Advances and Applications in Mathematical Sciences
Volume 21, Issue 4, February 2022, Pages 1651-1664

SEPARATION AXIOMS IN FUZZY DOUBLE TOPOLOGICAL SPACES

R. SOWMYA and A. KALAICHELVI

[1, 2]Research Scholar
Department of Mathematics
Avinashilingam Institute for Home Science
and Higher Education for Women
Coimbatore, India

Abstract

In this paper, the separation axioms fuzzy double W - T_o, fuzzy double W - T_1, fuzzy double W - T_2, fuzzy double K - T_o, fuzzy double K - T_1, fuzzy double K - T_2 are introduced and analysed.

1. Introduction

In 2007, Kandil, et al. [2] introduced the concept of double sets, double topological spaces and continuous functions between these spaces. They also introduced separation axioms in double topological spaces. In this paper, fuzzy double sets and fuzzy double topological spaces are introduced and studied. The definitions of fuzzy separation axioms T_0, T_1 and T_2 introduced by Ganter, Steingale, and Waren [1] and Katsaras [3] are extended to fuzzy double topological spaces. It is shown that the extended fuzzy separation axioms are hereditary and productive.

2. Preliminary Definitions

Definition 2.1. Let X be a non-empty set. A fuzzy double set $\underline{f}$ is an

2020 Mathematics Subject Classification: 03E72, 54A40.
Keywords: fuzzy double set, fuzzy double topology, fuzzy double W - T_0, fuzzy double W - T_1, fuzzy double W - T_2, fuzzy double K - T_0, fuzzy double K - T_1, fuzzy double K - T_2.
Received October 5, 2020; Accepted November 10, 2020

ordered pair $(f_1, f_2) \in I^X \times I^X$ such that $f_1 \leq f_2$, where I is the closed unit interval $[0, 1]$.

The family of fuzzy double sets is denoted by $FD(X)$.

Definition 2.2. The fuzzy double null set denoted by $\underline{0} = (0, 0)$ is an ordered pair in X defined as $0(x) = 0$ and $0(x) = 0$ for each $x \in X$.

Definition 2.3. The fuzzy double universal set denoted by $\underline{1} = (1, 1)$ is an ordered pair in X defined as $1(x) = 1$ and $1(x) = 1$ for each $x \in X$.

Definition 2.4. Let X be a non-empty set. Let $\underline{f} = (f_1, f_2)$, $\underline{g} = (g_1, g_2)$ be fuzzy double sets in X. Then,

(i) $\underline{f}$ is a subset of $\underline{g}$ denoted by $\underline{f} \leq \underline{g}$ is defined as $f_1(x) \leq g_1(x)$ and $f_2(x) \leq g_2(x)$ for each $x \in X$. The complement of $\underline{f}$ is denoted by $(\underline{f})^c = (f_2^c, f_1^c)$ is a fuzzy double set in X defined as $(f_1)^c(x) = 1 - f_1(x)$ and $(f_2)^c(x) = 1 - f_2(x)$ for each $x \in X$.

(ii) The union of $\underline{f}$ and $\underline{g}$ is denoted by $\underline{f} \vee \underline{g}$ is a fuzzy double subsets in X denoted by $(\underline{f} \vee \underline{g})(x) = (f_1(x) \vee g_1(x), f_2(x) \vee g_2(x))$ for each $x \in X$.

(iii) The intersection of $\underline{f}$ and $\underline{g}$ is denoted by $\underline{f} \wedge \underline{g}$ is a fuzzy double subsets in X denoted by $(\underline{f} \wedge \underline{g})(x) = (f_1(x) \wedge g_1(x), f_2(x) \wedge g_2(x))$ for each $x \in X$.

(iv) The union of $((\underline{f})_\lambda)_{\lambda \in \wedge}$, a collection of fuzzy double sub-sets in X denoted by $\underset{\lambda \in \wedge}{\vee} \underline{f_\lambda}$ is a fuzzy double set in X defined as $(\underset{\lambda \in \wedge}{\vee} (\underline{f_\lambda}))(x) = (\underset{\lambda \in \wedge}{\vee} (f_1)_\lambda(x), \underset{\lambda \in \wedge}{\vee} (f_2)_\lambda(x))$ for each $x \in X$.

(v) The intersection of $((\underline{f})_\lambda)_{\lambda \in \wedge}$, a collection of fuzzy double sub-sets in X denoted by $\underset{\lambda \in \wedge}{\wedge} \underline{f_\lambda}$ is a fuzzy double set in X defined as $(\underset{\lambda \in \wedge}{\wedge} (\underline{f_\lambda}))(x) = (\underset{\lambda \in \wedge}{\wedge} (f_1)_\lambda(x), \underset{\lambda \in \wedge}{\wedge} (f_2)_\lambda(x))$ for each $x \in X$.

