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SOME FIXED POINT THEOREMS IN 2-FUZZY 2-HILBERT SPACE

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Abstract

In this paper, the concept of 2-fuzzy 2-Hilbert space is introduced and some fixed point theorems are developed.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [17] in 1965. The concept of 2-inner product space was introduced by C. R. Dimminie, S. Gahler and A. White [4]. Further various author gave definitions of fuzzy inner product space [5, 11, 12] and fuzzy normed linear space [6, 7, 12, 13, 15]. Further some applications of fixed points of various type of contractive mapping in Hilbert-2 and Banach-2 spaces were obtained among others by Browder [1], Browder and Petryshyn [2, 3], Hicks and Huffman [8], Huffman [9], Koparde and Waghmode [10], Smita Nair and Shalu Shrivastava [16]. Mukherjee and Bag [14] discussed some properties of fuzzy inner product space and established some fixed point theorems.

In this paper, the concept of 2-fuzzy 2-Hilbert space is introduced and some fixed point theorems are developed.

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2. Preliminaries

Definition 2.1. A fuzzy set is defined as $\widetilde{A} = \{x, \mu_A(x) : x \in X\}$, with a membership function $\mu_A(x) : X \to [0, 1]$, where $\mu_A(x)$ denotes the degree of membership of the element x to the set A.

Definition 2.2. Let X be a non empty and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{(x, \mu)/x \in X$ and $\mu \in (0, 1]\}$. Clearly f is bounded function for $|f(x)| \leq 1$. Let K be the space of real numbers then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by

$$f + g = \{(x, \mu) + (y, \eta)\} = \{(x + y), (\mu, \eta)/(x, \mu) \in f \text{ and } (y, \eta) \in g\}$$

and $kf = \{(kf, \mu)/(x, \mu) \in f\}$ where $k \in K$.

The linear space $F(X)$ is said to be normed space if for every $f \in F(X)$ there is associated a non-negative real number $\| f \|$ called the norm of f in such a way,

(1) $\| f \| = 0$ if and only if $f = 0$.

For,

$$\| f \| = 0 \Leftrightarrow \{\| (x, \mu) \| / (x, \mu) \in f\} = 0$$

$$\Leftrightarrow x = 0, \mu \in (0, 1] \Leftrightarrow f = 0$$

(2) $\| kf \| = | k | \| f \|$, $k \in K$.

For

$$\| kf \| = \{\| k(x, \mu) \| / (x, \mu) f, k \in K\}$$

$$= \{| k | \| x, \mu \| / (x, \mu) \in f\} = | k | \| f \|$$

(3) $\| f + g \| < \| f \| + \| g \|$ for every $f, g \in F(X)$.

For,

$$\| f + g \| = \{\| (x, \mu) + (y, \eta) \| / x, y \in X, \mu, \eta \in (0, 1]\}$$

$$= \{\| (x + y), (\mu \wedge \eta) \| / x, y \in X, \mu, \eta \in (0, 1]\}$$

$$\leq \{\| (x, \mu \wedge \eta) \| \| (y, \mu \wedge \eta) \| / (x, \mu) \in f \text{ and } (y, \eta) \in g\}$$

$$= \| f \| + \| g \|$$

Then $(F(X), \| \cdot \|)$ is a normed linear space.

Definition 2.3. A 2-fuzzy set on X is a fuzzy set on $F(X)$.

Definition 2.4. Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$ (R, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,

(N1) for all $t \in R$ with $t \leq 0$, $N(f_1, f_2, t) = 0$.

(N2) for all $t \in R$ with $t \geq 0$, $N(f_1, f_2, t) = 1$ if and only if f_1 and f_2 are linearly dependent.

(N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(N4) for all $t \in R$, with $t \geq 0$, $N(f_1, cf_2, t) = N(f_1, cf_2, t/|c|)$ if $c \neq 0$, $c \in K$ (field).

(N5) for all $s, t \in R$, $N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$.

(N6) $N(f_1, f_2, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

(N7) $\lim_{t \to \infty} N(f_1, f_2, t) = 1$.

Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Definition 2.5. A sequence $\{f_n\}$ in a 2-fuzzy normed linear space $(F(X), N)$ is said to be a convergent sequence if for a given $t > 0$ and $0 < r < 1$ there exist a positive number $n_0 \in N$ such that

$N(f_n - f, g, t) > 1 - r$ for $g \in F(X)$ and for every $n \geq n_0$.

