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SOME SEPARATION AXIOMS IN $Ng\delta s$- CLOSED SETS

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Abstract

The basic objective of this paper is to introduce and investigate the properties of few $Ng\delta s$-separation in nano topological spaces and studied some of its relation between the existing sets.

1. Introduction

In 1970, Levine [1] introduced the concept of generalized closed sets in topological spaces. The notion of Nano topology was introduced by Lellis Thivagar [2] which was defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it and also defined Nano closed sets, Nano-interior and Nano-closure of a set. Sakthivel and Devaki introduced nano generalized δs-closed sets in nano topological spaces. The basic definitions are recalled from the following papers [1], [2], [3], [4], [5], [6], [7], [8], [9].

2. Separation Axioms

In this section, we introduce and study weak separation axioms such as $Ng\delta s$-T_0, $Ng\delta s$-T_1 and $Ng\delta s$-T_2 spaces and obtain some of their properties.

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Definition 2.1. A nano topological space U is said to be $Ng\delta s$-T_0 space if for each pair of distinct points x and y of U, there exists a $Ng\delta s$-open set containing one point but not the other.

Theorem 2.2. *A nano topological space U is a $Ng\delta s$-T_0 space if and only if $Ng\delta s$ closures of distinct points are distinct.*

Proof. Let x and y be distinct points of U. Since U is $Ng\delta s$-T_0 space, there exists a $Ng\delta s$-open set G such that $x \in G$ and $y \notin G$. Consequently, $U - G$ is a $Ng\delta s$-closed set containing y but not x. But $Ng\delta s\mathrm{cl}\,\{y\}$ is the intersection of all $Ng\delta s$-closed sets containing y. Hence $y \in Ng\delta s\mathrm{cl}\,\{y\}$ but $x \notin Ng\delta s\mathrm{cl}\,\{y\}$ as $x \notin U - G$. Therefore, $Ng\delta s\mathrm{cl}\,\{x\} \neq Ng\delta s\mathrm{cl}\,\{y\}$.

Conversely, let $Ng\delta s\mathrm{cl}\,\{x\} \neq Ng\delta s\mathrm{cl}\,\{y\}$ for $x \neq y$. Then there exists at least one point $z \in U$ such that $z \in Ng\delta s\mathrm{cl}\,\{x\}$ but $z \notin Ng\delta s\mathrm{cl}\,\{y\}$. We claim $x \notin Ng\delta s\mathrm{cl}\,\{y\}$, because if $x \in Ng\delta s\mathrm{cl}\,\{y\}$ then $\{x\} \subset Ng\delta s\mathrm{cl}\,\{y\}$ implies $Ng\delta s\mathrm{cl}\,\{x\} \subset Ng\delta s\mathrm{cl}\,\{y\}$. So $z \in Ng\delta s\mathrm{cl}\,\{y\}$ which is a contradiction. Hence $x \notin Ng\delta s\mathrm{cl}\,\{y\}$. which implies $x \in U - Ng\delta s\mathrm{cl}\,\{y\}$, which is a $Ng\delta s$-open set containing x but not y. Hence U is $Ng\delta s$- T_0 space.

Theorem 2.3. *If $f : U \to V$ is a bijection strongly $Ng\delta s$-open and U is $Ng\delta s$- T_0 space, then V is also $Ng\delta s$- T_0 space.*

Proof. Let y_1 and y_2 be two distinct points of V. Since f is bijective there exist distinct points x_1 and x_2 of U such that $Ng\delta s$- T_0 and $f(x_1) = y_1$ and $f(x_2) = y_2$. Since U is $Ng\delta s$- T_0 space there exists a $Ng\delta s$-open set G such that $x_1 \in G$ and $x_2 \notin G$.

Therefore $y_1 = f(x_1) \in f(G)$ and $y_2 = f(x_2) \notin f(G)$. Since f being strongly $Ng\delta s$-open function, $f(G)$ is $Ng\delta s$-open in V. Thus, there exists a $Ng\delta s$-open set $f(G)$ in V such that $y_1 \in f(G)$ and $y_2 \notin f(G)$. Therefore V is $Ng\delta s$- T_0 space.