Definition 2.5. Let X be a non-empty set. A collection $\underline{\delta}$ of fuzzy double sets on X defines a fuzzy double topology on X if the following conditions are satisfied

(i) $\underline{0}, \underline{1}, \in \underline{\delta}$

(ii) $\underline{f_\lambda} \in \underline{\delta}$ for each $\lambda \in \wedge \Rightarrow \vee_{\lambda \in \wedge} \underline{f_\lambda} \in \underline{\delta}$

(iii) $\underline{f_i} \in \underline{\delta}$ for $i = 1, 2, \ldots n \Rightarrow \wedge_{i=1}^{n} \underline{f_i} \in \underline{\delta}$

The pair $(X, \underline{\delta})$ is called a fuzzy double topological space.

Definition 2.6. Let $(X, \underline{\delta})$ be a fuzzy double topological space. Let $Y \subseteq X$. Let $\underline{f} = (f_1, f_2) \in \underline{\delta}$. Define $\underline{f}/Y = (f_1/Y, f_2/Y)$ such that $(f_1/Y)(Z) = f_1(Z)$ and $(f_2/Y)(Z) = f_2(Z)$ for all $Z \in Y$. Define $(\underline{\delta}/Y) = \{(\underline{f}/Y) \mid \underline{f} \in \underline{\delta}\}$. Then $(\underline{\delta}/Y)$ is called the fuzzy double subspace topology on Y and $(Y, \underline{\delta}/Y)$ is called a fuzzy double subspace of $(X, \underline{\delta})$.

Definition 2.7. Let $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ be fuzzy double sets on I^X and I^Y respectively. The Cartesian product $\underline{f} * \underline{g}$ of $\underline{f}$ and $\underline{g}$ is a fuzzy double set on $I^X \times I^Y$ defined by

$$\begin{aligned}(\underline{f} * \underline{g})(x, y) &= \min(\underline{f}(x), \underline{g}(y)) \\ &= \min((f_1, f_2)(x), (g_1, g_2)(y)) \\ &= (\min\{(f_1(x), g_1(y))\}, \min\{(f_2(x), g_2(y))\})\end{aligned}$$

For each $(x, y) \in I^X \times I^Y$

Definition 2.8. Let $(X, \underline{\delta_1})$ and $(Y, \underline{\delta_2})$ be two fuzzy double topological spaces. Then the product fuzzy double topology $\underline{\delta_1} \times \underline{\delta_2}$ on $I^X \times I^Y$ is the fuzzy double topology having the collection $\{\underline{f} * \underline{g} \mid \underline{f} = (f_1, f_2) \in \underline{\delta_1}, \underline{g} = (g_1, g_2) \in \underline{\delta_2}\}$ as a basis.

Definition 2.9. Let $\{(x_\lambda, \underline{\delta_\lambda}) \mid \lambda \in \wedge\}$ be a family of fuzzy double topological spaces and $X = \prod_{\lambda \in \wedge} X_\lambda$. Let $\{\underline{f_\lambda} = ((f_1)_\lambda, (f_2)_\lambda | \lambda \in \wedge$ and $\underline{f_\lambda}$ is fuzzy double set in X_λ. Then their product $\prod_{\lambda \in \wedge} \underline{f_\lambda}$ is a fuzzy double set in $\prod_{\lambda \in \wedge} \underline{f_\lambda}$ defined as $\prod_{\lambda \in \wedge} \underline{f_\lambda} = (\prod_{\lambda \in \wedge} (f_1), \prod_{\lambda \in \wedge} (f_2)_\lambda)$ where $\prod_{\lambda \in \wedge} (f_1)_\lambda(x) = \min \{(f_1)_\lambda(x_\lambda)\}$, for a $x \in \prod x_\lambda \in \wedge$ and $\prod_{\lambda \in \wedge} (f_2)_\lambda(x) = \min \{(f_2)_\lambda(x_\lambda)\}$, for all $x \in \prod x_\lambda \in \wedge$.

The product topology on X is the one with the basic fuzzy double open sets of the form $\prod_{\lambda \in \wedge} \underline{f_\lambda}$ where $\underline{f_\lambda} \in \underline{\delta}$ and $\underline{f_\lambda} = \underline{1}$ except for finitely many $\lambda's$.

3. Separation Axioms

Definition 3.1. A Fuzzy double topological space $(X, \underline{\delta})$ is called a fuzzy double $W\text{-}T_0$ if for any two distinct points $x, y \in X$ there exists a fuzzy double open set $\underline{f} = (f_1, f_2)$ such that either $f_1(x) = 1, f_2(x) = 1, f_1(y) = 0, f_2(y) = 0$ or $f_1(y) = 1, f_2(y) = 1, f_1(x) = 0, f_2(x) = 0$.