Definition 2.6. A sequence $\{f_n\}$ is said to be a Cauchy sequence in a 2-fuzzy normed linear space $F(X)$ if for a given $r > 0$ with $0 < r < 1$, $t > 0$

there exist a positive number n_0 such that $N(f_n - f_m, g, t) > 1 - r$ for $g \in F(X)$ and for every $n, m \geq n_0$.

Definition 2.7. A 2-fuzzy 2-normed linear space (X, N) is said to be complete if every Cauchy sequence in X converge to some point in X.

Definition 2.8. A complete 2-fuzzy 2-normed linear space is a 2-fuzzy 2-Banach space.

Definition 2.9. A 2-fuzzy 2-normed linear space (X, N) is said to be complete if every Cauchy sequence in X converge to some point in X.

Definition 2.10. A complete 2-fuzzy 2-normed linear space is a 2-fuzzy 2-Banach space.

3. 2-Fuzzy 2-Hilbert Space

Definition 3.1. Let $F(X)$ be a linear space over the complex field $\mathbb{C}$. Define a fuzzy subset μ as a mapping from $F(X) \times F(X) \times F(X) \times \mathbb{C} \to [0, 1]$ such that $f_1 \in F(X)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying the following conditions

(I_1) For $f, g, h \in F(X)$ and $s, t \in \mathbb{C}$

$$\mu(f + g, h, f_1, |t| + |s|) \geq \min\{\mu(f, h, f_1, |t|), \mu(g, h, f_1, |s|)\}$$

(I_2) For $s, t \in \mathbb{C}$, $\mu(f, g, h, |st|) \geq \min\{(f, f, h, |s|^2), \mu(g, g, h, |t|^2)\}$

(I_3) For $t \in \mathbb{C}$, $\mu(f, g, h, |t|) = \mu(g, f, h, |t|)$

(I_4) For $\alpha_1, \alpha_2 \in \mathbb{C}$ with $\alpha_1 \neq 0, \alpha_2 \neq 0, \mu(\alpha_1 f, \alpha_2 f, h, t)$ $= \mu\left(f, g, h, \frac{t}{|\alpha_1 \alpha_2|}\right)$

(I_5) $\mu(f, f, h, t) = 0 \ \forall t \in \mathbb{C}/R^+$

$\mu(f, f, h, t) = 1 \ \forall t > 0$ if and only if f, h are linearly dependent.

(I_6) $\mu(f, g, h, t)$ is invariant under any permutation.

(I_7) $\forall t > 0, \mu(f, f, h, t) = \mu(g, g, h, t)$

(I_8) $\mu(f, g, h, t)$ is monotonic non-decreasing function of $\mathbb{C}$ and $\lim_{t \to \infty} \mu(f, g, h, t) = 1$.

Then μ is said to be the 2-fuzzy 2-inner product on $F(X)$ and the pair $(F(X), \mu)$ is called 2-fuzzy 2-inner product space.

Definition 3.2. A sequence $\{f_n\}$ in 2-fuzzy 2-inner product space $F(X)$ is said to be a convergent sequence if for a given $t > 0$ and $0 < r < 1$ there exist a positive number $n_0 \in \mu$ such that

$$\mu(f_n - f, f_n - f, h, t) > 1 - r$$

for $h \in F(X)$ and for every $n \geq n_0$, where $0 < t \leq 1$ and $r \in (0, 1)$.

Definition 3.3. A sequence $\{f_n\}$ is said to be a cauchy sequence in a 2-fuzzy 2-inner product space $F(X)$ if for a given $r > 0$ with $0 < t < 1, t > 0$, there exist a positive number n_0 such that

$$\mu(f_n - f_m, f_n - f_m, h, t) > 1 - r$$

for $h \in F(X)$ and for every $n, m \geq n_0$.

Definition 3.4. A 2-fuzzy 2-inner product space $F(X)$ is said to be complete if every cauchy sequence in $F(X)$ converges to some point in $F(X)$.

Definition 3.5. A complete 2-fuzzy 2-inner product space is a 2-fuzzy 2-Hilbert space.