Definition 2.4. A nano topological space U is said to be $Ng\delta s$- T_1 space if for any pair of distinct points x and y, there exist a $Ng\delta s$-open sets G and H such that $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$.

Theorem 2.5. *A nano topological space U is $Ng\delta s$- T_1 space if and only if singletons are $Ng\delta s$- closed sets.*

Proof. Let U be a $Ng\delta s$- T_1 space and $x \in K$. Let $y \in K - \{x\}$. Then for $x \neq y$, there exists $Ng\delta s$-open set K_y such that $y \in K_y$ and $x \notin K_y$. Consequently, $y \in K_y \subset U - \{x\}$. That is $U - \{x\} = [fK_y : y \in K - \{x\}g]$, which is $Ng\delta s$- open set. Hence $\{x\}$ is $Ng\delta s$- closed set.

Conversely, suppose $\{x\}$ is $Ng\delta s$-closed set for every $x \in U$. Let x and $y \in U$ with $x \neq y$. Now $x \neq y$ implies $y \in K - \{x\}$. Hence $K - \{x\}$ is $Ng\delta s$-open set containing y but not x. Similarly, $K - \{y\}$ is $Ng\delta s$-open set containing x but not y. Therefore U is $Ng\delta s$- T_1 space.

Theorem 2.6. *The property being $Ng\delta s$- T_1 space is preserved under bijection and strongly $Ng\delta s$- open function.*

Proof. Let $f : U, V$ be bijective and strongly $Ng\delta s$-open function. Let U be a $Ng\delta s$- T_1 space and y_1, y_2 be any two distinct points of V. Since f is bijective there exist distinct points x_1, x_2 of U such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now U being a $Ng\delta s$- T_1 space, there exist $Ng\delta s$-open sets G and H such that $x_1 \in G, x_2 \notin G$ and $x_1 \notin H, x_2 \in H$. Therefore $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ and $y_1 = f(x_1) \notin f(H)$. Now f being strongly $Ng\delta s$-open, $f(G)$ and $f(H)$ are $Ng\delta s$-open subsets of V such that $y_1 \in f(G)$ but $y_2 \notin f(G)$ and $y_2 \in f(H)$ and $y_1 \notin f(H)$. Hence V is $Ng\delta s$- T_1 space.

Theorem 2.7. *Let $f : U \to V$ be bijective and $Ng\delta s$-open function. If U is $Ng\delta s$- T_1 and $TNg\delta s$ space, then V is $Ng\delta s$- T_1 space.*

Proof. Let y_1, y_2 be any two distinct points of V. Since f is bijective there exist distinct points x_1, x_2 of U such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ Now U being a $Ng\delta s$- T_1 space, there exist $Ng\delta s$-open sets G and H such that $x_1 \in G, x_2 \notin G$ and $x_1 \notin H, x_2 \in H$. Therefore $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ and $y_1 = f(x_1) \notin f(H)$. Now U is

$Ng\delta s$-space which implies G and H are nano open sets in U and f is $Ng\delta s$-open function, $f(G)$ and $f(H)$ are $Ng\delta s$-open subsets of V. Thus there exist $Ng\delta s$-open sets such that $y_1 \in f(G)$ but $y_2 \notin f(G)$ and $y_2 \in f(H)$ and $y_1 \notin f(H)$. Hence V is $Ng\delta s$-T_1 space.

Theorem 2.8. *If* $f : U \to V$ *is* $Ng\delta s$-*continuous injection and V is* NT_1 *then U is* $Ng\delta s$-T_1 *space.*

Proof. Let $f : U \to V$ be $Ng\delta s$-continuous injection and V be NT_1. For any two distinct points x_1, x_2 of U there exist distinct points y_1, y_2 of V such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since V is NT_1 space there exist nano open sets G and H in V such that $y_1 \in G$, $y_2 \notin G$ and $y_1 \notin H$, $y_2 \in H$. That is $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Since f is $Ng\delta s$-continuous $f^{-1}(G)$, $f^{-1}(H)$ are $Ng\delta s$-open sets in U. Thus, for two distinct points x_1, x_2 of U there exist $Ng\delta s$-open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Therefore U is $Ng\delta s$-T_1 space.

