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ON *k*-*m*-ZUMKELLER NUMBERS

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Abstract

In this paper we introduce the concept of k-m-Zumkeller number by partitioning the set of positive divisors of an integer n into k disjoint subsets such that the product of each subsets are equal and establish some properties on these numbers. Relations of these numbers with perfect numbers and k-m-perfect numbers are also obtained. We also define k-half-m-Zumkeller numbers and establish results on them. We also define k-T^*-m-Zumkeller numbers and illustrate these numbers with suitable examples.

1. Introduction

A positive integer n is said to be a perfect number if the sum of all the proper positive divisors of n is equal to n. In 2003 R. H. Zumkeller published in Sloane's sequence of integers A083207 where he generalized the concept of perfect numbers by partitioning the set of positive divisors of an integer into two disjoint subsets such that the sum of the two subsets is equal. In [5] Clark et al. called those numbers as Zumkeller numbers [6]. We define a new type of Zumkeller numbers termed as k-m-Zumkeller numbers by using the notion of partitioning the set of positive divisors of an integer n into k disjoint subsets such that they have equal product. We also define a new type of m-Zumkeller number as k-T^*-m-Zumkeller number by partitioning the unitary divisors of an integer.

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In section 2 of this paper we study the properties of k-m-Zumkeller numbers and also study the relation of these numbers with perfect numbers and k-m-perfect numbers. In section 3 we define k-half-m-Zumkeller numbers and study various properties on these numbers and also establish relations between k-half-m-Zumkeller numbers and m-perfect numbers. We also establish relations between k-m-Zumkeller numbers and k-half-m-Zumkeller numbers. In section 4 we define k-T^*-m-Zumkeller numbers with examples.

2. *k*-*m*-Zumkeller Numbers

Definition 2.1. A positive integer n is said to be k-m-Zumkeller number if set of all the positive divisors of n can be partitioned into k disjoint subsets of equal product.

The numbers 6, 8, 10, 14, 15, 16, 21, 22 are the first few 2-m-Zumkeller numbers. The numbers 12, 18, 20, 28, 32, 36, 44, 45 etc are the first few 3-m-Zumkeller numbers. Similarly 24, 30, 40, 42, 54, 56, 66, 70 etc are 4-m-Zumkeller numbers; 48, 80 etc. are 5-m-Zumkeller, numbers; 60, 72, 84, 90, 96 etc. are 6-m-Zumkeller numbers; 192 is the first 7-m-Zumkeller number; 120, 168 etc. are 8-m-Zumkeller numbers and 180 is the first 9-m-Zumkeller number and 240 is the first 10-m-Zumkeller number.

Here $\tau(n)$ denotes the number of positive divisors of n and $T(n)$ denotes the product of all the positive divisors of n.

Proposition 2.1 [3]**.** *Let the prime factorization of n be* $\prod_{i=1}^{r} p_i^{\alpha_i}$. *Then*

$$T(n) = n^{\frac{\tau(n)}{2}} \text{ and } \tau(n) = \prod_{j=1}^{r} (\alpha_j + 1).$$

Proposition 2.2. *If n is a k-m-Zumkeller number, then* $\tau(n) \geq 2k$.

Proof. Let n be a k-m-Zumkeller number with partition subset $\{A_1, A_2, \ldots, A_k\}$. Then the product of each of the partition subset is at least n. If the product of each of the partition subset is greater than n then clearly each of them contain at least 2 elements. Hence, $\tau(n) \geq 2k$.

If the product of each of the partition subset is equal to n then without loss of generality we may assume that $1, n \in A_1$. Then clearly each of the remaining subsets $A_2, A_3, \ldots A_k$ must contain at least two elements. Hence $\tau(n) \geq 2k$. ∎

The above proposition gives the necessary condition for a positive integer n to be an k-m-Zumkeller number.

The following proposition gives a necessary and sufficient condition for a positive integer n to be an k-m-Zumkeller number.

Proposition 2.3. *A positive integer n is a k-m-Zumkeller number if and only if $\tau(n) \geq 2k$ and $2k/\alpha_i \tau(n) \, \forall i = 1, 2, \ldots, r$ where r is the number of distinct prime divisors of n.*

Proof. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

then

$$T(n) = \left(\prod_{i=1}^{r} p_i^{\alpha_i}\right)^{\frac{\tau(n)}{2}}$$

$$= \prod_{i=1}^{r} p_i^{\frac{\alpha_i \tau(n)}{2}}. \tag{2.1}$$

Let n be a k-m-Zumkeller number then by Proposition 2.2 we have $\tau(n) \geq 2k$ and from equation (2.1) we have

$$k \mid \frac{\alpha_i \tau(n)}{2} \; \forall i = 1, 2, \ldots, r.$$

$$\Rightarrow 2k \mid \alpha_i \tau(n) \; \forall i = 1, 2, \ldots, r.$$

Converse part holds clearly.