Definition 3.6. A point $f \in F(X)$ is called a coincidence point of S and A if $Sf = Af$ and h only if and is said to be the point of coincidence of A and S if $h = Sf = Af$.

Theorem 3.7. *Let* S, G *and* T *be continuous self mappings* C *of a closed subset of a* 2-*fuzzy* 2-*Hilbert space* H *satisfying*

$$SG = GS, GT = TG, G(X) \subset T(X) \tag{1}$$

$\mu(Sf - Sg, Sf - Sg, h, t^2)$

$$\geq \min\{k_1 \frac{\mu(Sf - Gf, Sf - Gf, h, t^2)\mu(Gg - Tg, Gg - Tg, h, t^2)}{\mu(Sf - Tg, Sf - Tg, h, t^2)},$$

$$K_2 \frac{\mu(Gf - Tf, Gf - Tf, h, t^2)\mu(Sg - Gg, Sg - Gg, h, t^2)}{\mu(Gf - Tg, Gf - Tg, h, t^2)}\} \text{ for all } f, g \in C \quad (2)$$

Then S, G and T have a unique common fixed point.

Proof. Let $f \in C$, by (1), define sequence $\{g_n\}$ in such that

$$g_{2n} = Sf_{2n}, g_{2n+1} = Tf_{2m+1} \text{ and } g_{2n-1} = Gf_{2n} \quad (3)$$

From (2),

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) = \mu(sf_{2n} - sf_{2n+1}, sf_{2n} - sf_{2n+1}, h, t^2)$$

$$\geq \min\{k_1$$

$$\frac{\mu(Sf_{2n} - Gf_{2n}, Sf_{2n} - Gf_{2n}, h, t^2)\mu(Gf_{2n+1} - Tf_{2n+1}, Gf_{2n+1} - Tf_{2n+1}, h, t^2)}{\mu(Sf_{2n} - Tf_{2n+1}, Sf_{2n} - Tf_{2n+1}, h, t^2)},$$

$$k_2 \frac{\mu(Gf_{2n} - Tf_{2n}, Gf_{2n} - Tf_{2n}, h, t^2)\mu(Sf_{2n+1} - Gf_{2n+1}, Sf_{2n+1} - Gf_{2n+1}, h, t^2)}{\mu(Gf_{2n} - Tf_{2n+1}, Gf_{2n} - Tf_{2n+1}, h, t^2)}\}$$

$$\geq \min\{k_1$$

$$\frac{\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2)}{\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2)},$$

$$k_2 \frac{\mu(g_{2n-1} - g_{2n}, g_{2n-1} - g_{2n}, h, t^2)\mu(g_{2n+1} - g_{2n}, g_{2n+1} - g_{2n}, h, t^2)}{\mu(g_{2n-1} - g_{2n}, g_{2n-1} - g_{2n}, h, t^2)}\}$$

$$\geq (k_1 + k_2)\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)$$

Therefore,

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq (k_1 + k_2)$$

$$\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)$$

i.e.,

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq k\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)$$

where $k = k_1 + k_2$

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq k\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\geq k^n\mu(g_0 - g_1, g_0 - g_1, h, t^2)$$

For every integer $l > 0$,

$$\mu(g_n - g_{n+l}, g_n - g_{n+l}, h, t^2) \geq \min\{\mu(g_n - g_{n+1}, g_n - g_{n+1}, h, t^2),$$

$$\mu(g_{n+1} - g_{n+2}, g_{n+1} - g_{n+2}, h, t^2)$$

$$\mu(g_{n+p-1} - g_{n+p}, g_{n+p-1} - g_{n+p}, h, t^2)$$

$$\geq (1 + k + k^2 + \ldots + k^{l-1})\mu(g_n - g_{n+l}, g_n - g_{n+l}, h, t^2)$$

$$\geq \frac{k^l}{1-k}\mu(g_n - g_{n+l}, g_n - g_{n+l}, h, t^2)$$

As $n \to \infty$, $\{g_n\}$ is a Cauchy Sequence in C and as C is closed $g_n \to r \in C$. Now as $\{Sf_{2n}\}, \{Gf_{2n+1}\}, \{Tf_{2n+1}\}$ are also subsequences of $\{g_n\}$ so they will also converges to r.