Definition 3.2. A fuzzy double topological space $(X, \underline{\delta})$ is called a fuzzy double $W\text{-}T_1$, if for any two distinct points $x, y \in X$, there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(x) = 1, f_2(x) = 1, f_1(y) = 0, f_2(y) = 0$ and $g_1(x) = 0, g_2(x) = 0, g_1(y) = 1, g_2(y) = 1$.

Definition 3.3. A fuzzy double topological space $(X, \underline{\delta})$ is called a fuzzy double $W\text{-}T_2$ or fuzzy double W-Hausdorff, if for any two distinct points $x, y \in X$, there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(x) = 1, f_2(x) = 1, f_1(y) = 0, f_2(y) = 0$ and $g_1(x) = 0, g_2(x) = 0, g_1(y) = 1, g_2(y) = 1$ and $\underline{f} \wedge \underline{g} = \underline{0}$.

Definition 3.4. A fuzzy double topological space $(X, \underline{\delta})$ is called a fuzzy double $K\text{-}T_0$, if for any two distinct points $x, y \in X$, there exists a fuzzy double open set $\underline{f} = (f_1, f_2)$ such that either $f_1(x) > 0, f_2(x) > 0, f_1(y) = 0, f_2(y) = 0$ or $f_1(y) > 0, f_2(y) > 0, f_1(x) = 0, f_2(x) = 0$.

Definition 3.5. A fuzzy double topological space $(X, \underline{\delta})$ is called a fuzzy double K - T_1, if for any two distinct points $x, y \in X$, there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(x) > 0$, $f_2(x) > 0$, $g_1(y) > 0$, $g_2(y) > 0$, $f_1(y) = 0$, $f_2(y) = 0$, $g_1(x) = 0$, $g_2(x) = 0$.

Definition 3.6. A fuzzy double topological space $(X, \underline{\delta})$ is called a fuzzy double K - T_2, or fuzzy double K-Hausdorff, if for any two distinct points $x, y \in X$, there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(x) > 0$, $f_2(x) > 0$, $g_1(y) > 0$, $g_2(y) > 0$, $f_1(y) = 0$, $f_2(y) = 0$, $g_1(x) = 0$, $g_2(x) = 0$ and $\underline{f} \wedge \underline{g} = \underline{0}$.

Theorem 3.7. *Subspace of a fuzzy double W-Hausdorff space is a fuzzy double W-Hausdorff space.*

Proof. Let $(x, \underline{\delta})$ be a fuzzy double W-Hausdorff space. Let Y be a subspace of X. To prove: $(Y, \underline{\delta}/Y)$ is a fuzzy double W-Hausdorff space. Consider $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Then $y_1, y_2 \in X$, there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(y_1) = 1$, $f_2(y_1) = 1$, $g_1(y_2) = 1$, $g_2(y_2) = 1$, $f_1(y_2) = 0$, $f_2(y_2) = 0$, $g_1(y_1) = 0$, $g_2(y_1) = 0$ and $\underline{f} \wedge \underline{g} = 0$ i.e.) $f_1 \wedge g_1 = 0$ and $f_2 \wedge g_2 = 0$. Since Y is a subspace of X, $\underline{f} / Y, \underline{g} / Y \in \underline{\delta} / Y$ where $\underline{f}/Y = (f_1/Y, f_2/Y)$ and $\underline{g}/Y = (g_1/Y, g_2/Y)$

Therefore $(f_1/Y)(y_1) = f_1(y_1) = 1$

$$(f_2/Y)(y_1) = f_2(y_1) = 1$$

$$(g_1/Y)(y_2) = g_1(y_2) = 1$$

$$(g_2/Y)(y_2) = g_2(y_2) = 1$$

Consider $(\underline{f}/Y) \cap (\underline{g}/Y) = (f_1/Y) \wedge (g_1/Y), (f_2/Y) \wedge (g_2/Y)$

$((f_1/Y) \wedge (g_1/Y))(y) = (f_1/Y)(y) \wedge (g_1/Y)(y)$, for all $y \in Y \subseteq X$

$= f_1(y) \wedge g_1(y)$, for all $y \in Y \subseteq X$

$= (f_1 \wedge g_1)(y)$, for all $y \in Y \subseteq X$

$= 0(y)$, for all $y \in Y \subseteq X$

$(f_1/Y) \wedge (g_1/Y) = 0$

$((f_2/Y) \wedge (g_2/Y))(y) = (f_2/Y)(y) \wedge (g_2/Y)(y)$, for all $y \in Y \subseteq X$

$= f_2(y) \wedge g_2(y)$, for all $y \in Y \subseteq X$

$= (f_2 \wedge g_2)(y)$, for all $y \in Y \subseteq X$

$= 0(y)$, for all $y \in Y \subseteq X$

$(f_2/Y) \wedge (g_2/Y) = 0$

$\Rightarrow (\underline{f}/Y) \cap (\underline{g}/Y) = (0, 0) = \underline{0}$

Therefore, subspace of a fuzzy double W-Hausdorff space is a fuzzy double W-Hausdorff space.

