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FIXED POINT THEOREMS IN *b*-METRIC SPACE

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Abstract

In this article basic definitions of b-metric space and complete metric space are presented. Some fixed-point theorems in b-metric space have been derived which are considered as the extension of the results obtained by the Hardy Rogers and Reich.

1. Introduction

In the development of non-linear analysis, fixed point theory plays a awfully vital role. Also, it's been wide utilized in completely different branches of engineering and sciences. Metric mounted purpose theory is a vital a part of mathematical analysis as a result of its applications in several areas like variation and linear inequalities, improvement and approximation theory. The mounted purpose theorem in metric areas plays a major role to construct ways to resolve the issues in arithmetic and sciences. Though metric mounted purpose theory may be a huge field of study and is capable of resolution several equations. To beat the matter of measurable functions with

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relevance a live and their convergence, Czerwik [7-10] desires associate degree extension of mathematical space. Mistreatment this concept, he conferred a generalization of the famous Banach mounted purpose theorem within the b-metric areas. Several researchers studied the b-metric areas like Chug [6], Aydi [1], Du [16], Boriceanu [2-4], Kir [12], Bota [5], Pacurar [15], Shi [15].

2. Preliminaries

The basic properties and definitions of b-metric spaces are highlighted by Czerwik [7]

Definition 1. T is the set having $s \geq 1 \in R$ then the self-map ρ on T is a b-metric if

1. $\rho(a, b) = 0$ iff $a = b$.

2. $\rho(a, b) = \rho(b, a)$

3. $\rho(a, c) \leq s[\rho(a, b) + \rho(b, c)]$ for all $a, b, c \in T$.

The pair (T, ρ) is b-metric space.

When, $s = 1$ it trim down as the metric space in standard form.

Definition 2 [7]**.** Let (T, ρ) be the b-metric spaces then $\{a_n\}$ in T is

1. If and providing for each $\in > 0$ there exists $n(\in) \in N,$ so that for each $n, m \geq n(\in)$ we have $\rho(a_n, a_m) < \in$ for the Cauchy sequence.

2. If and providing for each $a \in T$ so that for all $\in > 0$ there exists $n(\in) \in N,$ so that for every $n \leq n(\in)$ we have $\rho(a_n, a_m) < \in$ for the convergent sequence.

Definition 3 [7]**.**

1. If (T, ρ) be the b-metric spaces then the subset $L \in\subset T$ is

(i) If and providing for each progression of essentials of L there are a subsequence which converges to the component of L is compact.

(ii) If and providing for every progression $\{a_n\}$ in the L then converge to an single element a, $a \in L$ is closed.

2. If all the progression are convergent then the b-metric space will be complete.

3. Main Results

Theorem 1. *If T is the complete metric with the metric namely ρ and $M : T \to T$ is the function through the following property*

$$\rho(Ma, Mb) \leq r\rho(a, Ma) + s\rho(b, Mb) + t\rho(a, b)$$

Every $a, b \in T$ while r, s, t are the non-negative along with persuade $r + s + t < 1.$ Hence M has a distinctive fixed point.

Theorem 2. *T be the absolute b-metric space along the metric namely ρ and $M : T \to T$ is the function through the following*

$$\rho(Ma, Mb) \leq r\rho(a, Ma) + s\rho(b, Mb) + t\rho(a, b)$$

Every $a, b \in T$ while r, s, t are non-negative along with persuade $r + u(s + t) < 1$ for $u \geq 1$ then M has a unique fixed point.

Proof of Theorem 2. When $a_o \in T$ and $\{a_n\}$ be the progression within T, such that

$$a_n = Ma_{n-1} = M^n a_o$$

At present

$$\rho(a_{n+1}, a_n) = \rho(Ma_n, Ma_{n-1})$$

$$\leq r\rho(a_n, Ma_n) + s\rho(a_{n-1}, Ma_{n-1}) + t\rho(a_n, a_{n-1})$$

$$= r\rho(a_n, a_{n+1}) + s\rho(a_{n-1}, a_n) + t\rho(a_n, a_{n-1})$$

$$\Rightarrow (1 - r)\rho(a_{n+1}, a_n) \leq (s + t)\rho(a_n, a_{n-1})$$

$$\Rightarrow \rho(a_{n+1}, a_n) \leq \frac{(s + t)}{(1 - r)}\rho(a_n, a_{n-1}) = v\rho(a_n, a_{n-1})$$

Where $v = \dfrac{(s + t)}{(1 - r)} < \dfrac{1}{u}$

Enduring it gives $\rho(a_{n+1}, a_n) \leq p^n \rho(a_0, a_1)$

$\Rightarrow$ M is the contraction mapping.

Now verify $\{a_n\}$ is the Cauchy progression in T.

If $m, n > 0$ while $m > n$

$$\rho(a_n, a_m) \leq u[\rho(a_n, a_{n+1}) + \rho(a_{n+1}, a_m)]$$

$$\leq u\rho(a_n, a_{n+1}) + u^2\rho(a_{n+1}, a_{n+2}) + u^3\rho(a_{n+2}, a_{n-3}) + \cdots$$

$$\leq up^n\rho(a_0, a_1) + u^2p^{n+1}\rho(a_0, a_1) + u^3p^{n+2}\rho(a_0, a_1) + \cdots$$

$$= up^n\rho(a_0, a_1)[1 + up + (up)^2 + (up)^3 + \cdots] = \frac{(up)^n}{1 - sp}\rho(a_0, a_1)$$

Infer (T, p) is the b-metric when $\{b_n\}$ is the progression in T so that

$$\rho(b_{n+1}, b_{n+2}) \leq \lambda\rho(b_n, b_{n+1}), n = 0.1, \cdots$$

If $0 \leq \lambda < 1$, then the progression $\{b_n\}$ is a Cauchy progression in T endow with $u.\lambda < 1$.

