u_n \|$$

$$= \| x_0 - P^p(x_0), x_{2n} \| + \| x_0 - x_{2n}, u_2, u_3, \ldots, u_n \|$$

$$+ \| P^p(x_{2n}) - P^p(x_0), u_2, u_3, \ldots, u_n \|$$

$$\to 0 \text{ as } n \to \infty.$$

Therefore, $\| x_0 - P^p(x_0), u_2, u_3, \ldots, u_n \| = 0 \; \forall u_2, u_3, \ldots, u_n \in X$, thus x_0 is a fixed point of P^p.

Similarly, x_0 is also a fixed point of Q^q i.e., x_0 is the common fixed point of P^p and Q^q.

Next let $k > r$ and to prove the uniqueness of a common fixed point of P^p and Q^q with $x_0 \neq y_0$. Then, $\| x_0 - y_0, u_2, u_3, \ldots, u_n \| \neq 0$, for all $u_2, u_3, \ldots u_n$ in X,

$$\min\{\| P^p(x_0) - Q^q(y_0), u_2, u_3, \ldots, u_n \|, \| x_0 - Q^q(y_0), u_2, u_3, \ldots, u_n \|,$$

$$\| x_0 - P^p(y_0), u_2, u_3, \ldots, u_n \|,$$

$$\| P^p(x_0) - Q^q P^p(x_0), u_2, u_3, \ldots, u_n \|, \| x_0 - Q^q P^p(x_0), u_2, u_3, \ldots, u_n \|\}$$

$$+ k \min\{\| x_0 - Q^q(y_0), u_2, u_3, \ldots, u_n \|, \| y_0 - P^p(x_0), u_2, u_3, \ldots, u_n \|,$$

$$\| x_0 - P^p Q^q(y_0), u_2, u_3, \ldots, u_n \|,$$

$$\| Q^q(y_0) - Q^q P^p(x_0), u_2, u_3, \ldots, u_n \|\} \leq r \| x_0 - y_0, u_2, u_3, \ldots, u_n \|$$

or

$$k \| x_0 - y_0, u_2, u_3, \ldots, u_n \| \leq r \| x_0 - y_0, u_2, u_3, \ldots, u_n \|$$

$$\text{i.e., } \| x_0 - y_0, u_2, u_3, \ldots, u_n \| \leq \frac{r}{k} \| x_0 - y_0, u_2, u_3, \ldots, u_n \|,$$

which is impossible.

This proves that P^p and Q^q have a unique common fixed point. $P^p(P(x_0)) = P^p(P(x_0)) = P(x_0)$, but x_0 is the unique fixed point of $P^p(x_0)$.

So, $P(x_0) = x_0$.

Similarly, $Q(x_0) = x_0$, and also x is the unique fixed point of P and Q.

Now, $\| x_0 - Tx_0, u_2, u_3, \ldots, u_n \| = \| P^p(x_0) - Q^q(Tx_0), u_2, u_3, \ldots, u_n \|$.

So,

$$\min\{\| P^p(x_0) - Q^q(Tx_0), u_2, u_3, \ldots, u_n \|, \| Tx_0 - P^p(Tx_0), u_2, u_3, \ldots, u_n \|,$$

$$\| Q^q(x_0) - Q^q(T^q x_0), u_2, u_3, \ldots, u_n \|,$$

$$\| P^p(Tx_0) - Q^q P^p(Tx_0), u_2, u_3, \ldots, u_n \|, \| T^q x_0 - Q^q P^p(Tx_0), u_2, u_3, \ldots, u_n \|\}$$

$$+ k \min\{\| Tx_0 - Q^q(T^q x_0), u_2, u_3, \ldots, u_n \|, \| T^q x_0 - P^p(Tx_0), u_2, u_3, \ldots, u_n \|,$$

$$\| Tx_0 - P^p Q^q(T^q x_0), u_2, u_3, \ldots, u_n \|,$$

$$\| Q^q(T^q x_0) - Q^q P^p(Tx_0), u_2, u_3, \ldots, u_n \|\} \le r \| x_0 - Tx_0, u_2, u_3, \ldots, u_n \|$$

or,

$$k \| Tx_0 - T^q x_0, u_2, u_3, \ldots, u_n \| \le r \| x_0 - Tx_0, u_2, u_3, \ldots, u_n \|$$

or

$$k \| Tx_0 - T^q x_0, u_2, u_3, \ldots, u_n \| \le \frac{r}{k} \| x_0 - Tx_0, u_2, u_3, \ldots, u_n \|$$

which gives,

$$\| x_0 - Tx_0, u_2, u_3, \ldots, u_n \| = 0.$$

Thus,

$$x_0 = Tx_0.$$

Hence x_0 is the unique common fixed point of T, P and Q.

Corollary 2.1. *If I, P and Q are three operators mapping a complete n-normed space $(X, \| \cdot, \ldots, \cdot \|)$ to itself be sequentially continuous and if for all $x, y, u_2, u_3, \ldots, u_n$ in X*

(i) $$\min\{\| P^p(x) - Q^q(y), u_2, u_3, \ldots, u_n \|, \| I_x - P^p(I_x), u_2, u_3, \ldots, u_n \|,$$

$$\| I_y - Q^q(I_y), u_2, u_3, \ldots, u_n \|,$$

$$\| P^p(I_x) - Q^q P^p(I_x), u_2, u_3, \ldots, u_n \|, \| I_y - Q^q P^p(I_x), u_2, u_3, \ldots, u_n \|\}$$

$$+ k \min\{\| I_x - Q^q(I_y), u_2, u_3, \ldots, u_n \|, \| I_y - P^p(I_x), u_2, u_3, \ldots, u_n \|,$$

$$\| I_x - P^p Q^q(I_y), u_2, u_3, \ldots, u_n \|, \| Q^q(I_y) - Q^q P^p(I_x), u_2, u_3, \ldots, u_n \|\}$$

$$\le r \| x - y, u_2, u_3, \ldots, u_n \|$$

where $r \in (0, 1)$ *and* k *is a real number and* I *is an Identity operator.*

(ii) $\| I_x - I_y, u_2, u_3, \ldots, u_n \| \leq \| x - y, u_2, u_3, \ldots, u_n \|$

(iii) $IP^P = P^p I$

$IQ^q = Q^q I$

then there exists a unique common fixed point of I, P *and* Q *if* $k > r$.

Proof. If $I(x) = x \; \forall x \in X$, and we take $T = I$ theorem reduces to

$$\min\{\| P^p(x) - Q^q(x), u_2, u_3, \ldots, u_n \|, \| x - P^p(x), u_2, u_3, \ldots, u_n \|,$$

$$\| y - Q^q(y), u_2, u_3, \ldots, u_n \|,$$

$$\| P^p(x) - Q^q P^p(x), u_2, u_3, \ldots, u_n \|, \| y - Q^q P^p(x), u_2, u_3, \ldots, u_n \|\}$$

$$+ k \min\{\| y - Q^q(y), u_2, u_3, \ldots, u_n \|, \| y - P^p(x), u_2, u_3, \ldots, u_n \|,$$

$$\| x - P^p Q^q(x), u_2, u_3, \ldots, u_n \|, \| Q^q(x) - Q^q P^p(x), u_2, u_3, \ldots, u_n \|\}$$

$$\leq r \| x - y, u_2, u_3, \ldots, u_n \|.$$

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