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OPEN NEIGHBORHOOD COLORING OF SPLITTING GRAPH AND TENSOR PRODUCT OF GRAPHS

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Abstract

In this paper, we determine the open neighborhood chromatic number of splitting graph of some special graph like path graph, cycle graph, star graph and ladder graph in detail. Also, we obtain the open neighborhood chromatic number of the tensor products $P_m \otimes P_n$ and $C_3 \otimes K_{1, n}$.

1. Introduction

All the graphs considered here are simple, finite and undirected graph $G = (V(G), E(G))$. For every vertex $u, v \in V(G)$, the edge connecting two vertices is denoted by $uv \in E(G)$. For all other standard concepts of graph theory, we see [1], [2], [5]. A proper coloring of a graph is an assignment of colors to the vertices such that adjacent vertices receive different colors. The minimum number of colors required to color the vertices is called the chromatic number of the graph denoted by $\chi(G)$.

An open neighborhood coloring of a graph $G(V, E)$ as a coloring $f : V(G) \to Z^{+}$, such that for each $w \in V(G)$ and $\forall\ u, v \in N(w), f(u) \neq f(v)$. An open neighborhood k-coloring of a graph $G(V, E)$ is a k-coloring $f : V(G) \to \{1, 2, \ldots, k\}, k \in Z^{+}$ which admits the conditions of an open neighborhood coloring. The minimum value of k for which G admits an open

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neighborhood k-coloring is called the open neighborhood chromatic number of G and denoted by $\chi_{onc}(G)$.

[3], [4] K. N. Geetha, et al. introduced the notion of open neighborhood coloring and they have discussed the open neighborhood chromatic number of some standard graphs.

For a graph G the splitting graph of G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$. The resultant graph is denoted as $Sp(G)$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. [6], [7] The tensor product of G_1 and G_2 denoted by $G = G_1 \otimes G_2$ is the graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u, v)$ adjacent to $(u', v') \mid d_{G_1}(u, u') = 1$ and $d_{G_2}(v, v') = 1\}$. In this section, an open neighborhood coloring of Splitting graph of path graph P_n, cycle graph C_n, star graphs $K_{1,n}$ and ladder graph L_n are discussed.

Theorem 1.1. *For splitting graph of path graph* $Sp(P_n)$, $\chi_{onc} Sp(P_n) = 4$, *for* $n \geq 3$.

Proof. Let we denote $\{v_1, v_2, \ldots, v_n\}$ as the vertices of P_n, and $\{v'_1, v'_2, \ldots, v'_n\}$ as the new vertices.

The vertex set and edge set of $Sp(P_n)$ are

$$V[Sp(P_n)] = \left\{\left(\bigcup_{i=1}^{n} v_i \cup v'_i\right)\right\} \text{ and } E[Sp(P_n)] = \left\{\left(\bigcup_{i=1}^{n-1} (v_i v_{i+1} \cup v_i v'_{i+1} \cup v'_i v_{i+1})\right)\right\}$$

Hence $|V(Sp(P_n))| = 2n$ and $|E(Sp(P_n))| = 3(n-1)$. Now we define $f : V(Sp(P_n)) \to Z^+$ given by for $n \geq 3$,

$$\text{For } 1 \leq i \leq n,\ f(v_i) = \begin{cases} \{1\}, & \text{if } i \equiv 1, 2 \pmod 4 \\ \{2\}, & \text{if } i \equiv 0, 3 \pmod 4 \end{cases}$$

$$f(v'_i) = \begin{cases} \{3\}, & \text{if } i \equiv 1, 2 \pmod 4 \\ \{4\}, & \text{if } i \equiv 0, 3 \pmod 4. \end{cases}$$

It is easy to verify that f is an open neighborhood 4-coloring of $Sp(P_n)$.

$$\therefore \chi_{onc}(Sp(P_n)) = 4,\ n \geq 3.$$

Hence the theorem. □

Theorem 1.2. *For splitting graph of cycle graph* $Sp(C_n)$, $\chi_{onc}(Sp(C_n))$

$$= \begin{cases} \{4\}, & \text{if } n \equiv 0 \pmod 4 \\ \{6\}, & \text{for } n = 6 \\ \{5\}, & \text{otherwise.} \end{cases}$$

Proof. Let we denote $\{v_1, v_2, \ldots, v_n\}$ as the vertices of C_n and $\{v_1', v_2', \ldots, v_n'\}$ as the new vertices.

