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CHANGING AND UNCHANGING OF MAJORITY DOMINATING CHROMATIC NUMBER WHEN REMOVAL OFA SINGLE VERTEX

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Abstract

In this article, how the removal of a single vertex from a graph G can change the majority dom-chromatic number is determined for any graph. A graph is majority dom-chromatic critical if the removal of any vertex decreases or increases its majority dom-chromatic number. There are two types namely *CVR* and *UVR* with respect to majority dom-chromatic sets of a graph. Also the vertex classification $V^0_{M\chi}(G)$, $V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ are studied and its characterisation theorems are determined.

1. Introduction

Let G be a finite and simple graph with p vertices and q edges. A subset D of vertices in a graph $G = (V, E)$ is called a dominating set [1] of G if every vertex in $(V - D)$ is adjacent to some vertex in D. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a

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minimal dominating set in G. A set $S \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a majority dominating set [4] of G if at least half of the vertices of $V(G)$ are either in S or adjacent to the elements of S. A majority dominating set S is minimal if no proper subset of S is a majority dominating set of a graph G. The minimum cardinality of a minimal majority dominating set is called majority domination number of G, is denoted by $\gamma_M(G)$. It is the minimum majority dominating set of G.

A dominating set $S \subseteq V(G)$ such that the induced sub graph $\langle S \rangle$ satisfies the property $\chi(\langle S \rangle) = \chi(G)$ is called as dom-chromatic set [2] of a graph G. The minimum cardinality of a dominating chromatic set is called dom-chromatic number and it is denoted by $\gamma_{ch}(G)$ or $\gamma_\chi(G)$. A dom-chromatic set S of G such that $|S| = \gamma_{ch}(G)$ is the minimum dom-chromatic set of a graph G.

[6] For any graph G, CVR and UVR with respect to domination numbers are defined by, $CVR : \gamma(G - v) \neq \gamma(G)$, for all $v \in V(G)$ and $UVR : \gamma(G - v) = \gamma(G)$, for all $v \in V(G)$.

[5] For any graph G, CVR_M and UVR_M with respect to majority domination numbers are defined by, $CVR_M : \gamma_M(G - v) \neq \gamma_M(G)$, for all $v \in V(G)$ and $UVR_M : \gamma_M(G - v) \neq \gamma_M(G)$, for all $v \in V(G)$.

[2] A graph G said to be a CVR-graph if $\gamma_{ch}(G - u) \neq \gamma_{ch}(G)$, for all $u \in V(G)$ and a graph G said to be a UVR-graph if $\gamma_{ch}(G - u) \neq \gamma_{ch}(G)$, for all $u \in V(G)$.

A set $S \subseteq V(G)$ is said to be a chromatic preserving set or a cp-set if $\chi(\langle S \rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by $cpn(G)$.

[1] The private neighbour set of u with respect to S denoted by $pn[u, S]$ is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$

2. $CVR_{M\chi}$ and $UVR_{M\chi}$ Graphs

Definition 2.1 [3]. A subset S of $V(G)$ is said to be Majority Dominating Chromatic set (MDC set) if S is a majority dominating set and S satisfies $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a majority dominating chromatic set of G is called a majority dominating chromatic number and is denoted by $\gamma_{M\chi}(G)$.

Definition 2.2. For any graph G, the vertex set can be partitioned with respect to MDC sets into three sets $V^0_{M\chi}(G)$, $V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ and is defined by,

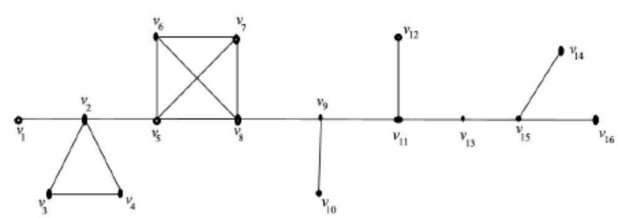
$$V^0_{M\chi}(G) = \{v \in V(G) / \gamma_{M\chi}(G - v) = \gamma_{M\chi}(G)\},$$

$$V^-_{M\chi}(G) = \{v \in V(G) / \gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)\} \text{ and}$$

$$V^+_{M\chi}(G) = \{v \in V(G) / \gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)\}.$$

Definition 2.3. A graph G is said to be a $CVR_{M\chi}$-graph if $\gamma_{M\chi}(G - v) \neq \gamma_{M\chi}(G)$, for every $v \in V(G)$. A graph G is said to be a $UVR_{M\chi}$-graph if $\gamma_{M\chi}(G - v) = (G)$, for every $v \in V(G)$.