Now as S, G and T are continuous such that $Sr = Gr : Gr = Tr$

Again from (2),

$$\mu(SSf_{2n} - Gf_{2n+1}, SSf_{2n} - Gf_{2n+1}, h, t^2) \geq$$

$$\min\{k_1$$

$$\frac{\mu(SSf_{2n} - Gf_{2n}, SSf_{2n} - Gf_{2n}, h, t^2)\mu(Gf_{2n+1} - Tf_{2n+1}, Gf_{2n+1} - Tf_{2n+1}, h, t^2)}{\mu(SSf_{2n} - Tf_{2n+1}, SSf_{2n} - Tf_{2n+1}, h, t^2)},$$

$$k_2 \frac{\mu(Gf_{2n} - Tf_{2n}, Gf_{2n} - Tf_{2n}, h, t^2)\mu(SSf_{2n+1} - Gf_{2n+1}, SSf_{2n+1} - Gf_{2n+1}, h, t^2)}{\mu(Gf_{2n} - Tf_{2n+1}, Gf_{2n} - Tf_{2n+1}, h, t^2)}\}$$

As $n \to \infty$

$$\mu(Sr - r, Sr - r, h, t^2) \geq \min\{k_1 \frac{\mu(Sr - r, Sr - r, h, t^2)\mu(r - r, r - r, h, t^2)}{\mu(Sr - r, Sr - r, h, t^2)},$$

$$k_2 \frac{\mu(r - r, r - r, h, t^2)\mu(Sr - r, Sr - r, h, t^2)}{\mu(r - r, r - r, h, t^2)}\}$$

tends to zero

Therefore, $Sr = Gr = Tr = r$.

Uniqueness:

To prove the uniqueness of fixed point, let 'q' be the another fixed point of and then by using (2)

$$\mu(r - q, r - q, h, t^2) = \mu(Sr - Gq, Sr - Gq, h, t^2)$$

$$\geq \min\{k_1 \frac{\mu(Sr - Gr, Sr - Gr, h, t^2)\mu(Gq - Tq, Gq - Tq, h, t^2)}{\mu(Sr - Tq, Sr - Tq, h, t^2)},$$

$$k_2 \frac{\mu(Gr - Tr, Gr - Tr, h, t^2)\mu(Sq - Gq, Sq - Gq, h, t^2)}{\mu(Gr - Tq, Gr - Tq, h, t^2)}\}$$

$$\geq \min\{k_1 \frac{\mu(r - r, r - r, h, t^2)\mu(q - q, q - q, h, t^2)}{\mu(r - q\ r - q, h, t^2)},$$

$$k_2 \frac{\mu(r - r, r - r, h, t^2)\mu(q - r, q - r, h, t^2)}{\mu(r - q, r - q, h, t^2)}\}$$

tends to zero.

Therefore, $r = q$. Thus r is the unique common fixed point of S, G and T.

This completes the proof.

Theorem 3.8. *Let* S, G, T *and* K *be continuous self mappings* C *of a* *closed subset of 2-fuzzy 2-Hilbert space H satisfying*

$$SK = KS,\ TG = GT,\ S(X) \subset G(X) \text{ and } T(X) \subset K(X) \tag{4}$$

$$\mu(Sf - Sg,\ Sf - Sg,\ h,\ t^2)$$

$$\geq \min\{k_1 \frac{\mu(Kf - Sf, Kf - Sf, h, t^2)\mu(Tg - Gg, Tg - Gg, h, t^2)}{\mu(Kf - Tg, Kf - Tg, h, t^2)},$$

$$K_2 \frac{\mu(Sf - Tg, Sf - Tg, h, t^2)\mu(Kg - Sg, Kg - Sg, h, t^2)}{\mu(Tf - Gg, Tf - Gg, h, t^2)},$$

$$K_3 \frac{\mu(Tf - Gg, Tf - Gg, h, t^2)\mu(Kg - Tg, Kg - Tg, h, t^2)}{\mu(Tf - Sg, Tf - Sg, h, t^2)}\} \text{ for all } f,\ g \in C \tag{5}$$

