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COMMON FIXED POINT RESULTS IN $\mathcal{V}$-FUZZY METRIC SPACES USING w-COMPATIBLE MAPS

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Abstract

The aim of this paper to prove a common fixed point theorem for two compatible maps in generalized $\mathcal{V}$-fuzzy metric spaces. The fundamental outcome is likewise outlined by a guide to exhibit the level of legitimacy of our speculation.

1. Introduction

Mustafa and Sims [8] brought the however of the thought of G-metric spaces as a speculation of metric spaces. Besides, Sedghi et al. [9] presented the idea of S-metric spaces as one of the speculations of the metric spaces. Abbas et al. [2] broadened the thought of S-metric spaces to A-metric space by stretching out the definition to n-tuple. In 1965, Zadeh [13] at first presented the idea of fuzzy sets. From that point forward, a few powerful mathematicians thought about the idea of fuzzy sets to present many

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energizing ideas in the field of science, like fuzzy differential equations, fuzzy logic and fuzzy metric spaces. A fuzzy metric space is notable to be a significant speculation of the metric space. In 1975, kramosil and Michalek [7] utilized the idea of fuzzy sets to present the thought of fuzzy metric spaces. George and Veeramani [3] modified the idea of fuzzy metric spaces in the feeling of Kramosil and Michalek [7]. Sun and Yang [11] begat the possibility of G-fuzzy metric spaces. Vishal Gupta and Ashima Kanwar [12] introduce the $\mathcal{V}$-fuzzy metric space. In 1986, Jungck [6] introduced the concept of compatible maps in metric spaces. We prove common fixed point theorem for w-compatible in $\mathcal{V}$-fuzzy metric spaces.

2. Preliminaries

Definition 2.1 [10]. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a * b < c * d$ whenever $a < c$ and $b < d$, and $a, b, c, d \in [0, 1]$.

Definition 2.2 [12]. Consider $\mathcal{X}$ be a non empty set. A triple $(\mathcal{X}, \mathcal{V}, *)$ is said to be $\mathcal{V}$-Fuzzy Metric Spaces $(\mathcal{V} - FMS)$ where $*$ is a continuous norm and $\mathcal{V}$ is a fuzzy set on $\mathcal{X}^n \times (0, \infty)$ satisfying the following conditions for all $t, s > 0$.

(V-1) $\mathcal{V}(v, v, \ldots v, u, t) > 0$ for all $v, u \in \mathcal{X}$ with $v \neq u$,

(V-2) $\mathcal{V}(v_1, v_1, \ldots, v_1, v_2, t) \geq \mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t)$ for all $v_1, v_2, \ldots, v_{n-1}, v_n \in \mathcal{X}$ with $v_2 \neq v_3 \neq \ldots \neq v_n$,

(V-3) $\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t) = 1 \Leftrightarrow v_1 = v_2 = \ldots = v_n$,

(V-4) $\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t) = \mathcal{V}(p\{v_1, v_2, \ldots, v_{n-1}, v_n\}, t)$ where p is a permutation function,

(V-5) $\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t+s) \geq \mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, \ell, t)$
$$* \mathcal{V}(\ell, \ell, \ldots, \ell, v_n, s),$$

(V-6) $\lim_{t\to\infty} \mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t) = 1,$

(V-7) $\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.3. Consider $\mathcal{X} = \mathbb{R}$ and $(\mathcal{X}, \mathcal{A})$ be a $\mathcal{A}$ metric spaces. Define $\mathcal{V} : \mathcal{X}^n \times (0, \infty) \to [0, 1]$ such that

$$\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t) = e^{-\frac{\mathcal{A}(v_1, v_2, \ldots, v_{n-1}, v_n)}{t}}$$

for all $v_1, v_2, \ldots, v_{n-1}, v_n \in \mathcal{X}$ and $t > 0$. Then $(\mathcal{X}, \mathcal{V}, *)$ is a $\mathcal{V}-FMS$.