Example 2.4. Consider the graph G with $p = 16$.

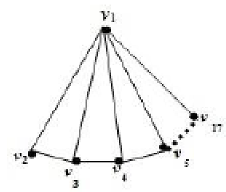


***G*. Figure - (i)**

In the graph G, $S = \{v_5, v_6, v_7, v_8, v_{15}\}$ is the $\gamma_{M\chi}$-set of G. Then $\gamma_{M\chi}(G) = 5$. For the graph $G - \{v_5\}$, $\gamma_{M\chi}(G - \{v_5\}) = |\{v_2, v_3, v_4, v_8\}| = 4$. Therefore $\gamma_{M\chi}(G - v_5) < \gamma_{M\chi}(G)$. Hence $v_5 \in V^-_{M\chi}(G)$. For the graph

$G - \{v_8\}$, $\gamma_{M\chi}(G - v_2) = |\{v_5, v_6, v_7, v_8, v_{15}\}| = 5$. Therefore $\gamma_{M\chi}(G - v_8) = \gamma_{M\chi}(G)$. It implies that $v_8 \in V^0_{M\chi}(G)$.

Example 2.5. Consider the graph $G = F_p$, $p = 17$ a Fan.



***G*. Figure- (ii)**

In this graph G, $\gamma_{M\chi}(G) = |\{v_1, v_2, v_3\}| = 3$. For $G - \{v_2\}$, $\gamma_{M\chi}(G - v_2) = |\{v_1, v_3, v_4\}| = 3$. Therefore $\gamma_{M\chi}(G - v_2) = \gamma_{M\chi}(G)$ and $v_2 \in V^0_{M\chi}(G)$. For the graph $\{G - v_1\}$, $\gamma_{M\chi}(G - v_1) = |\{v_3, v_4, v_7, v_{10}\}| = 4$. Hence $\gamma_{M\chi}(G - v_2) > \gamma_{M\chi}(G)$ and $v_1 \in V^+_{M\chi}(G)$.

Theorem 2.6. *If a graph G is a vertex color critical then* $G \in CVR_{M\chi}$.

Proof. Since the graph G is vertex color critical, $\gamma_{M\chi}(G) = p$. If the removal of any vertex v from $V(G)$, $\chi(G - v) \neq \chi(G)$. It implies that $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$, for every vertex $v \in V(G)$. Hence $G \in CVR_{M\chi}$.

Corollary 2.7. *Let* $G = K_p$, $p \geq 2$. *Then* $G \in CVR_{M\chi}$.

Proof. By the result (3.1) [3], $\gamma_{M\chi}(G) = p$. For the graph $\gamma_{M\chi}(G - v_1) = p - 1$. Hence $\gamma_{M\chi}(G - v_1) < \gamma_{M\chi}(G)$. Therefore $v_1 \in V^-_{M\chi}(G)$. For every vertex $v \in V(G)$, $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$ and $G \in CVR_{M\chi}$.

3. Results on $V^+_{M\chi}(G)$

Proposition 3.1. *Let* $G = K_{1,\, p-1}$. *Then* $v_1 \in V^+_{M\chi}(G)$ *and* $v_i \in V^0_{M\chi}(G)$ *where* v_1 *is a central vertex and* v_i*'s are pendants.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$, where v_1 is the central vertex and others are pendants. The set $S = \{v_1, v_2\}$ is the MDC set of G and $\gamma_{M\chi}(G) = 2$. For a graph $G - \{v_1\}$, $\gamma_{M\chi}(G - v_1) = \left\lceil \frac{p-1}{2} \right\rceil$. Therefore $\gamma_{M\chi}(G - v_1) > \gamma_{M\chi}(G)$. Hence $v_1 \in V^{+}_{M\chi}(G)$. Suppose any pendant v_i, $i = 2, \ldots, p$, $\gamma_{M\chi}(G - v_i) = 2 = \gamma_{M\chi}(G)$. Therefore $v_i \in V^{0}_{M\chi}(G)$, where v_i's are pendants.