Theorem 2.9. *If* $f : U \to V$ *is* $Ng\delta s$-*irresolute injective function and V is* $Ng\delta sT_1$ *space then U is* $Ng\delta s$-T_1 *space.*

Proof. Let x_1, x_2 be pair of distinct points in U. Since f is injective there exist distinct points y_1, y_2 of V such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since V is $Ng\delta s$-T_1 space there exist $Ng\delta s$-open sets G and Y in V such that $y_1 \in G$, $y_2 \notin G$ and $y_1 \notin H$, $y_2 \in H$. That is $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_1 \notin f^{-1}(G)$. Since f is $Ng\delta s$-irresolute $f^{-1}(G)$, $f^{-1}(H)$ are $Ng\delta s$-open sets in U. Thus, for two distinct points x_1, x_2 of U there exist $Ng\delta s$-open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Therefore U is $Ng\delta s$-T_1 space.

Definition 2.10. A nano topological space U is said to be $Ng\delta s$-T_2 space if for any pair of distinct points x and y, there exist disjoint $Ng\delta s$-open sets G and H such that $x \in G$ and $y \in H$.

Theorem 2.11. *If $f : U \to V$ is $Ng\delta s$- continuous injection and V is NT_2 then U is $Ng\delta s$- T_2 space.*

Proof. Let $f : U \to V$ be $Ng\delta s$- continuous injection and V be NT_2. For any two distinct points x_1, x_2 of U there exist distinct points y_1, y_2 of V such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since V is NT_2 space there exist disjoint nano open sets G and H in V such that $y_1 \in G$ and $y_2 \in H$. That is $x_1 \in f^{-1}(G)$ and $x_2 \in f^{-1}(H)$. Since f is $Ng\delta s$- continuous $f^{-1}(G), f^{-1}(H)$ are $Ng\delta s$- open sets in U. Further f is injective, $f^{-1}(G) f^{-1}(H) = f^{-1}(G H)$ $= f^{-1}(\emptyset) = \emptyset$. Thus, for two disjoint points x_1, x_2 of U there exist disjoint $Ng\delta s$- open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$ and $x_2 \in f^{-1}(H)$. Therefore U is $Ng\delta s$-T_2 space.

Theorem 2.12. *If $f : U \to V$ is $Ng\delta s$- irresolute injective function and V is $Ng\delta s$-T_2 space then U is $Ng\delta s$-T_2 space.*

Proof. Let x_1, x_2 be pair of distinct points in U. Since f is injective there exist distinct points y_1, y_2 of V such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since V is $Ng\delta s$-T_2 space there exist disjoint $Ng\delta s$- open sets G and H in V such that $y_1 \in G$ and $y_2 \in H$. That is $x_1 \in f^{-1}(G)$ and $x_2 \in f^{-1}(H)$. Since f is $Ng\delta s$- irresolute injective $f^{-1}(G), f^{-1}(H)$ are distinct $Ng\delta s$- open sets in U. Thus, for two disjoint points x_1, x_2 of U there exist disjoint $Ng\delta s$- open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$ and $x_2 \in f^{-1}(H)$. Therefore U is $Ng\delta s$-T_2 space.

Definition 2.13. A nano topological space U is called $N\delta$-$T_3/4$ if every $Ng\delta s$- closed set in it is $N\delta$- closed.

Theorem 2.14. *For a nano topological space U, if U is a $N\delta$-$T_3/4$ space, then every nano singleton set $\{x\}$ is $N\delta$- open or $N\delta$- closed.*

Proof. Suppose U is a $N\delta$-$T_3/4$ space. If $\{x\}$ is not $N\delta$- closed, then $U - \{x\}$ is not $N\delta$- open. Then the only $N\delta$- open set containing $U - \{x\}$ is U.