Theorem 3.8. *Product of two fuzzy double W-Hausdorff spaces is a fuzzy double W-Hausdorff space in the product topology*.

Proof. Let $(X, \underline{\delta_1})$ and $(Y, \underline{\delta_2})$ be two fuzzy double W-Hausdorff spaces.

To prove: $(X \times Y, \underline{\delta_1} \times \underline{\delta_2})$ is a fuzzy double W-Hausdorff space.

Consider two distinct points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Either $x_1 \neq x_2$ or $y_1 \neq y_2$ Assume $x_1 \neq x_2$, therefore there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(x_1) = 1, f_2(x_1) = 1, g_1(x_2) = 1$, $g_2(x_2) = 1, f_1(x_2) = 0, f_2(x_2) = 0, g_1(x_1) = 0, g_2(x_1) = 0$ and $\underline{f} \wedge \underline{g} = \underline{0}$, where $\underline{0}$ is a fuzzy double null set in X $\underline{f} * \underline{1} \in \underline{\delta_1} \times \underline{\delta_2}$, since $\underline{f} \in \underline{\delta_1}, \underline{1} \in \underline{\delta_2}$ and $\underline{g} * \underline{1} \in \underline{\delta_1} \times \underline{\delta_2}$, since $\underline{g} \in \underline{\delta_1}, \underline{1} \in \underline{\delta_2}$ where $\underline{f} * \underline{1} = (f_1 * 1, f_2 * 1)$ and $\underline{g} * \underline{1} = (g_1 * 1, g_2 * 1)$ Consider $(f_1 * 1)(x_1, y_1) = \min\{f_1(x_1), 1(y_1)\} = \min\{1, 1\} = 1$

$(f_2 * 1)(x_1, y_1) = \min\{f_2(x_1), 1(y_1)\} = \min\{1, 1\} = 1$

$(g_1 * 1)(x_2, y_2) = \min\{g_1(x_2), 1(y_2)\} = \min\{1, 1\} = 1$

$(g_2 * 1)(x_2, y_2) = \min\{g_2(x_2), 1(y_2)\} = \min\{1, 1\} = 1$

Also $\underline{f} \wedge \underline{g} = \underline{0}$

$\Rightarrow (f_1 \wedge g_1, f_2 \wedge g_2) = \underline{0}$

$\Rightarrow (f_1 \wedge g_1)(x) = 0(x)$ and $(f_2 \wedge g_2)(x) = 0(x)$, for all $x \in X$

$\Rightarrow f_1(x) \wedge g_1(x) = 0(x)$ and $f_2(x) \wedge g_2(x) = 0(x)$, for all $x \in X$

$\Rightarrow$ either $f_1(x) = 0(x)$ or $g_1(x) = 0(x)$ and $f_2(x) = 0(x)$ or $g_2(x) = 0(x)$, for all $x \in X$

$\Rightarrow$ either $f_1(x) \wedge 1(y) = 0$ or $g_1(x) \wedge 1(y) = 0$ and

either $f_2(x) \wedge 1(y) = 0$ or $g_2(x) \wedge 1(y) = 0$ for all $x \in X$ and $y \in X$

$\Rightarrow$ either $(f_1 * 1)(x, y) = 0$ or $(g_1 * 1)(x, y) = 0$ and either $(f_2 * 1)(x, y) = 0$ or $(g_2 * 1)(x, y) = 0$ for all $(x, y) \in X \times Y$

$\Rightarrow (f_1 * 1) \wedge (g_1 * 1)(x, y) = 0$ and $(f_2 * 1) \wedge (g_2 * 1)(x, y) = 0$, for all $(x, y) \in X \times Y$

$\Rightarrow (\underline{f} * \underline{1}) \cap (\underline{g} * \underline{1}) = \underline{0}$.

Therefore product of two fuzzy double W-Hausdorff spaces is a fuzzy double W-Hausdorff space in the product topology.

Theorem 3.9. *Arbitrary product of fuzzy double W-Hausdorff spaces is a fuzzy double W-Hausdorff space in the product topology.*

Proof. Let $\{(X_\lambda, \underline{\delta_\lambda}) \mid \lambda \in \wedge\}$ be a collection of fuzzy double W-Hausdorff spaces. Let $X = \prod_{\lambda \in \wedge} X_\lambda$ in the product topology. Consider two distinct points

$(x_\lambda)_{\lambda \in \wedge}, (y_\lambda)_{\lambda \in \wedge} \in \prod_{\lambda \in \wedge} X_\lambda$.