Proposition 3.2. *If G has exactly one full degree vertex and other vertices are of degree* $d(v_i) < \frac{p-1}{2}$ *then* $|V^{+}_{M\chi}(G)| = 1$.

Proof. Let G be a graph which contains a full degree vertex v such that $d(v) = p - 1$. Let S be a MDC set of G. Since $d(v) = p - 1$, v must be in majority dominating set S and minimal cp-set of G. Then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Let S' be the $\gamma_{M\chi}$-set of $G' = \{G - v\}$ and $\{G - v\}$ contains isolates, then $\gamma_{M\chi}(G') > |S| = \gamma_{M\chi}(G)$. It implies that $v \in V^{0}_{M\chi}(G)$. If $\{G - v\}$ contains the vertices v_i with $d(v_i) \leq \left\lceil \frac{p-1}{2} \right\rceil$ then $|S'| \geq 2$. Therefore $|S'| \geq |S| + 1$. It implies that $\gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)$ and $v \in V^{+}_{M\chi}(G)$. Thus, all other vertices are $V^{0}_{M\chi}(G)$. Hence $|V^{+}_{M\chi}(G)| = 1$.

Proposition 3.3. *Let T be a tree with p vertices. If a vertex* $v \in V(T)$ *satisfies one of the following conditions.*

(i) *v is in a dominating edge* $e = \{uv\}$ *with* $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$ *and* $d(u) < \left\lceil \frac{p}{2} \right\rceil - 1$.

(ii) *v is a vertex with degree* $d(v) = p - 1$ *and others pendants.*

(iii) *v is in every* $\gamma_{M\chi}$*-set of T. Then* $V^{+}_{M\chi}(T)$.

Proof. Let T be a tree with p vertices.

Case (i) Let $e = \{uv\}$ is a dominating edge with $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and $d(u) < \left\lceil \frac{p}{2} \right\rceil - 1$. Since $\chi(G) = 2$, $S = \{u, v\}$ be a $\gamma_{M\chi}$-set of T. Let $S_1 = \{u, u_1, v_i\}$ be a set of $T - \{v\}$, where u and u_1 are adjacent and v_i's are isolates such that $|N[S_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ with $|S_1| > |S|$. Then $\chi(T) = \chi(\langle S_1 \rangle) = \chi(T - v)$. Thus S_1 is a MDC set of $T - \{v\}$ and $\gamma_{M\chi}(T - v) \leq |S_1|$. Since $|S_1| > |S|$, $\gamma_{M\chi}(T - v) > |S| = \gamma_{M\chi}(T)$. Hence $v \in V^+_{M\chi}(T)$.

Case (ii) Let $d(v) = p - 1$ and $d(v_i) = 1$, for all $v_i \in V(T)$. Then $\gamma_{M\chi}(T) = |\{v, v_1\} = 2|$, for some v_1 such that $d(v_1) = 1$. Since v is adjacent to all vertices v_i of T, $\langle T - \{v\} \rangle$ is disconnected with only isolates. Now, there exists a MDC set S in $T - \{v\}$ with only isolates and $|S| = \left\lceil \frac{p-1}{2} \right\rceil$. It implies that $|S| = \gamma_{M\chi}(T - \{v\}) > \gamma_{M\chi}(T)$ and $v \in V^+_{M\chi}(T)$.

Case (iii) If the vertex v is in every minimum MDC set of T, then v is in a dominating edge $e = uv$ or v is a full degree vertex of T. It implies that $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, $d(u) < \left\lceil \frac{p}{2} \right\rceil - 1$ and other vertices v_i's are of degree with $d(v_i) < \left\lceil \frac{p}{2} \right\rceil - 1$. By Case (i), the vertex $v \in V^+_{M\chi}(T)$.

Proposition 3.4. *For any graph* G, $|V^+_{M\chi}(G)| \leq \gamma_{M\chi}(G)$.

Proof. Let S be a $\gamma_{M\chi}$- set of G. Let $v \in V^+_{M\chi}(G)$. By proposition (3.3), v is in every $\gamma_{M\chi}$-set S of G. Then $v \in S$ and $V^+_{M\chi}(G) \subseteq S$. Hence $|V^+_{M\chi}(G)| \leq |S| = \gamma_{M\chi}(G)$.