Therefore $U - \{x\}$ is $Ng\delta s$-closed set of U. Since U is a $N\delta\text{-}T_3/4$ space, $U - \{x\}$ is $N\delta$-closed, which implies $\{x\}$ is $N\delta$-open.

Theorem 2.15. *Every $N\delta\text{-}T_3/4$ space is $NT_3/4$ space.*

Proof. Follows from the fact that every $N\delta g$-closed set is $Ng\delta s$-closed set.

Definition 2.16. A subset A of U is $N\delta$-nowhere dense if $\text{Nint}\,(Ncl(A)) = \emptyset$.

Lemma 2.17. *For a nano topological space U the following are valid*

(1) *Every nano singleton set is $N\delta$-pre closed or $N\delta$-open in U.*

(2) *Every nano singleton set is $N\delta$-nowhere dense or $N\delta$-pre open in U.*

Theorem 2.18. *For a nano topological space the following are equivalent U is $N\delta\text{-}T_3/4$ space.*

(1) *Every $N\delta$-pre closed singleton set of U is $N\delta$-closed.*

(2) *Every non $N\delta$-open singleton set of U is $N\delta$-closed.*

Proof (i) $\Rightarrow$ (ii) Let $x \in U$ and $\{x\}$ be $N\delta$-pre closed in U. By above lemma, $\{x\}$ is not $N\delta$-open and hence by theorem 2.3.2, $\{x\}$ is $N\delta$-closed. (ii) $\Rightarrow$ (i) If $\{x\}$ is not $N\delta$-open for some $x \in U$, then by lemma 2.3.5, it is $N\delta$-pre closed and by (ii) it is $N\delta$-closed. Hence U is $N\delta\text{-}T_3/4$ space.

(ii) $\Rightarrow$ (iii) If $\{x\}$ is not $N\delta$-open for some $x \in U$, by lemma 2.3.5, $\{x\}$ is $N\delta$-pre closed and by (ii), it is $N\delta$-closed.

(iii) $\Rightarrow$ (ii) Let $\{x\}$ be $N\delta$-pre closed in $\{x\}$. By lemma 2.3.5, $\{x\}$ is not $N\delta$-open and hence by (iii), it is $N\delta$-closed.

Definition 2.19. A nano topological space U is called $N\delta\text{-}T_{1/2}$ space if every $Ng\delta s$-closed set in it is nano semi closed.

Theorem 2.20. *For a nano topological space U the following conditions are equivalent*

(1) *U is* $N\delta\text{-}T_{1/2}$ *space.*

(2) *Every nano singleton set is either* $N\delta$- *closed or nano semi open.*

Proof (i) $\Rightarrow$ (ii)**.** If $\{x\}$ is not $N\delta$-closed, then $U - \{x\}$ is not $N\delta$-open. Then the only $N\delta$-open set containing $U - \{x\}$ is U. Therefore $U - \{x\}$ is $Ng\delta s$-closed set in U. By (i), $U - \{x\}$ is nano semi closed, which implies $\{x\}$ is nano semi open.

(ii) $\Rightarrow$ (i) Let $A \subseteq U$ be $Ng\delta s$-closed set and $x \in NScl\,(A)$. Then consider the following cases

Case (1). Let $\{x\}$ be $N\delta$-open. Since $x \in NScl\,(A)$, then $\{x\} \cap NScl\,(A) \neq \emptyset$. This implies $x \in A$.

Case (2): Let $\{x\}$ be $N\delta$-closed. Assume that $x \notin A$, then $x \in NScl\,(A) - A$, which implies $\{x\} \subset NScl\,(A) - A$. This is not possible according to Theorem 2.2.9. This shows that, $x \in A$.

So in both cases, $NScl\,(A) \subset A$. Since the reverse inclusion is trivial, implies $NScl\,(A) = A$. Therefore A is nano semi closed.