Therefore $x_\mu \neq y_\mu$ for some $\mu \in \wedge$. Therefore there exists two fuzzy double open sets, $(\underline{f})_\mu = ((f_1)_\mu, (f_2)_\mu)$ and $(\underline{g})_\mu = ((g_1)_\mu, (g_2)_\mu) \in (\underline{\delta})_\mu$ such that

$(f_1)_\mu(x_\mu) = 1(f_2)_\mu(x_\mu) = 1,\ (g_1)_\mu(y_\mu) = 1,\ (g_2)_\mu(y_\mu) = 1,$

$(f_1)_\mu(y_\mu) = 0,\ (f_2)_\mu(y_\mu) = 0,\ (g_1)_\mu(x_\mu) = 0,\ (g_2)_\mu(x_\mu) = 0$ and

$(\underline{f})_\mu \cap (\underline{g})_\mu = (\underline{0})_\mu$

Let $\underline{f} = \prod_{\lambda\in\wedge}(\underline{f})_\lambda$, where $(\underline{f})_\lambda = (\underline{1})_\lambda$ for $\lambda \neq \mu$ and

$\underline{g} = \prod_{\lambda\in\wedge}(\underline{g})_\lambda$, where $(\underline{g})_\lambda = (\underline{1})_\lambda$ for $\lambda \neq \mu$

Then $\underline{f}, \underline{g} \in \prod_{\lambda\in\wedge}(\underline{g})_\lambda$

$$\underline{f} = \prod_{\lambda\in\wedge}(\underline{f})_\lambda = \Big(\prod_{\lambda\in\wedge}(f_1)_\lambda, \prod_{\lambda\in\wedge}(f_2)_\lambda\Big) \text{ and}$$

$$\underline{g} = \prod_{\lambda\in\wedge}(\underline{g})_\lambda = \Big(\prod_{\lambda\in\wedge}(g_1)_\lambda, \prod_{\lambda\in\wedge}(g_2)_\lambda\Big)$$

$$\prod_{\lambda\in\wedge}(f_1)_\lambda(x_\lambda) = \min\{(f_1)_\lambda(x_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (f_1)_\mu(x_\mu) \text{ for some } \mu \in \wedge$$

$$= 1$$

$$\prod_{\lambda\in\wedge}(f_2)_\lambda(x_\lambda) = \min\{(f_2)_\lambda(x_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (f_2)_\mu(x_\mu) \text{ for some } \mu \in \wedge$$

$$= 1$$

$$\prod_{\lambda\in\wedge}(g_1)_\lambda(y_\lambda) = \min\{(g_1)_\lambda(y_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (g_1)_\mu(y_\mu) \text{ for some } \mu \in \wedge$$

$$= 1$$

$$\prod_{\lambda\in\wedge}(g_2)_\lambda(y_\lambda) = \min\{(g_2)_\lambda(y_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (g_2)_\mu(y_\mu) \text{ for some } \mu \in \wedge$$

$= 1.$

Consider $\prod_{\lambda \in \wedge}(\underline{f})_\lambda \cap \prod_{\lambda \in \wedge}(\underline{g})_\lambda = (\prod_{\lambda \in \wedge}(f_1)_\lambda, \prod_{\lambda \in \wedge}(f_2)\lambda) \cap (\prod_{\lambda \in \wedge}(g_1)_\lambda, \lambda \prod_{\lambda \in \wedge}(g_2)\lambda)$

$= (\prod_{\lambda \in \wedge}(f_1)_\lambda \wedge \prod_{\lambda \in \wedge}(g_1)_\lambda, \prod_{\lambda \in \wedge}(f_2)_\lambda \wedge \prod_{\lambda \in \wedge}(g_2)_\lambda)$

Then, $(\prod_{\lambda \in \wedge}(f_1)_\lambda \wedge \prod_{\lambda \in \wedge}(g_1)_\lambda)(x_\lambda)$

$= (\prod_{\lambda \in \wedge}(f_1)_\lambda(x_\lambda)) \wedge (\prod_{\lambda \in \wedge}(g_1)_\lambda(x_\lambda))$ for all $\lambda \in \wedge$

$= (\min\{(f_1)_\lambda(x_\lambda)\}) \wedge (\min\{(g_1)_\lambda(x_\lambda)\})$ for all $\lambda \in \wedge$

$= (f_1)_\mu(x_\mu) \wedge (g_1)_\mu(x_\mu)$

$= ((f_1)_\mu \wedge (g_1)_\mu)(x_\mu)$

$= 0$

$(\prod_{\lambda \in \wedge}(f_2)_\lambda \wedge \prod_{\lambda \in \wedge}(g_2)_\lambda)(y_\lambda)$

$= (\prod_{\lambda \in \wedge}(f_2)_\lambda(y_\lambda)) \wedge (\prod_{\lambda \in \wedge}(g_2)_\lambda(y_\lambda))$ for all $\lambda \in \wedge$

$= (\min\{(f_2)_\lambda(y_\lambda)\}) \wedge (\min\{(g_2)_\lambda(y_\lambda)\})$ for all $\lambda \in \wedge$

$= (f_2)_\mu(y_\mu) \wedge (g_2)_\mu(y_\mu)$

$= ((f_2)_\mu \wedge (g_2)_\mu)(y_\mu)$

$= 0.$

Therefore, arbitrary product of two fuzzy double K-Hausdorff spaces is a fuzzy double K-Hausdorff space in the product topology.