Theorem 3.5. *If* $v \in V^+_{M\chi}(G)$ *and* v *is in every minimal* cp *-set of* G *then* $|p_n[v, S]| \geq 2$, *for all* $\gamma_{M\chi}$ *-set of* G.

Proof. Let S be a $\gamma_{M\chi}$-set of G. Let v be a vertex in every minimal cp-set of G. Let v be a vertex in every minimal cp-set of G. Then $\chi(\langle S - v \rangle) = \chi(G - v) < \chi(G)$. Let $Pn[v, S] = \phi$. Then $\{S - v\}$ is a $\gamma_{M\chi}$- set of $\{G - v\}$. It is a contradiction to $v \in V^{+}_{M\chi}(G)$. Suppose $|Pn[v, S]| = \{v\}$. Then v is an isolated vertex in S and hence $v \in V^{0}_{M\chi}(G)$. It is a contradiction to $v \in V^{+}_{M\chi}(G)$. If $|Pn[v, S]| = \{u\}$ then $\{S - v\} \cup \{u\}$ is a $\gamma_{M\chi}$-set of $\{G - v\}$. Thus $\gamma_{M\chi}(G - v) \leq |S| = \gamma_{M\chi}(G)$. It is a contradiction to $v \in V^{+}_{M\chi}(G)$.

Hence, $|Pn[v, S]| \geq 2$.

Proposition 3.6. *For any graph G with an isolate, there exists a $\gamma_{M\chi}$-set of G not containing that isolate.*

Proof. Let v be an isolate of G. If S is a $\gamma_{M\chi}$-set of G containing v then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = \chi(G)$.

Case (i) If $|N[S]| > \left\lceil \frac{p}{2} \right\rceil$ then $|N[S - \{v\}]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S - v \rangle) = \chi(G)$. It implies that $S - \{v\}$ is a $\gamma_{M\chi}$-set. Hence $S - \{v\} = S'$ is a $\gamma_{M\chi}$-set of G without an isolate v.

Case (ii) If $|N[S]| = \left\lceil \frac{p}{2} \right\rceil$ then $|N[S - \{v\}]| \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and $v \neq N[S]$. Now, if $|N[S - \{v\}] \cup \{v_1\}| \geq \left\lceil \frac{p}{2} \right\rceil$, for any $v_1 \in V(G)$ then $S' = S - \{v\} \cup \{v_1\}$. Also, $\chi(\langle S' \rangle) = \chi(\langle S \rangle)$ and $|S'| = |S| = \gamma_{M\chi}(G)$. Hence S' is a $\gamma_{M\chi}$-set of G without an isolate v.

4. Results on $V^{0}_{M\chi}(G)$ and $V^{-}_{M\chi}(G)$

Proposition 4.1. *If G is a graph with $\gamma_{M\chi}(G) = |V(G)|$ then $\in CVR_{M\chi}$.*

Proof. Let G be a graph with p vertices and $\gamma_{M\chi}(G) = |V(G)| = p$. Then

G is a vertex color critical graph. Therefore, for any vertex $v \in V(G)$, the graph $G' = G - \{v\}$ has the value $\gamma_{M\chi}(G') < p$. It implies that $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$, for every $v \in V(G)$. Hence $G \in CVR_{M\chi}$.

Proposition 4.2. *If G is a vertex color critical graph then* $V(G) = V^{-}_{M\chi}(G)$ *but the converse is not true.*

Proof. By Proposition (4.1), $\gamma_{M\chi}(G) = p$ and for all v, $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G) \Rightarrow V^{-}_{M\chi}(G) = V(G)$. For the converse, Let $G = P_p$, $p = 9$. Then $\gamma_{M\chi}(G) = 3$. For any vertex v, $P_9 - \{v\} = P_8$ and $\gamma_{M\chi}(P_8) = 2$. Hence $V(G) = V^{-}_{M\chi}(G)$ and $G \in CVR_{M\chi}$ but $G = P_9$ is not a vertex color critical graph.

Proposition 4.3. *Any Path* P_p, $p \equiv 3 (\mathrm{mod}\ 6)$ *is a* $CVR_{M\chi}$ *graph.*

Proof. Let $G = P_p$, $p = 6k + 3$, $k \geq 1$. Then by the result (3.3) [3], $\gamma_{M\chi}(G) = k + 2$. For each vertex $v \in V(G)$, $\gamma_{M\chi}(G - v) = k + 1$, where $p \equiv 6k + 2$. Hence $P_p \in CVR_{M\chi}$ where $p \equiv 3 (\mathrm{mod}\ 6)$.