Lemma 2.4 [12]. *Consider* $(\mathcal{X}, \mathcal{V}, *)$ *be a* $\mathcal{V}-FMS$. *Then* $\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t)$ *is non-decreasing with respect to t.*

Lemma 2.5 [12]. *Consider* $(\mathcal{X}, \mathcal{V}, *)$ *be a* $\mathcal{V}-FMS$ *such that*

$$\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, kt) \geq \mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t)$$

with $k \in (0, 1)$. *Then* $v_1 = v_2 = \ldots = v_n$.

Definition 2.6 [12]. Consider $(\mathcal{X}, \mathcal{V}, *)$ be a $\mathcal{V}-FMS$.

(i) A sequence $\{v_m\}$ is said to be convergent to v if

$$\lim_{m\to\infty} \mathcal{V}(v_m, v_m, \ldots, v_m, v, t) = 1.$$

(ii) A sequence $\{v_m\}$ is said to be a Cauchy sequence if

$$\lim_{m,\, n\to\infty} \mathcal{V}(v_m, v_m, \ldots, v_m, v_n, t) = 1.$$

(iii) $(\mathcal{X}, \mathcal{V}, *)$ is said to be complete if every Cauchy sequence in $\mathcal{X}$ is convergent.

Definition 2.7 [12]. Let $\mathcal{P}$ and $\mathcal{Q}$ be two self maps on a $\mathcal{V}-FMS$ $(\mathcal{X}, \mathcal{V}, *)$. If

$$\lim_{m\to\infty} \mathcal{V}(\mathcal{P}\mathcal{Q}v_m, \mathcal{P}\mathcal{Q}v_m, \ldots, \mathcal{P}\mathcal{Q}v_m, \mathcal{Q}\mathcal{P}v_m, t) = 1$$

where $\{v_m\}$ is a sequence in $\mathcal{X}$ which satisfies $\lim_{m\to\infty} \mathcal{P}v_m = \lim_{m\to\infty} \mathcal{Q}v_m = x$ for some point $x \in \mathcal{X}$ and $t > 0$, the maps $\mathcal{P}$ and $\mathcal{Q}$ are said to be w-compatible.

3. Main Results

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{V}, *)$ be a complete $\mathcal{V} - FMS$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two self maps from $\mathcal{X}$ into itself such that*

$$\mathcal{P}(\mathcal{X}) \subset \mathcal{Q}(\mathcal{X}) \tag{3.1.1}$$

$\mathcal{P}$ and $\mathcal{Q}$ is *continuous* (3.1.2)

$\mathcal{V}(\mathcal{P}v_1, \mathcal{P}v_2, \ldots, \mathcal{P}v_{n-1}, \mathcal{P}v_n, \rho t) \geq \mathcal{V}(\mathcal{Q}v_1, \mathcal{Q}v_2, \ldots, \mathcal{Q}v_{n-1}, \mathcal{Q}v_n, t)$ *for each* $v_1, v_2, \ldots, v_{n-1}, v_n \in \mathcal{X}$ *and* $0 \leq \rho < 1$ (3.1.3)

$\mathcal{P}$ and $\mathcal{Q}$ are w-compatible maps. (3.1.4)

Then, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point in $\mathcal{X}$.

Proof. Let v_0 be an arbitrary point in $\mathcal{X}$. Choose a point $v_1 \in \mathcal{X}$ such that $\mathcal{P}v_0 = \mathcal{Q}v_1$ with $\mathcal{P}(\mathcal{X}) \subset \mathcal{Q}(\mathcal{X})$. Construct a sequence $\{v_h\}$ in $\mathcal{X}$ as follows:

$$u_h = \mathcal{P}v_h = \mathcal{Q}v_{h+1},\ h = 0, 1, \ldots$$

From (3.1.3), we have

$$\begin{aligned}
\mathcal{V}(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+1}, t) &\geq \mathcal{V}\left(\mathcal{Q}v_h, \mathcal{Q}v_h, \ldots, \mathcal{Q}v_h, \mathcal{Q}v_{h+1}, \frac{t}{\rho}\right) \\
&= \mathcal{V}\left(\mathcal{P}v_{h-1}, \mathcal{P}v_{h-1}, \ldots, \mathcal{P}v_{h-1}, \mathcal{P}v_h, \frac{t}{\rho}\right) \\
&\geq \mathcal{V}\left(\mathcal{Q}v_{h-1}, \mathcal{Q}v_{h-1}, \ldots, \mathcal{Q}v_{h-1}, \mathcal{Q}v_h, \frac{t}{\rho^2}\right) \\
&= \mathcal{V}\left(\mathcal{P}v_{h-2}, \mathcal{P}v_{h-2}, \ldots, \mathcal{P}v_{h-2}, \mathcal{P}v_h, \frac{t}{\rho^2}\right)
\end{aligned}$$

$$\vdots$$

$$\mathcal{V}(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+1}, t) \geq \mathcal{V}\left(\mathcal{P}v_0, \mathcal{P}v_0, \ldots, \mathcal{P}v_0, \mathcal{P}v_1, \frac{t}{\rho^h}\right) \quad (3.1.5)$$

For any $m \in \mathbb{N}$, $t = \frac{t}{m} + \frac{t}{m} + \ldots + \frac{t}{m}$ and using (V-5) frequently many times,

$$\mathcal{V}(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+m}, t)$$

$$= \mathcal{V}\left(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+m}, \frac{t}{m} + \frac{(m-1)t}{m}\right)$$

$$\geq \mathcal{V}\left(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+1}, \frac{t}{m}\right)$$

$$* \mathcal{V}\left(\mathcal{P}v_{h+1}, \mathcal{P}v_{h+1}, \ldots, \mathcal{P}v_{h+1}, \mathcal{P}v_{h+m}, \frac{(m-1)t}{m}\right)$$

$$= \mathcal{V}\left(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+1}, \frac{t}{m}\right)$$

$$* \mathcal{V}\left(\mathcal{P}v_{h+1}, \mathcal{P}v_{h+1}, \ldots, \mathcal{P}v_{h+1}, \mathcal{P}v_{h+m}, \frac{t}{m} + \frac{(m-2)t}{m}\right)$$

$$* \mathcal{V}\left(\mathcal{P}v_{h+2}, \mathcal{P}v_{h+2}, \ldots, \mathcal{P}v_{h+2}, \mathcal{P}v_{h+m}, \frac{(m-2)t}{m}\right)$$

$$\vdots$$

$$\geq \mathcal{V}\left(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+1}, \frac{t}{m}\right) * \mathcal{V}\left(\mathcal{P}v_{h+1}, \mathcal{P}v_{h+1}, \ldots, \mathcal{P}v_{h+1}, \mathcal{P}v_{h+2}, \frac{t}{m}\right)$$

$$* \ldots * \mathcal{V}\left(\mathcal{P}v_{h+m-1}, \mathcal{P}v_{h+m-1}, \ldots, \mathcal{P}v_{h+m-1}, \mathcal{P}v_{h+m} \frac{t}{m}\right).$$

Using (3.1.5) we obtain,

$$\mathcal{V}(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+m}, t)$$

$$\geq \mathcal{V}\left(\mathcal{P}v_0, \mathcal{P}v_0, \ldots, \mathcal{P}v_0, \mathcal{P}v_1, \frac{t}{\rho^h m}\right) * \mathcal{V}\left(\mathcal{P}v_0, \mathcal{P}v_0, \ldots, \mathcal{P}v_0, \mathcal{P}v_1, \frac{t}{\rho^{h+1} m}\right)$$

$$* \ldots * \mathcal{V}\left(\mathcal{P}v_0, \mathcal{P}v_0, \ldots, \mathcal{P}v_0, \mathcal{P}v_1, \frac{t}{\rho^{h+m-1}m}\right).$$