The vertex set and edge set of $Sp(C_n)$ is given by

$$V[Sp(C_n)] = \left\{\left(\bigcup_{i=1}^{n} v_i \cup v_i'\right)\right\} \text{ and}$$

$$E[Sp(C_n)] = \left\{\left(\bigcup_{i=1}^{n-1} (v_iv_{i+1} \cup v_iv_{i+1}' \cup v_i'v_{i+1})\right) \cup (v_nv_1 \cup v_n'v_1 \cup v_nv_1')\right\}$$

Then $|V(Sp(C_n))| = 2n$ and $|E(Sp(C_n))| = 3n$. Define $f : V(Sp(C_n)) \to Z^+$ as follows.

Case 1. When $n \equiv 0 \pmod 4$. For $1 \leq i \leq n$

$$\text{For } 1 \leq i \leq n,\ f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1, 2 \pmod 4 \\ 2, & \text{if } i \equiv 0, 3 \pmod 4 \end{cases}$$

$$f(v_i') = \begin{cases} 3, & \text{if } i \equiv 1, 2 \pmod 4 \\ 4, & \text{if } i \equiv 0, 3 \pmod 4. \end{cases}$$

Case 2. When $n \equiv 1 \pmod 4$. For $1 \leq i \leq n - 1$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1, 2 \pmod 4 \\ 2, & \text{if } i \equiv 0, 3 \pmod 4 \end{cases} \quad f(v_i') = \begin{cases} 4, & \text{if } i \equiv 2, 3 \pmod 4 \\ 5, & \text{if } i \equiv 0, 1 \pmod 4 \end{cases}$$

and $f(v_n) = f(v_i') = 3$.

Case 3. When $n \equiv 2 \pmod 4$. For $1 \leq i \leq n-2$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1, 2 \pmod 4 \\ 2, & \text{if } i \equiv 0, 3 \pmod 4 \end{cases}$$

$f(v_{n-1}) = f(v_n) = 3.$

For $1 \leq i \leq n-6$,

$$f(v'_i) = \begin{cases} 3, & \text{if } i \equiv 0, 3 \pmod 4 \\ 4, & \text{if } i \equiv 1, 2 \pmod 4 \end{cases}$$

$f(v'_{n-1}) = f(v'_n) = f(v'_{n-4}) = f(v'_{n-5}) = 5$ and $f(v'_{n-2}) = f(v'_{n-3}) = 4.$

Case 4. When $n \equiv 2 \pmod 4$. For $1 \leq i \leq n-2$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1, 2 \pmod 4 \\ 2, & \text{if } i \equiv 0, 3 \pmod 4 \end{cases}$$

$f(v_{n-1}) = 3$, $f(v_n) = 2$ and $f(v'_1) = 4$

$$\text{For } 2 \leq i \leq n-2, \; f(v'_i) = \begin{cases} 3, & \text{if } i \equiv 2, 3 \pmod 4 \\ 4, & \text{if } i \equiv 0, 1 \pmod 4 \end{cases}$$

and $f(v'_{n-1}) = f(v'_n) = 5$. It can be easily seen that f is an open neighborhood coloring of C_n so that $\chi_{onc}(Sp(C_n)) = \begin{cases} \{4\}, & \text{if } n \equiv 0 \pmod 4 \\ \{6\}, & \text{for } n = 6 \\ \{5\}, & \text{otherwise} \end{cases}$

Hence the theorem. □

Theorem 1.3. *For splitting graph of star graph* $Sp(K_{1,n})$,

$$\chi_{onc} Sp(K_{1,n}) = 2n, \; n \geq 2.$$

Proof. Let $\{v, v_1, v_2, \ldots, v_n\}$ be the vertices of star $K_{1,n}$ and $\{v', v'_1, v'_2, \ldots, v'_n\}$ be the new vertices of $K_{1,n}$. Here v is the apex vertex. The vertex set and edge set of $Sp(K_{1,n})$ is given by

$$V[Sp(K_{1,n})] = \left\{\left(\bigcup_{i=1}^{n} v_i \cup v_i'\right) \cup v \cup v'\right\} \text{ and}$$

$$E[Sp(K_{1,n})] = \left\{\left(\bigcup_{i=1}^{n-1} (vv_i \cup vv_i' \cup v'v_i)\right)\right\}$$

Then $|V(Sp(K_{1,n}))| = 2(n+1)$ and $|E(Sp(K_{1,n}))| = 3n$. We define $f: V(Sp(K_{1,n})) \to Z^+$ given by