Proposition 4.4. *If G is a* $CVR_{M\chi}$ *graph then* $V^{-}_{M\chi}(G) \neq \phi$.

Proof. Since G is a $CVR_{M\chi}$ graph, $V = V^{+}_{M\chi} \cup V^{-}_{M\chi}$.

Suppose $V^{-}_{M\chi}(G) \neq \phi$. (1)

Then $V = V^{+}_{M\chi}(G)$ and $\gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)$, for all $v \in V(G)$. Let S be a $\gamma_{M\chi}$-set with $|S| = p - 1$ of G. Then $V - S \neq \phi$. Let $u \in V - S$ and $\{u\} \subseteq V(G) - S$. It implies that $S \subseteq V(G) - \{u\} = G - u$. Since G is a $CVR_{M\chi}$ graph, $\chi(\langle S \rangle) = \chi(G)$ and $\chi(\langle S \rangle) = \chi(\langle G - u \rangle)$. It implies that S is a $\gamma_{M\chi}$- set of $(G - u)$ and $\gamma_{M\chi}(G - u) \leq |S| = \gamma_{M\chi}(G)$. Therefore $u \in V^{-}_{M\chi}(G)$, it is a contradiction to (1). Hence $V^{-}_{M\chi}(G) \neq \phi$, for any $CVR_{M\chi}$ graph G.

Proposition 4.5. *A Wheel graph* $G = W_p$, $p > 5$ *is a* $CVR_{M\chi}$ *graph when p is even.*

Proof. Let $G = W_p$, $C_{p-1} \vee K_1$. By the result (3.5) [3], $\gamma_{M\chi}(G) = p$, when p is even. Let $V(G) = \{v_1, v_2, \dots, v_{p-1}, v_p\}$ where $v_i \in C_{p-1}$, $i = 1, 2, \dots, p-1$ and $v_p \in k_1$. Suppose $G' = G - \{v_k\}$.

Case (i) Let $\{v_k\}$ be the central vertex of G. Then $G - \{v_k\} = G' = C_{p-1}$. Since p is even, C_{p-1} is an odd cycle. By the result (3.2) [3], $\gamma_{M\chi}(G') = p-1$, Therefore $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$.

Case (ii) Suppose $\{v_t\}$ be any vertex in C_{p-1}. Then the graph G becomes a Fan $G' = (G - \{v_t\}) = P_{p-2} \vee K_1$. By the result (3.6) [3], $\gamma_{M\chi}(G') = 3$. Hence $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$.

In these two cases, the removal of any vertex $\{v_i\}$ in G,

$\gamma_{M\chi}(G - v_i) < \gamma_{M\chi}(G)$. Hence $G \in CVR_{M\chi}$.

Proposition 4.6. *Let* $G = W_p$, p *is odd. Then*

(i) $v_i \in V^0_{M\chi}(G)$ *if* $v_i \in C_{p-1}$.

(ii) $v_i \in V^+_{M\chi}(G)$ *if v is a central vertex of G and* $p \geq 17$.

Proof. For $G = W_p$, $C_{p-1} \vee K_1$, p is odd, $V(G) = \{v_1, v_2, \dots, v_{p-1}, v_p\}$. By the result (3.5) [3], $\gamma_{M\chi}(G) = 3$. (1) The removal of any vertex v from $V(G)$, there exists two cases.

Case (i) Suppose any vertex $v_i \in C_{p-1}$. Then $G' = G - \{v_i\}$ and $G = F_{p-1} = p_{p-2} \vee K_1$, where $(p-1)$ is even. By the result (3.6) [3], $\gamma_{M\chi}(G') = 3$. By the result (1), $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$ and $v_i \in V^0_{M\chi}(G)$, for any vertex $v_i \in C_{p-1}$.