Theorem 3.10. *Subspace of a fuzzy double K-Hausdorff space is a fuzzy double K-Hausdorff space.*

Proof. Let $(x, \underline{\delta})$ be a fuzzy double K-Hausdorff space. Let Y be a subspace of X. To prove: $(Y, \underline{\delta}/Y)$ is a fuzzy double W-Hausdorff space.

Consider $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Then $y_1, y_2 \in X$, there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(y_1) > 0, f_2(y_1) > 0, g_1(y_2) > 0, g_2(y_2) > 0$ and $\underline{f} \wedge \underline{g} = 0$ i.e.) $f_1 \wedge g_1 = 0$ and $f_2 \wedge g_2 = 0$ Since Y is a subspace of X, $\underline{f} \mid Y, \underline{g} \mid Y \in \underline{\delta} \mid Y$ where $\underline{f} \mid Y = (f_1 \mid Y, f_2 \mid Y)$ and $\underline{g} \mid Y = (g_1 \mid Y, g_2 \mid Y)$

Therefore $(f_1 \mid Y)(y_1) = f_1(y_1) > 0$

$(f_2/Y)(y_1) = f_2(y_1) > 0$

$(g_1/Y)(y_2) = g_1(y_2) > 0$

$(g_2/Y)(y_2) = g_2(y_2) > 0$

Consider $(\underline{f}/Y) \cap (\underline{g}/Y) = (f_1/Y) \wedge (g_1/Y), (f_2/Y) \wedge (g_2/Y)$

$(f_1/Y) \wedge (g_1/Y)(y) = (f_1/Y)(y) \wedge (g_1/Y)(y)$, for all $y \in Y \subseteq X$

$= f_1(y) \wedge g_1(y)$ for all $y \in Y \subseteq X$

$= (f_1 \wedge g_1)(y)$ for all $y \in Y \subseteq X$

$= 0(y)$, for all $y \in Y \subseteq X$

$(f_1/Y) \wedge (g_1/Y) = 0$,

$((f_2/Y) \wedge (g_2/Y)(y) = (f_2/Y)(y) \wedge (g_2/Y)(y)$, for all $y \in Y \subseteq X$

$= f_2(y) \wedge g_2(y)$, for all $y \in Y \subseteq X$

$= (f_2 \wedge g_2)(y)$, for all $y \in Y \subseteq X$

$= 0(y)$, for all $y \in Y \subseteq X$

$(f_2/Y) \wedge (g_2/Y) = 0$

$\Rightarrow (\underline{f}/Y) \cap (\underline{g}/Y) = (0, 0) = \underline{0}$.

Therefore, subspace of a fuzzy double K-Hausdorff space is a fuzzy double K-Hausdorff space.

Theorem 3.11. *Product of two fuzzy double K-Hausdorff spaces is a fuzzy double K-Hausdorff space in the product topology*.

Proof. Let $(X, \underline{\delta_1})$ and $(Y, \underline{\delta_2})$ be two fuzzy double K-Hausdorff spaces. To prove: $(X \times Y, \underline{\delta_1} \times \underline{\delta_2})$ is a fuzzy double K-Hausdorff space. Consider two distinct points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Either $x_1 \neq x_2$ or $y_1 \neq y_2$ Assume $x_1 \neq x_2$, therefore there exists two fuzzy double open sets $\underline{f} = (f_1, f_2)$ and $\underline{g} = (g_1, g_2)$ such that $f_1(x_1) > 0, f_2(x_1) > 0, g_1(x_2) > 0, g_2(x_2) > 0$ and $\underline{f} \wedge \underline{g} = \underline{0}$, where $\underline{0}$ is a fuzzy double null set in X $\underline{f} * \underline{1} \in \underline{\delta_1} \times \underline{\delta_2}$, since $\underline{f} \in \underline{\delta_1}, \underline{1} \in \underline{\delta_2}$ and $\underline{\delta} * \underline{1} \in \underline{\delta_1} \times \underline{\delta_2}$, since $\underline{g} \in \underline{\delta_1}, \underline{1} \in \underline{\delta_2}$ where $\underline{f} * \underline{1} = (f_1 * 1, f_2 * 1)$ and $\underline{g} * \underline{1} = (g_1 * 1, g_2 * 1)$