Hence by taking $\lim_{n\to\infty} \rho(a_n, a_m) = 0$

$\Rightarrow \{a_n\}$ is a Cauchy progression in T.

If T is complete, then $\{a_n\}$ converges to a^*

Now a^* is the fixed point of M. While

$$\rho(a^*, Ma^*) \leq u[\rho(a^*, a_n) + \rho(a_n, Ma^*)]$$

$$\leq u[\rho(a^*, a_n) + \rho(Ma_{n-1}, Ma^*)]$$

$$\leq \frac{u}{(1 - xu)}[\rho(a^*, a_n) + y\rho(a_{n-1}, a_n) + z\rho(a_{n-1}, a^*)]$$

$$\leq \frac{u}{(1 - xu)}[\rho(a^*, a_n) + yp^n\rho(a_{n-1}, a_n) + z\rho(a_{n-1}, a^*)]$$

Considering $\lim n \to \infty$

$$\lim_{n\to\infty} \rho(a^*, Ma^*) = 0$$

$$\Rightarrow a^* = Ma^*$$

$\Rightarrow a^*$ is a fixed point of M. To prove the distinctiveness of the fixed point, consider a and b, 2 fixed points of M.

$$a = Ma = My$$

$$\rho(a, b) = \rho(Ma, Mb) \leq r\rho(a, Ma) + s\rho(b, Mb) + t\rho(a, b)$$

$$\rho(a, b) \leq c\rho(a, b)$$

Which is not true. Hence the evidence.

Theorem 3. *If* (T, p) *is the complete b-metric spaces and the mapping* $M : T \to T$ *gratify the subsequent conditions in every* $a, b \in T$

$$\rho(Ma, Mb) \leq r\rho(a, Ma) + s\rho(b, Mb) + t\rho(a, Mb) + e\rho(b, Ma) + f\rho(a, b)$$

while $r, s, t, e, f \in T$ *are nonnegative and we set* $a = r + s + t + e + f$, *such that* $a \in \left(o, \frac{1}{2u}\right)$, *for* $u \geq 1$ *then M has a unique fixed point.*

Proof of Theorem 3. If the condition (1) hold on (T, ρ) for a self map M on it. Then $a \in \left(o, \frac{1}{2u}\right)$ there exists $\beta < \frac{1}{2u}$ such that

$$\rho(Ma, M^2a) \leq \beta\rho(a, Ma)$$

Let $a_o \in T$ and $\{a_n\}$ be a sequence in T, such that $a_n = Ma_{n-1}, M^n a_0$

Using (2) we get

$$\rho(a_{n+1}, a_n) \leq \beta^n \rho(a_0, a_1)$$

Now we show that $\{a_n\}$ is a Cauchy sequence in T.

Let $m, n > 0$, with $m > n$

$$\rho(a_n, a_m) \leq u[\rho(a_n, a_{n+1}) + \rho(a_{n+1}, a_m)]$$

$$\leq u\rho(a_n, a_{n+1}) + u^2\rho(a_{n+1}, a_{n+2}) + u^3\rho(a_{n+2}, a_{n-3}) + \cdots$$

$$\leq u\beta^n\rho(a_0, a_1) + u^2\beta^{n+1}\rho(a_0, a_1) + u^3\beta^{n+2}\rho(a_0, a_1) + \cdots$$

Considering $\lim n \to \infty$ results $\lim_{n\to\infty} \rho(a_n, a_m) = 0$

This implies $\{a_n\}$ is the Cauchy progression in T.

If T is complete, then $\{a_n\}$ converges to a^*.

To prove a^* is fixed point of M.

While

$$\rho(a^*, Ma^*) \leq u[\rho(a^*, a_n) + \rho(a_n, Ma^*)] \leq u[\rho(a^*, a_n) + \rho(Ma_{n-1}, Ma^*)]$$

$$\leq u[\rho(a^*, a_n) + r\rho(a_{n-1}, Ma_{n-1}) + s\rho(a^*, Ma^*)$$

$$+ \rho(a^*, Ma^*) \leq u[\rho(a^*, a_n) + r\rho(a_{n-1}, Ma_{n-1}) + s\rho(a^*, Ma^*)$$

$$t\rho(a_{n-1}, Ma^*) + e\rho(a^*, Ma_n) + f\rho(a_{n-1}, a^*)]$$

$$+ t\rho(a_{n-1}, Ma^*) + e\rho(a^*, Ma_{n-1}) + f\rho(a_{n-1}, Ma^*)]$$

Considering $\lim n \to \infty$ implies $\rho(a^*, Ma^*) \leq u(s + t)\rho(a^*, Ma^*)$

It is a contradiction until $a^* = Ma^*$.

To prove fixed point is exclusive. Let us consider a and b be 2 fixed points of M, so that $a = Ma, b = Mb$.

Again

$$\rho(a, b) = \rho(Ma, Mb)$$

$$\leq r\rho(a, Ma) + s\rho(b, Mb) + t\rho(a, My) + e\rho(b, Ma) + f\rho(a, b)$$

$$\leq (t + e + f)\,\rho\,(a, b)$$

Which is a contradiction. Hence the proof.

4. Conclusion

In this article, some basic definitions and fixed-point results in the complete b-metric spaces are presented. The results are obtained by extending the results mentioned by the authors Reich [14] and Hardy Rogers [11] in their articles for complete metric spaces.

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