Case (ii) Suppose v_p is a central vertex and $p \geq 17$. The $\gamma_{M\chi}$-set of G is

$S = \{v_1, v_2, v_p\}$. Then $\gamma_{M\chi}(G) = |S| = 3$. If the removal of a central vertex v_p, $G' = G - \{v_p\}$ and G' becomes C_{p-1} even cycle. By the proposition (3.2) [3],

$$\gamma_{M\chi}(G') = \begin{cases} p, & \text{if } p \text{ is odd} \\ \left\lceil \frac{p}{6} \right\rceil, & \text{if } p \equiv (\text{mod } 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1, & \text{if } p \equiv 0,\ 4(\text{mod } 6). \end{cases} \quad (2)$$

For $p \leq 16$, by the result (2), $\gamma_{M\chi}(G') = |S'| = 3$. If $p \geq 17$, by the result (2), $\gamma_{M\chi}(G') = |S'| \geq 4$. Therefore, by the result (1), $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$ and $v_p \in V^{+}_{M\chi}(G)$. Hence $v_p \in V^{+}_{M\chi}(G)$, if $p \geq 17$.

Theorem 4.7. *Let G be a $CVR_{M\chi}$ graph with p vertices. Then $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$-set of G. If G is a $CVR_{M\chi}$-graph then $\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$. Suppose $|S| = \gamma_{M\chi}(G) = p$. Then $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$ holds. Suppose $|S| = \gamma_{M\chi}(G) < p$. Then $V - S \neq \phi$. Now choose any vertex $v \in V - S$. Since $\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$, $v \in V^{-}_{M\chi}(G)$. Therefore $V - S \subseteq V^{-}_{M\chi}(G)$. It implies that $|V - S| \leq |V^{-}_{M\chi}(G)|$. Hence $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$.

Theorem 4.8. *Let $\gamma_{M\chi}(G)$ be the $\gamma_{M\chi}$-number of a graph G. If $\gamma_{M\chi}(G) = |V(G)|$ then $V(G) = V^{-}_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$-set of G and $\gamma_{M\chi}(G) = |V(G)| = p$. Then G is a vertex color critical graph. For any $v \in V(G)$, $\chi(G-v) < \chi(G)$ and it implies that $\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$. Hence $v \in V^{-}_{M\chi}(G)$. For every $v \in S$,

$\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$ is true. Hence $V^{-}_{M\chi}(G) = |V(G)|$.

Theorem 4.9. *Let G be a connected* CVR_{M_χ} *graph with* $\chi(G) \geq 3$. *Then G has a unique* γ_{M_χ}*-set of G if and only if* $\gamma_{M_\chi}(G) = |V(G)|$.

Proof. Let the graph G have a unique γ_{M_χ}-set S. Then we claim that

$$V(G) - S = \phi. \tag{3}$$

Suppose $V - S = \phi$. Since G is a CVR_{M_χ} graph, $\gamma_{M_\chi}(G - u) < \gamma_{M_\chi}(G)$, for every $v \in V - S$. Then for each $v \in V - S$, $\chi(\langle S - v \rangle) = \chi(\langle S \rangle)$ and the induced sub graph $\langle S \rangle$ is a vertex color critical. Hence for any $u \in V - S$, S is a MDC set of $G - \{u\}$, which is a contradiction to the assumption (1). Therefore there exist $v \in V - S$ such that $\chi(\langle S - v \rangle) = \chi(\langle S \rangle)$. Then $Pn[u, S] \neq \phi$, for any $u \in S$,

Case (i) Let $|Pn[u, S]| = 1$. If $Pn[u, S] = \{u\}$ then u is an isolate in $\langle S \rangle$. Since G is connected, $N(u) \neq \phi$ and $N(u) \subseteq V - S$. Also some vertex $w \in V - S$ is adjacent to any vertex in S. Let $w \in N(u)$. Then $(S - u) \cup \{w\}$ is a γ_{M_χ}-set of G, which is a contradiction to the assumption (1). So $Pn[u, S] = \{v\}$. Then $(S - u) \cup \{v\}$ is γ_{M_χ}-set of G, which is a contradiction to (1). Hence $V - S = \phi$. Thus $|V(G)| = \gamma_{M_\chi}(G)$.

Case (ii) Suppose $|Pn[v, S]| \geq 2$. Let $v \in Pn[v, S]$. Then there exists a $w \neq v$ such that $w \in Pn[v, S]$. It implies that $(S - u) \cup \{w\}$ is a γ_{M_χ}-set of G, which is a contradiction to (1). Let $x, w \in Pn[v, S]$. Then $(S - u) \cup \{w\}$ is a γ_{M_χ}-set of $G - x$. Thus, $|V(G)| = |S| = \gamma_{M_\chi}(G)$.