For $1 \leq i \leq n$,

$$f(v_i) = i + n,\ f(v_i') = i,\ f(v) = 1,\ f(v') = 2$$

It can be easily seen that f is an open neighborhood coloring of $K_{1,n}$ so that

$$\chi_{onc} Sp(K_{1,n}) = 2n,\ n \geq 2.$$

Hence the theorem. □

Theorem 1.4. *For ladder graph* L_n,

$$\chi_{onc}(L_n) = 3,\ n \geq 3.$$

Proof. The ladder graph denoted by L_n is obtained from two path P_n with $\{v_1, v_2, \ldots, v_n\}$ and $\{v_1, v_2, \ldots, u_n\}$ vertices by joining the vertices $v_i u_i$ for $1 \leq i \leq n$.

$$V[L_n] = \left\{\left(\bigcup_{i=1}^{n} v_i \cup v_i'\right)\right\} \text{ and } E[L_n] = \left\{\left(\bigcup_{i=1}^{n-1} (v_i v_{i+1} \cup u_i u_{i+1})\right) \cup \left(\bigcup_{i=1}^{n} u_i v_i\right)\right\}$$

we have $|V(L_n)| = 2n$ and $|E(L_n)| = 3n - 2$. We color the graph L_n by defining a function $f: V(L_n) \to \{1, 2, 3\}$.

For $1 \leq i \leq n$,

$$f(v_i) = f(u_i) = \begin{cases} \{1\}, & \text{for } i \equiv 1 \pmod 3 \\ \{2\}, & \text{for } i \equiv 2 \pmod 3 \\ \{3\}, & \text{for } i \equiv 0 \pmod 3 \end{cases}$$

It can be easily seen that *f* is an open neighborhood coloring of L_n so that

$$\chi_{onc}(L_n) = 2n,\ n \geq 3.$$

Hence the theorem. □

Theorem 1.5. *For splitting graph of ladder graph* $Sp(l_n)$, $\chi_{onc}(Sp(l_n)) = 6,\ n \geq 3$.

Proof. Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ are the vertices of l_n and $\{v'_1, v'_2, \ldots, v'_n, u'_1, u'_2, \ldots, u'_n\}$ are the new vertices. The vertex set and edge set of $Sp(l_n)$ are

$$V[Sp(l_n)] = \left\{\left(\bigcup_{i=1}^{n} v_i \cup v'_i \cup u_i \cup u'_i\right)\right\} \text{ and}$$

$$E[Sp(l_n)] = \left\{\left(\bigcup_{i=1}^{n-1} (v_iv_{i+1} \cup u_iu_{i+1} \cup v_iv'_{i+1} \cup v'_iv_{i+1} \cup u_iu'_{i+1} \cup u'u_{i+1_i})\right)\right.$$

$$\left.\cup \left(\bigcup_{i=1}^{n} v_iu_i \cup u_iv_i \cup v_iu'_i)\right)\right\}$$

Hence $|V(Sp(l_n))| = 4n$ and $|E(Sp(l_n))| = 3(3n-2)$. Now we define $f : V(Sp(l_n)) \to \{1, 2, \ldots, 6\}$ given by for $n \geq 3$,

$$\text{For } 1 \leq i \leq n,\ f(v_i) = f(u_i) = \begin{cases} \{1\}, & \text{for } i \equiv 1 \pmod 3 \\ \{2\}, & \text{for } i \equiv 2 \pmod 3 \\ \{3\}, & \text{for } i \equiv 0 \pmod 3 \end{cases} \text{ and } f(v'_1) = 5.$$

$$\text{For } 2 \leq i \leq n,\ f(v'_i) = \begin{cases} \{6\}, & \text{for } i \equiv 2, 3 \pmod 6 \\ \{4\}, & \text{for } i \equiv 4, 5 \pmod 6 \\ \{5\}, & \text{for } i \equiv 0, 1 \pmod 6 \end{cases}$$

$$\text{For } 1 \leq i \leq n,\ f(u'_i) = \begin{cases} \{4\}, & \text{for } i \equiv 1, 2 \pmod 6 \\ \{5\}, & \text{for } i \equiv 3, 4 \pmod 6 \\ \{6\}, & \text{for } i \equiv 0, 5 \pmod 6). \end{cases}$$

It can be easily seen that *f* is an open neighborhood coloring of $Sp(L_n)$ so that $\chi_{onc}(Sp(l_n)) = 6,\ n \geq 3.$ Hence the theorem.