Consider $(f_1 * 1)(x_1, y_1) = \min\{f_1(x_1), 1(y_1)\} = \min\{f_1(x_1)\} > 0$

$(f_2 * 1)(x_1, y_2) = \min\{f_2(x_1), 1(y_1)\} > 0$

$(g_1 * 1)(x_2, y_2) = \min\{g_1(x_2), 1(y_2)\} > 0$

$(g_2 * 1)(x_2, y_2) = \min\{g_2(x_2), 1(y_2)\} > 0$

Also $\underline{f} \wedge \underline{g} = \underline{0}$

$\Rightarrow (f_1 \wedge g_1, f_2 \wedge g_2) = \underline{0}$

$\Rightarrow (f_1 \wedge g_1)(x) = 0(x)$ and $(f_2 \wedge g_2)(x) = 0(x)$, for all $x \in X$

$\Rightarrow f_1(x) \wedge g_1(x) = 0(x)$ and $f_2(x) \wedge g_2(x) = 0(x)$, for all $x \in X$

$\Rightarrow$ either $f_1(x) = 0(x)$ or $g_1(x) = 0(x)$ and either $f_2(x) = 0(x)$ or $g_2(x) = 0(x)$, for all $x \in X$

$\Rightarrow$ either $f_1(x) \wedge 1(y) = 0$ or $g_1(x) \wedge 1(y) = 0$ and

either $f_2(x) \wedge 1(y) = 0$ or $g_2(x) \wedge 1(y) = 0$ for all $x \in X$ and $y \in X$

$\Rightarrow$ either $(f_1 * 1)(x, y) = 0$ or $(g_1 * 1)(x, y) = 0$ and either $(f_2 * 1)(x, y) = 0$ or $(g_2 * 1)(x, y) = 0$ for all $(x, y) \in X \times Y$

$\Rightarrow (f_1 * 1) \wedge (g_1 * 1)(x, y) = 0$ and $(f_2 * 1) \wedge (g_2 * 1)(x, y) = 0$, for all

$(x,\ y) \in X \times Y$

$\Rightarrow (\underline{f} * \underline{1}) \cap (\underline{g} * \underline{1}) = \underline{0}.$

Therefore product of two fuzzy double K-Hausdorff spaces is a fuzzy double K-Hausdorff space in the product topology.

Theorem 3.12. *Arbitrary product of fuzzy double K-Hausdorff spaces is a fuzzy double K-Hausdorff space in the product topology.*

Proof. Let $\{(X_\lambda,\ \underline{\delta_\lambda}) \mid \lambda \in \wedge\}$ be a collection of fuzzy double K-Hausdorff spaces. Let $X = \prod_{\lambda \in \wedge} X_\lambda$ in the product topology Consider two distinct points $(x_\lambda)_{\lambda \in \wedge},\ (y_\lambda)_{\lambda \in \wedge},\ \in \prod_{\lambda \in \wedge} X_\lambda$ Therefore $x_\mu \neq y_\mu$ for some $\mu \in \wedge$. Therefore there exist two fuzzy double open sets, $(\underline{f})_\mu = ((f_1)_\mu,\ (f_2)_\mu)$ and $(\underline{g})_\mu = ((g_1)_\mu,\ (g_2)_\mu) \in (\underline{\delta})_\mu$ such that $(f_1)_\mu(x_\mu) > 0, (f_2)_\mu(x_\mu) > 0, (g_1)_\mu(y_\mu) > 0,$ $(g_2)_\mu(y_\mu) > 0$ and $(\underline{f})_\mu \wedge (\underline{g})_\mu = (\underline{0})_\mu$.

Let $\underline{f} = \prod_{\lambda \in \wedge} (\underline{f})_\lambda$, where $(\underline{f})_\lambda = (\underline{1})_\lambda$ for $\lambda \neq \mu$ and $\underline{g} = \prod_{\lambda \in \wedge} (\underline{g})_\lambda$, where $(\underline{g})_\lambda = (\underline{1})_\lambda$ for $\lambda \neq \mu$.