Note. For $n = p^k$, where k is odd we have $2k/k(k+1)$ but n is not a k-m-Zumkeller number because in this case $\tau(n) = k + 1 < 2k$.

Remark 2.1. If p, q, r are distinct primes, then the integers of the form

i. $pq, p^3, p^4, p^3q, pqr, p^2qr, p^3q^2, p^5q$ etc. are 2-m-Zumkeller numbers.

ii. $p^2q, p^5, p^6, p^2q^2, p^3q^2, p^5q$ etc. are 3-m-Zumkeller numbers.

iii. p^3q, pqr, p^7, p^8 etc. are 4-m-Zumkeller numbers.

iv. $p^4q, p^4q^3, p^9q, p^9, p^{10}$ etc. are 5-m-Zumkeller numbers.

v. $p^5q, p^3q^2, p^2qr, p^{11}, p^{12}$ etc. are 6-m-Zumkeller numbers.

vi. $p^6q, p^6qr, p^{13}q, p^6q^3, p^{13}, p^{14}$ etc. are 7-m-Zumkeller numbers.

vii. $p^3qr, p^3q^3, p^{15}, p^{16}$ etc. are 8-m-Zumkeller numbers.

viii. $p^2q^2r, p^5q^2, p^8q, p^{17}, p^{18}$ etc. are 9-m-Zumkeller numbers.

ix. $p^4qr, p^3q^4, p^9q, p^{19}, p^{20}$ etc. are 10-m-Zumkeller numbers.

Corollary 2.4. *The integer* $n = p_1p_2, \ldots, p_r$ *(where* p_i*'s are distinct primes,* $r \geq 2$*) is* 2^{r-1}*-Zumkeller number.*

Proof. Since $n = p_1p_2, \ldots, p_r$

$\therefore \tau(n) = 2^r.$

Clearly, $2.2^{r-1} \mid 2^r.$

Hence n is a 2^{r-1}-m-Zumkeller number.

Corollary 2.5. *If p is a prime, then* p^α *is a k-m-Zumkeller number for any positive integer* α *where* $\alpha \equiv 0 \pmod{2k}$ *or* $\alpha \equiv 2k - 1 \pmod{2k}$.

Proof. p^α is a k-m-Zumkeller number

$\Leftrightarrow 2k \mid \alpha(\alpha + 1)$

$\Leftrightarrow \alpha \equiv 0 \pmod{2k}$ or $\alpha \equiv 2k - 1 \pmod{2k}$.

Corollary 2.6. *All even perfect numbers* $n = 2^{p-1}(2^p - 1)$, *where p is a prime and* 2^{p-1} *is a Mersenne prime, are p-m-Zumkeller numbers.*

Proof. $n = 2^{p-1}(2^p - 1)$ where 2^{p-1} and $2^p - 1$ are relatively prime.

$$\therefore \tau(n) = \tau(2^{p-1})\tau(2^p - 1) = 2p.$$

Hence by Proposition 2.3 we have n is a p-m-Zumkeller number. □

Note. [7] The highest known even perfect number (till March 2019) for $p = 82, 589, 933$ is $n = 2^{82, 589, 932}(2^{82, 589, 933} - 1) = 110847779864, \ldots,$ 007191207936 contains 49, 724, 095 digits. So by above corollary, n is 82, 589, 933-m-Zumkeller number; i.e. we can partitioned the positive divisors of this n into $p = 82, 589, 933$ disjoint subsets of equal product as $\{1, n\}, \left\{2, \frac{n}{2}\right\}, \left\{4, \frac{n}{4}\right\}$ etc.

Proposition 2.7. *If n is a k-m-Zumkeller number and p is a prime such that* $(n, p) = 1$ *then* np^l *is also a k-m-Zumkeller number for any positive integer l if* $k/\tau(n)$.

Proof. Since $(n, p) = 1$

$$\therefore \tau(np^l) = \tau(n)(l + 1).$$

Clearly, np^l is a k-m-Zumkeller number if $2k/l(l + 1)\tau(n)$ i.e. if $k/\tau(n)$ for any positive integer l. □

Definition 2.2 [4]. A positive integer n is said to be k-m-perfect number if $T(n) = n^k$.

Proposition 2.8. *All the k-m-perfect numbers are k-m-Zumkeller numbers.*

Proof. Let n be a k-m-perfect number then,

$$T(n) = n^k$$

$$\Rightarrow n^{\frac{\tau(n)}{2}} = n^k$$

$$\Rightarrow \tau(n) = 2k.$$

Hence, n is a k-m-Zumkeller number.

3. Half-*m*-Zumkeller Numbers

Definition 3.1. A positive integer n is called k-half-m-Zumkeller number if set of all the proper positive divisors of n can be partitioned into k disjoint non-empty subsets of equal product.