Then S, G, T *and K have a unique common fixed point.*

Proof. Let $f \in C$, by (4), define sequence $\{g_n\}$ in C such that

$g_{2n} = Tf_{2n+1} = Sf_{2n},\ g_{2n+1} = Gf_{2n+1}$ and $Kf_{2n} = g_{2n-1}$ for all $n = 0, 1, 2, \ldots$ (6)

From (5),

$$\mu(g_{2n} - g_{2n+1},\ g_{2n} - g_{2n+1}, h,\ t^2) = \mu(Sf_{2n} - Sf_{2n+1},\ Sf_{2n} - Sf_{2n+1},\ h,\ t^2)$$

$$\geq \min\{k_1$$

$$\frac{\mu(Kf_{2n} - Sf_{2n}, Kf_{2n} - Sf_{2n}, h, t^2)\mu(Tf_{2n+1} - Gf_{2n+1}, Tf_{2n+1} - Gf_{2n+1}, h, t^2)}{\mu(Kf_{2n} - Tf_{2n+1}, Kf_{2n} - Tf_{2n+1}, h, t^2)},$$

$$k_2 \frac{\mu(Sf_{2n} - Tf_{2n+1}, Sf_{2n} - Tf_{2n+1}, h, t^2)\mu(Kf_{2n+1} - Sf_{2n+1}, Kf_{2n+1} - Sf_{2n+1}, h, t^2)}{\mu(Tf_{2n} - Gf_{2n+1}, Tf_{2n} - Gf_{2n+1}, h, t^2)}$$

$$k_3 \frac{\mu(Tf_{2n} - Gf_{2n+1}, Tf_{2n} - Gf_{2n+1}, h, t^2)\mu(Kf_{2n+1} - Tf_{2n+1}, Kf_{2n+1} - Tf_{2n+1}, h, t^2)}{\mu(Tf_{2n} - Sf_{2n+1}, Tf_{2n} - Sf_{2n+1}, h, t^2)}\}$$

$$\geq \min\{k_1 \frac{\mu(g_{2n-1} - g_{2n}, g_{2n-1} - g_{2n}, h, t^2)\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2)}{\mu(g_{2n-1} - g_{2n}, g_{2n-1} - g_{2n}, h, t^2)},$$

$$k_2 \frac{\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2)}{\mu(g_{2n-1} - g_{2n+1}, g_{2n-1} - g_{2n+1}, h, t^2)}$$

$$k_3 \frac{\mu(g_{2n-1} - g_{2n+1}, g_{2n-1} - g_{2n+1}, h, t^2)\mu(g_{2n+1} - g_{2n}, g_{2n+1} - g_{2n}, h, t^2)}{\mu(g_{2n-1} - g_{2n+1}, g_{2n-1} - g_{2n+1}, h, t^2)}$$

Therefore,

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq (k_1 + k_2)\mu(g_{2n-1} - g_{2n}, g_{2n-1} - g_{2n}, h, t^2)$$

$$+ k_3\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2)$$

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq \frac{(k_1 + k_2)}{1 - k_3}\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)$$

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq k\mu(g_{2n} - g_{2n-1}, g_{2n} - g_{2n-1}, h, t^2)$$

where $k = \dfrac{(k_1 + k_2)}{1 - k_3}$

$$\mu(g_{2n} - g_{2n+1}, g_{2n} - g_{2n+1}, h, t^2) \geq k\mu(g_{n-1} - g_n, g_{n-1} - g_n, h, t^2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\geq k^n\mu(g_0 - g_1, g_0 - g_1, h, t^2)$$

For every integer $l > 0$,

$$\mu(g_n - g_{n+l}, g_n - g_{n+l}, h, t^2) \geq \min\{\mu(g_n - g_{n+1}, g_n - g_{n+1}, h, t^2),$$

$$\mu(g_{n+1} - g_{n+2}, g_{n+1} - g_{n+2}, h, t^2), \ldots$$

$$\mu(g_{n+l-1} - g_{n+l}, g_{n+l-1} - g_{n+l}, h, t^2)\}$$

$$\geq (1 + k + k^2 + \ldots + k^{l-1})\mu(g_n - g_{n+l}, g_n - g_{n+l}, h, t^2)$$

$$\geq \frac{k^l}{1 - k}\mu(g_n - g_{n+l}, g_n - g_{n+l}, h, t^2)$$

As $n \to \infty, \{g_n\}$ is a Cauchy Sequence in C and as C is closed $g_n \to r \in C$. Now as $\{sf_{2n}\}, \{Gf_{2n+1}\}, \{Tf_{2n+1}\}, \{Kf_{2n}\}$ are also subsequences of $\{g_n\}$ so they will also converges to r.