Then $\underline{f},\ \underline{g} \prod_{\lambda \in \wedge} (\underline{\delta})_\lambda$

$$\underline{f} = \prod_{\lambda \in \wedge} (\underline{f})_\lambda = (\prod_{\lambda \in \wedge} (f_1)_\lambda,\ \prod_{\lambda \in \wedge} (f_2)_\lambda) \text{ and}$$

$$\underline{g} = \prod_{\lambda \in \wedge} (\underline{g})_\lambda = (\prod_{\lambda \in \wedge} (g_1)_\lambda,\ \prod_{\lambda \in \wedge} (g_2)_\lambda)$$

$$\prod_{\lambda \in \wedge} (f_1)_\lambda (x_\lambda) = \min\{(f_1)_\lambda (x_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (f_1)_\mu (x_\mu) > 0 \text{ for some } \mu \in \wedge$$

$$\prod_{\lambda \in \wedge} (f_1)_\lambda (x_\lambda) > 0$$

$$\prod_{\lambda\in\wedge}(f_2)_\lambda(x_\lambda) = \min\{(f_2)_\lambda(x_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (f_2)_\mu(x_\mu) > 0 \text{ for some } \mu \in \wedge$$

$$\prod_{\lambda\in\wedge}(f_2)_\lambda(x_\lambda) > 0$$

$$\prod_{\lambda\in\wedge}(g_1)_\lambda(y_\lambda) = \min\{(g_1)_\lambda(y_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (g_1)_\mu(y_\mu) > 0 \text{ for some } \mu \in \wedge$$

$$\prod_{\lambda\in\wedge}(g_1)_\lambda(y_\lambda) > 0$$

$$\prod_{\lambda\in\wedge}(g_2)_\lambda(y_\lambda) = \min\{(g_2)_\lambda(y_\lambda)\} \text{ for all } \lambda \in \wedge$$

$$= (g_2)_\mu(y_\mu) > 0 \text{ for some } \mu \in \wedge$$

$$\prod_{\lambda\in\wedge}(g_2)_\lambda(y_\lambda) > 0.$$

Consider

$$\prod_{\lambda\in\wedge}(\underline{f})_\lambda \cap \prod_{\lambda\in\wedge}(\underline{g})_\lambda$$

$$= (\prod_{\lambda\in\wedge}(f_1)_\lambda, \prod_{\lambda\in\wedge}(f_2)_\lambda) \cap (\prod_{\lambda\in\wedge}(g_1)_\lambda, \prod_{\lambda\in\wedge}(g_2)_\lambda)$$

$$= (\prod_{\lambda\in\wedge}(f_1)_\lambda \wedge \prod_{\lambda\in\wedge}(g_1)_\lambda, \prod_{\lambda\in\wedge}(f_2)_\lambda \wedge \prod_{\lambda\in\wedge}(g_2)_\lambda)$$

Then, $(\prod_{\lambda\in\wedge}(f_1)_\lambda \wedge \prod_{\lambda\in\wedge}(g_1)_\lambda)(x_\lambda)$

$$= (\prod_{\lambda\in\wedge}(f_1)_\lambda(x_\lambda)) \wedge (\prod_{\lambda\in\wedge}(g_1)_\lambda(x_\lambda)) \text{ for all } \lambda \in \wedge$$

$$= (\min\{(f_1)_\lambda(x_\lambda)\}) \wedge (\min\{(g_1)_\lambda(x_\lambda)\}) \text{ for all } \mu \in \wedge$$

$$= (f_1)_\mu(x_\mu) \wedge (g_1)_\mu(x_\mu)$$

$$= ((f_1)_\mu \wedge (g_1)_\mu)(x_\mu)$$

$$= 0$$

$$(\prod_{\lambda \in \wedge} (f_2)_\lambda \wedge \prod_{\lambda \in \wedge} (g_2)_\lambda)(y_\lambda)$$

$$= (\prod_{\lambda \in \wedge} (f_2)_\lambda (y_\lambda)) \wedge (\prod_{\lambda \in \wedge} (g_2)_\lambda (y_\lambda)) \text{ for all } \lambda \in \wedge$$

$$= (\min \{(f_2)_\lambda (y_\lambda)\}) \wedge (\min \{(g_2)_\lambda (y_\lambda)\}) \text{ for all } \mu \in \wedge$$

$$= (f_2)_\mu (y_\mu) \wedge (g_2)_\mu (y_\mu)$$

$$= ((f_2)_\mu \wedge (g_2)_\mu)(y_\mu)$$

$$= 0.$$

Therefore, arbitrary product of fuzzy double K-Hausdorff spaces is a fuzzy double K-Hausdorff space in the product topology.

Conclusion

In this paper, the separation axioms fuzzy double W - T_0, fuzzy double W - T_1, fuzzy double W-Hausdorff, fuzzy double K - T_0, fuzzy double K - T_1, fuzzy double K-Hausdorff are introduced and it is proved that these separation axioms are hereditary and productive.

References

[1] T. E. Gantner, R. C. Steinlage and R. H. Warren, Compactness in fuzzy topological spaces, J. Math. Anal. Appl. 62 (1978), 547-562.

[2] A. Kandil, O. A. E. Tantawy and M. Wafaie, On flou (Intutionistic) topological spaces, J. Fuzzy Math. 15(2) (2007), 1-23.

[3] A. K. Katsaras, Ordered fuzzy topological spaces, J. Math. Anal. Appl. 84 (1981), 44-58.