The numbers 12, 16, 18, 20 etc. are the first few 2-half-m-Zumkeller numbers. The numbers 24, 30, 40, 42, 54, 56, 64 etc. are the first few 3-half-m-Zukeller numbers. Similarly 48, 80 etc. are 4-half-m-Zumkeller numbers and 60, 72, 84, 90, 96 etc. are 5-half-m-Zumkeller numbers.

Let $T'(n)$ denotes the product of all the proper positive divisors of n then $T'(n) = \frac{n^{\frac{\tau(n)}{2}}}{n}$.

Proposition 3.1. *If n is a k-half-m-Zumkeller number then* $\tau(n) \geq 2k + 1$.

Proof. Let n be a k-half-m-Zumkeller number with partition subset $\{A_1, A_2, \ldots, A_k\}$. Then the product of each of the partition subset is at least $\frac{n}{2}$. If the product of each of the partition subset is greater than $\frac{n}{2}$ then clearly each of them contain at least 2 elements. Hence together with n itself, there exist at least $2k + 1$ positive divisors of n. Hence $\tau(n) \geq 2k + 1$.

If the product of each of the partition subset is equal to $\frac{n}{2}$ then without loss of generality we may assume that $1, \frac{n}{2} \in A_1$. Then clearly each of the remaining subsets $A_2, A_3, \ldots, A_k$ must contain at least two elements. Hence together with n itself, there exist at least $2k + 1$ positive divisors of n. Hence $\tau(n) \geq 2k + 1$. □

The above Proposition gives the necessary condition for an integer to be a k-half-m-Zumkeller number.

The following Proposition gives a necessary and sufficient condition for an integer n to be a k-half-m-Zumkeller number.

Proposition 3.2. *A positive integer n is a k-half-m-Zumkeller number if and only if* $\tau(n) \geq 2k+1$ *and* $2k| \alpha_i \,|[\tau(n)-2] \,\forall i = 1, 2, \ldots, r$ *where r is the number of distinct prime divisors of n.*

Proof. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

then

$$T'(n) = n^{\frac{\tau(n)-2}{2}}$$

$$= \prod_{i=1}^{r} p_i^{\frac{\alpha_i[\tau(n)-2]}{2}}. \tag{3.1}$$

Let n be a k-half-m-Zumkeller number then by Proposition 3.1 we have $\tau(n) \geq 2k+1$ and from equation (3.1) we have

$$k| \;\frac{\alpha_i[\tau(n)-2]}{2} \;\forall i = 1, 2, \ldots, r.$$

$$\Rightarrow 2k| \;\alpha_i[\tau(n)-2] \;\forall i = 1, 2, \ldots, r.$$

The converse part holds clearly.

Note. For $n = p^k$, where k is odd we have $2k| \;k \,|[k+1-2]$ but n is not a k-half-m-Zumkeller number because in this case $\tau(n) = k+1 < 2k+1$.

Remark 3.1. If p, q, r are distinct primes then the integers of the form

i. $p^2q, \; p^4q, \; p^6q$ etc. are 2-half-m-Zumkeller numbers.

ii. $p^3q, \; pqr, \; p^6$ etc. are 3-half-m-Zumkeller numbers.

iii. $p^4q,\ p^2q^2r$ etc. are 4-half-m-Zumkeller numbers.

iv. $p^3q^2,\ p^5q,\ p^2qr$ etc. are 5-half-m-Zumkeller numbers.

Proposition 3.3. *The k-m-perfect numbers are $(k-1)$-half-m-Zumkeller numbers.*

Proof. Let n be a k-m-perfect number, then

$$T(n) = n^k$$

$$\Rightarrow n^{\frac{\tau(n)}{2}} = n^k$$

$$\Rightarrow \tau(n) = 2k.$$

Hence, $2(k-1) | [\tau(n) - 2]$.

$\Rightarrow n$ is a $(k-1)$-half-m-Zumkeller number.

Proposition 3.4. *A positive integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is both k-m-Zumkeller and k-half-m Zumkeller number if*

i. *either k/α_i or $2k/\alpha_i \forall i = 1, 2, \ldots, r$ when k is odd.*

ii. $2k/\alpha_i \forall i = 1, 2, \ldots, r$ *when k is even.*

Proof. If n is a k-m-Zumkeller number, then

$$2k/\alpha_i\ \tau(n) \forall i = 1, 2, \ldots, r. \tag{3.2}$$

Again if n is a k-half-m-Zumkeller number, then

$$2k/\alpha_i\ [\tau(n) - 2] \forall i = 1, 2, \ldots, r. \tag{3.3}$$

From equations (3.2) and (3.3) we have,

$2k/2\alpha_i\ \forall i = 1, 2, \ldots, r.$

$\Rightarrow k/\alpha_i\ \forall i = 1, 2, \ldots, r.$

$\Rightarrow \alpha_i = kz$ where z is a positive integer.