Conversely, $\gamma_{M_\chi}(G) = |V(G)| = p$. It implies that the graph G have a unique MDC set of G.

Theorem 4.10. *If v is an isolated of G then* $v \in V^0_{M_\chi}(G)$.

Proof. Let v be an isolated vertex of G. Then v is not in minimal cp-set of G. Let S be a γ_{M_χ}-set of G and not containing the vertex v. Then

$| N[S] | \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Then $\gamma_{M\chi}(G) = | S |$. For the graph $\{G - v\}$, $\chi(\langle G - v \rangle) = \chi(G)$ and S is again the $\gamma_{M\chi}$-set of $\{G - v\}$. Therefore $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$ and $v \in V^0_{M\chi}(G)$.

Theorem 4.11. *If a vertex $v \in V(G)$ is not in any minimal cp-set of G then $v \in V^0_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$-set of G. Let v be a vertex which is not in any minimal cp-set of G. Then $\chi(\langle S - v \rangle) = \chi(G)$. Hence $Pn[v, S] \neq \phi$. Let $Pn[v, S] = 1$. If $Pn[v, S] = \{v\}$ then v is an isolated vertex in S. By the Proposition (4.10), $v \in V^0_{M\chi}(G)$.

Theorem 4.12. *Let v be a vertex of G with $v \in V^+_{M\chi}(G)$. Then there exists a vertex $u \in V(G)$ such that $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.*

Proof. Let S be as MDC set of G. Then $| N[S] | \geq \left\lceil \frac{p}{2} \right\rceil$.

Case (i) Suppose $| N[S] | \neq \vee(G)$. Then there exists a vertex $u \in N[S]$ and implies that $u \notin S$, $u \in \vee - N[S]$. Then $S \subseteq V - u$ and $| N[S] | \geq \left\lceil \frac{p}{2} \right\rceil$ and $| N_{G-u}[S] | \geq \left\lceil \frac{p}{2} \right\rceil$. It implies that $| N_{G-u}[S] | \geq \left\lceil \frac{p-1}{2} \right\rceil$. Therefore S is a MDC set of $\{G - v\}$. Then $\gamma_{M\chi}(G - u) \leq | S | = \gamma_{M\chi}(G)$. If $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$ then $u \in V^-_{M\chi}(G)$, which is a contradiction to $v \in V^+_{M\chi}(G)$.

Hence $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.

Case (ii) Suppose $N[S] = V(G)$. Let $u \notin S$ and $u \in N[S]$. Then $| N_{G-u}[S] | = p - 1 \geq \left\lceil \frac{p-1}{2} \right\rceil$. Therefore S is a MDC set of $\{G - v\}$. Then $\gamma_{M\chi}(G - u) \leq | S | = \gamma_{M\chi}(G)$. If $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$ then $u \in V^-_{M\chi}(G)$ and

$V(G) \in V^{-}_{M\chi}(G)$, which is a contradiction to $v \in V^{+}_{M\chi}(G)$. Hence $\gamma_{M\chi}(G-u) = \gamma_{M\chi}(G)$.

Case (iii) Suppose $|N[S]| \leq V(G)$. Then there exists a vertex $u \in S$ and $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$. For $S - \{u\}$, $\chi(\langle S - v \rangle) < \chi(\langle S \rangle) = \chi(G)$ and S is not a $\gamma_{M\chi}$-set of G. Therefore choose $S_1 = S - \{u\} \cup \{w\}$ where $w \in V - S$ such that $|N[S_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ and w is adjacent to any vertex of S with $|S_1| = |S|$. Hence S_1 is a $\gamma_{M\chi}$-set of $\{G - v\}$ and $\gamma_{M\chi}(G-u) = |S_1| = |S| = \gamma_{M\chi}(G)$.

5. Conclusion

In this article, it has been discussed that the removal of any vertex of a graph G how affects the majority dom-chromatic number of G. Also the vertex critical classifications $V^{0}_{M\chi}(G), V^{-}_{M\chi}(G)$ and $V^{+}_{M\chi}(G)$ are discussed. The characterisation theorems are also determined for $V^{0}_{M\chi}(G), V^{-}_{M\chi}(G)$ and $V^{+}_{M\chi}(G)$.

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