Now S, G, T as K and are continuous such that

Again from (5),

$$\mu(SSf_{2n} - Gf_{2n+1}, SSf_{2n+1} - Gf_{2n+1}, h, t^2) \geq$$

$\min\{k_1$

$$\frac{\mu(Kf_{2n} - SSf_{2n}, Kf_{2n} - SSf_{2n}, h, t^2)\mu(Tf_{2n+1} - Gf_{2n+1}, Tf_{2n+1} - Gf_{2n+1}, h, t^2)}{\mu(SSf_{2n} - Tf_{2n+1}, SSf_{2n} - Tf_{2n+1}, h, t^2)},$$

$$k_2 \frac{\mu(SSf_{2n} - Tf_{2n}, SSf_{2n} - Tf_{2n}, h, t^2)\mu(Kf_{2n+1} - SSf_{2n+1}, Kf_{2n+1} - SSf_{2n+1}, h, t^2)}{\mu(Tf_{2n} - Gf_{2n+1}, Tf_{2n} - Gf_{2n+1}, h, t^2)},$$

$$k_3 \frac{\mu(Tf_{2n} - Gf_{2n}, Tf_{2n} - Gf_{2n}, h, t^2)\mu(SSf_{2n+1} - Tf_{2n+1}, SSf_{2n+1} - Tf_{2n+1}, h, t^2)}{\mu(Tf_{2n} - SSf_{2n+1}, Tf_{2n} - SSf_{2n+1}, h, t^2)}\}$$

As $n \to \infty$

$$\mu(Sr - r, Sr - r, h, t^2) \geq \min\{k_1 \frac{\mu(r - Sr, r - Sr, h, t^2)\mu(r - r, r - r, h, t^2)}{\mu(Sr - r, Sr - r, h, t^2)},$$

$$k_2 \frac{\mu(Sr - r, Sr - r, h, t^2)\mu(r - Sr, r - Sr, h, t^2)}{\mu(r - r, r - r, h, t^2)}$$

$$k_3 \frac{\mu(r - r, r - r, h, t^2)\mu(Sr - r, Sr - r, h, t^2)}{\mu(r - Sr, r - Sr, h, t^2)}\}$$

tends to zero

Therefore, $Sr = Gr = Tr = Kr = r$

Uniqueness:

To prove the uniqueness of fixed point, let 'q' be the another fixed point of S, G, T and K then by using (5)

$$\mu(r - q, r - q, h, t^2) = \mu(Sr - Gq, Sr - Gq, h, t^2)$$

$$\geq \min \{k_1 \frac{\mu(Kr - Sr, Kr - Sr, h, t^2)\mu(Tq - Gq, Tq - Gq, h, t^2)}{\mu(Kr - Tq, Kr - Tq, h, t^2)},$$

$$k_2 \frac{\mu(Sr - Tq, Sr - Tq, h, t^2)\mu(Kq - Sq, Kq - Sq, h, t^2)}{\mu(Tr - Sq, Tr - Sq, h, t^2)}$$

$$k_3 \frac{\mu(Tr - Gq, Tr - Gq, h, t^2)\mu(Kq - Tq, Kq - Tq, h, t^2)}{\mu(Tr - Sq, Tr - Sq, h, t^2)}\}$$

$$\geq \min \{k_1 \frac{\mu(r - r, r - r, h, t^2)\mu(q - q, q - q, h, t^2)}{\mu(r - q, r - q, h, t^2)},$$

$$k_2 \frac{\mu(r - q, r - q, h, t^2)\mu(q - q, q - q, h, t^2)}{\mu(r - q, r - q, h, t^2)}$$

$$k_3 \frac{\mu(r - q, r - q, h, t^2)\mu(q - q, q - q, h, t^2)}{\mu(r - q, r - q, h, t^2)}\}$$

tends to zero.

Therefore, $r = q$. Thus r is the unique common fixed point of S, G, T and K.

This completes the proof.

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