$\therefore \tau(n) = (kz+1)^r = kl + 1$ where l is a positive integer.

i. Let k be odd. Then $\tau(n)$ is even if l is odd and $\tau(n)$ is odd if l is even hence from equations (3.2) and (3.3) we have

$$k/\alpha_i \ \forall i = 1, 2, \ldots, r, \text{ when } l \text{ is odd.}$$

and

$$2k/\alpha_i \ \forall i = 1, 2, \ldots, r, \text{ when } l \text{ is even.}$$

ii. Now let k be even then $\tau(n)$ is odd and hence from equations (3.2) and (3.3) we have

$$2k/\alpha_i \ \forall i = 1, 2, \ldots, r.$$

Proposition 3.5. *If a positive integer* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ *is a* $(k+1)$-m-*Zumkeller number, then* n *is also a* k-*half*-m-*Zumkeller number if* $\tau(n) = 2(k+1)$.

Proof. If n is a $(k+1)$-m-Zumkeller number, then

$2(k+1) \mid a_i \tau(n) \ \forall i = 1, 2 \ldots, r$

$\Rightarrow \alpha_i \tau(n) = (2k+2) z_i \ \forall i = 1, 2, \ldots, r,$ where z_i is an integer.

$\Rightarrow 2kz_i = \alpha_i \tau(n) - 2z_i \ \forall i = 1, 2, \ldots, r$

$\therefore 2k/\alpha_i [\tau(n) - 2]$ if $z_i = \alpha_i \ \forall i = 1, 2, \ldots, r$

Hence n is a k-half-m-Zumkller number if $\alpha_i = z_i = \dfrac{\alpha_i \tau(n)}{2(k+1)}$ $\forall i = 1, 2, \ldots, r.$

$$\Rightarrow \tau(n) = 2(k+1). \qquad \square$$

Example 3.1. 24 is a 4-m-Zumkller number as well as a 3-half-m-Zumkeller number because here $\tau(24) = 8 = (2 \times 3) + 2$.

Example 3.2. 36 is a 3-m-Zumkller number but is not a 2-half-m-Zumkeller number because here $\tau(36) = 9 \neq (2 \times 2) + 2$.

4. k-T^*-m-Zumkeller Numbers

Definition 4.1 [2]. A positive integer d is a unitary divisor of a positive integer n if $d \mid n$ and $\left(d, \frac{n}{d}\right) = 1$.

Let $\tau^*(n)$ denotes the number of unitary divisors of n and $T^*(n)$ denotes the product of all unitary divisors of n.

Proposition 4.1 [1]. *For any positive integers* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, *we have*

$$\tau^*(n) = 2^r \text{ and } T^*(n) = n^{\frac{\tau^*(n)}{2}}.$$

Definition 4.2. A positive integer n is said to be k-T^*-m-Zumkeller number if set of all the unitary divisors of n can be partitioned into k-disjoint subsets of equal product.

Proposition 4.2. *All the positive integer* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ *is* k-T^*-m-*Zumkeller number where* $k = 2^{r-1}, 2^{r-2}, \ldots 2$.

Proof. Since

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

$$\therefore T^*(n) = n^{\frac{2^r}{2}}$$

$$= n \times n \times n \times \ldots \times n \ (2^{r-1} \text{ times}) \quad (4.1)$$

$$= n^2 \times n^2 \times \ldots \times n^2 \ (2^{r-2} \text{ times}) \quad (4.2)$$

$$= \ldots \quad (4.3)$$

$$= \ldots \quad (4.4)$$

$$= n^{2^{r-2}} \times n^{2^{r-2}}. \quad (4.5)$$

Clearly, from the equations we have $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ is k-T^*-m-Zumkeller number where $k = 2^{r-1}, 2^{r-2}, \ldots, 2$. □

Example 4.1. $18 = 2 \times 3^2$ is a $2^{2-1} = 2 - T^*$-m-Zumkeller number with partition subsets $\{1, 18\}$ and $\{2, 9\}$.

Example 4.2. $84 = 2^2 \times 3 \times 7$ is a $2^{3-1} = 4$-T^*-m-Zumkeller number as well as a $2^{2-1} = 2$-T^*-m-Zumkeller number with partition subsets $\{1, 84\}, \{3, 28\}, \{4, 21\}, \{7, 12\}$ and $\{1, 3, 28, 84\}, \{4, 7, 12, 21\}$ respectively.

5. Conclusion

In this paper we have investigated various characteristics of k-m-Zumkeller numbers. This investigation can lead us to the study of Fibonacci primes and generalized perfect numbers like near perfect numbers and hyper perfect numbers.

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