2. Open Neighborhood Coloring of $P_m \otimes P_n$,

Now we start by investigating a tensor product of two path P_m and $P_n,\ m,\ n \in N,$ admits Open neighborhood coloring conjecture. Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices of P_m and $\{w_1, w_2, \ldots, w_n\}$ be the vertices of P_n.

Theorem 2.1. *For the tensor product* $P_m \otimes P_n$,

$$\chi_{onc}(P_m \otimes P_n) = 4,\ m,\ n \geq 3.$$

Proof. Let the vertex set of P_m and P_n are $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively. The vertex and edge set of $P_m \otimes P_n$ are given by

$$V(P_m \otimes P_n) = \{u_{i,\ j} \mid i = 1, 2, 3, \ldots, m,\ j = 1, 2, 3, \ldots, n\}$$

$$E(P_m \otimes P_n) = \{\left(\bigcup_{i=1}^{m-1} u_{i,\ j}u_{i+1,\ j+1}\right) \cup \left(\bigcup_{i=1}^{m-1} u_{i+1,\ j}u_{i,\ j+1}\right),$$

$$j = 1, 2, 3, \ldots, (n-1)\}.$$

Clearly, $P_m \otimes P_n$ has mn vertices and $2(m-1)(n-1)$ edges.

The vertices and edges are colored by defining $f : V(P_m \otimes P_n) \to \{1, 2, 3, 4\}$.

For $1 \leq i \leq m$ and $1 \leq i \leq n$,

If $i \equiv 1, 2 \pmod 4$, then

$$f(u_{i,\ j}) = \begin{cases} \{1\}, & \text{for } j \equiv 1, 2 \pmod 4 \\ \{2\}, & \text{for } j \equiv 0, 3 \pmod 4 \end{cases}$$

If $i \equiv 0, 3 \pmod 4$, then

$$f(u_{i,j}) = \begin{cases} \{3\}, & \text{for } j \equiv 1, 2 \pmod 4 \\ \{4\}, & \text{for } j \equiv 0, 3 \pmod 4 \end{cases}$$

It can be easily seen that *f* is an open neighborhood coloring of $P_m \otimes P_n$ so that

$$\chi_{onc}(P_m \otimes P_n) = 4,\ m, n \geq 3.$$

Theorem 2.2. *For the tensor product* $C_3 \otimes k_{1,n}$,

$$\chi_{onc}(C_3 \otimes k_{1,n}) = 3(n-1),\ n \geq 3.$$

Proof. Let $\{v_1, v_2, v_3\}$ be the vertices of C_3 and $\{w_1, w_2, \ldots, w_n\}$ be the vertices of $k_{1,n}$. The vertex and edge set of $C_3 \otimes k_{1,n}$ are given by

$$V(C_3 \otimes k_{1,n}) = \{u_{i,j} \mid i = 1, 2, 3,\ j = 1, 2, 3, \ldots, n\}$$

$$E(C_3 \otimes k_{1,n}) = \left\{ \left(\bigcup_{\substack{j=1 \\ i=1,2}}^{n-1} u_{1,1}u_{i+1,\,j+1} \right) \cup \left(\bigcup_{\substack{j=1 \\ i=1,3}}^{n-1} u_{2,1}u_{i,\,j+1} \right) \cup \left(\bigcup_{\substack{j=1 \\ i=1,2}}^{n-1} u_{3,1}u_{i,\,j+1} \right) \right\}$$

The vertices and edges are colored by defining $f : V(C_3 \otimes k_{1,n}) \to \{1, 2, 3, \ldots, k\}$.

For $1 \leq i \leq 3$, $f(u_{i,1}) = f(u_{i,2}) = i$

$$f(u_{i,j}) = i + 3j - 6,\ \text{for } j \geq 3.$$

It can be easily seen that *f* is an open neighborhood coloring of $C_3 \otimes k_{1,n}$ so that

$$\chi_{onc}(C_3 \otimes k_{1,n}) = 3(n-1),\ n \geq 2. \qquad \square$$

Conclusion

In this note, we have proved that the open neighborhood chromatic number for the splitting graph of some special graph like Path graph, cycle graph, star graph and ladder graph. Also, we found the open neighborhood coloring of tensor products $P_m \otimes P_n$ and $C_3 \otimes K_{1,\, n}$.

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