Theorem 2.21. *Every* $Ng\delta s\text{-}T_{1/2}$ *space is* $Ng\delta s\text{-}T_{1/2}$ *space.*

Proof. Let U be a $N\delta\text{-}T_{3/4}$ space. Then by Theorem 2.3.2, every nano singleton set of U is $N\delta$-open or $N\delta$-closed. But every $N\delta$-open set is nano semi open set. Thus every nano singleton set of U is nano semi open or $N\delta$-closed. By Theorem 2.3.8, U is $Ng\delta s\text{-}T_{1/2}$ space.

Theorem 2.22. *Every* $Ng\delta s\text{-}T_{1/2}$ *space is nano semi* $\text{-}T_{1/2}$ *space.*

Proof. Let A be Nsg-closed subset of U. Since every Nsg-closed set is $Ng\delta s$-closed and U is $Ng\delta s\text{-}T_{1/2}$ space, implies A is nano semi closed. Hence U is nano semi $\text{-}T_{1/2}$.

Theorem 2.23. *For any nano topological space U*

(1) $NSO\,(U) \subset Ng\delta sO(U)$.

(2) *A space U is* $N\delta\text{-}T_{1/2}$ *space if and only if* $NSO\,(U) = Ng\delta sO(U)$.

Proof. (i) if U is nano semi open, then $U - A$ is nano semi closed. So $U - A$ is $Ng\delta s$- closed, this implies A is $Ng\delta s$- open. Hence $SO(U) \subset Ng\delta sO(U)$. (ii) Let A be a $Ng\delta s\text{-}T_{1/2}$ space and $A \in Ng\delta sO(U)$. Then $U - A$ is $Ng\delta s$- closed set. By hypothesis, $U - A$ is nano semi closed and hence $A \in SO(U)$. Therefore $Ng\delta sO(U) \subset NSO(U)$. By (i), $NSO\,(U)C \subset Ng\delta sO(U)$. Therefore $NSO\,(U) = Ng\delta sO(U)$.

Conversely, let $NSO\,(U) = Ng\delta sO(U)$ and A be a $Ng\delta s$- closed set. Then $U - A$ is $Ng\delta s$- open. Hence $U - A$ is nano semi open, which implies A is nano semi closed. Thus every $Ng\delta s$- closed set is nano semi closed. Therefore U is $Ng\delta s\text{-}T_{1/2}$- space.

Lemma 2.24. *For a space U the following are equivalent*

(1) *Every* $N\delta$- *pre open singleton set is* $N\delta$- *closed.*

(2) *Every nano singleton set is* $N\delta$- *nowhere dense or* $N\delta$- *closed.*

Proof (i) $\Rightarrow$ (ii)**.** By Lemma 2.3.5, every nano singleton set is either $N\delta$- nowhere dense or $N\delta$- pre open. In the first case we are done and in the second case $N\delta$- closedness follows from the assumption.

(ii) $\Rightarrow$ (i) Let $\{x\}$ be $N\delta$- pre open. Assume that $\{x\}$ is not $N\delta$- closed. Then by (ii), it is $N\delta$- nowhere dense. Thus $\{x\} \subset N\,\text{int}\,(N\delta cl\,\{x\}) = \emptyset$, which is not possible. Hence $\{x\}$ is $N\delta$- closed.

Theorem 2.25. *For a space U the following are equivalent*

(1) *U is* $N\delta\text{-}T_1$ *space.*

(2) *U is* $N\delta\text{-}T_{3/4}$ *space and every singleton set is* $N\delta$- *nowhere dense or* $N\delta$- *closed.*

(3) *U is* $N\delta\text{-}T_{3/4}$ *space and every* $N\delta$- *pre open singleton set is* $N\delta$- *closed.*

Proof (i) $\Rightarrow$ (ii)**.** Obvious.

(ii) $\Rightarrow$ (i) If nano singleton set is not $N\delta$- closed, then it must be $N\delta$- open, since U is $N\delta$-$T_{3/4}$- space. But nano singleton set is $N\delta$- open if and only if it is nano regular open, which implies singleton set is nano regular open. Moreover, by rest of assumption U is $N\delta$- nowhere dense at the same time, U must be $N\delta$-T_1- space. (ii) $\Rightarrow$ (iii) Follows from Lemma 2.3.15.

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