Taking limit as $h, m \to \infty$, on both side

$$\lim_{h, m\to\infty} \mathcal{V}(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}v_{h+m}, t) = 1 * 1 * \ldots * 1 = 1.$$

Thus, $\{\mathcal{P}v_h\}$ is a Cauchy sequence in $\mathcal{X}$. Since $(\mathcal{X}, \mathcal{V}, *)$ is complete $\mathcal{V}-FMS$, it has a limit in $\mathcal{X}$ such that

$$\lim_{h\to\infty} u_h = \lim_{h\to\infty} \mathcal{P}v_h = \lim_{h\to\infty} \mathcal{Q}v_{h+1} = \mathfrak{z}.$$

Since the maps $\mathcal{Q}$ and $\mathcal{Q}$ is continuous (assume that $\mathcal{Q}$ is continuous), $\lim_{h\to\infty} \mathcal{Q}\mathcal{P}v_h = \mathcal{Q}\mathfrak{z}$. Further, the maps $\mathcal{P}$ and $\mathcal{Q}$ are w-compatible,

$$\lim_{h\to\infty} \mathcal{V}(\mathcal{Q}\mathcal{P}v_h, \mathcal{Q}\mathcal{P}v_h, \ldots, \mathcal{Q}\mathcal{P}v_h, P\mathcal{Q}v_h, t) = 1$$

Implies $\lim_{h\to\infty} \mathcal{P}\mathcal{Q}v_h = \mathcal{Q}\mathfrak{z}$. From (3.1.3), we have

$$\mathcal{V}(\mathcal{Q}\mathfrak{z}, \mathcal{Q}\mathfrak{z}, \ldots, \mathcal{Q}\mathfrak{z}, \mathfrak{z}, \rho t) = \mathcal{V}(\mathcal{P}\mathcal{Q}v_h, \mathcal{P}\mathcal{Q}v_h, \ldots, \mathcal{P}\mathcal{Q}v_h, \mathcal{P}v_h, \rho t)$$

$$\geq \mathcal{V}(\mathcal{Q}\mathcal{Q}v_h, \mathcal{Q}\mathcal{Q}v_h, \ldots, \mathcal{Q}\mathcal{Q}v_h, \mathcal{Q}\mathfrak{z}, t).$$

Proceeding limit as $h \to \infty$, we have $\mathcal{Q}\mathfrak{z} = \mathfrak{z}$. Again by (3.1.1), we obtain

$$\mathcal{V}(\mathcal{P}v_h, \mathcal{P}v_h, \ldots, \mathcal{P}v_h, \mathcal{P}\mathfrak{z}, \rho t) \geq \mathcal{V}(\mathcal{Q}v_h, \mathcal{Q}v_h, \ldots, \mathcal{Q}v_h, \mathcal{Q}\mathfrak{z}, t)$$

and taking limit as $h \to \infty$, we have $\mathcal{P}\mathfrak{z} = \mathfrak{z}$. Hence, $\mathcal{P}\mathfrak{z} = \mathcal{Q}\mathfrak{z} = \mathfrak{z}$ and $\mathfrak{z}$ is a common fixed point of $\mathcal{P}$ and $\mathcal{Q}$. Eventually, the uniqueness of $\mathfrak{z}$ as the common fixe point of $\mathcal{P}$ and $\mathcal{Q}$ as follows:

Suppose that $z(\neq \cdot\mathfrak{z})$ be another common fixed point of $\mathcal{P}$ and $\mathcal{Q}$. Then

$$\mathcal{V}(\mathfrak{z}, \mathfrak{z}, \ldots, \mathfrak{z}, z, t) = \mathcal{V}(\mathcal{P}\mathfrak{z}, \mathcal{P}\mathfrak{z}, \ldots, \mathcal{P}\mathfrak{z}, \mathcal{P}z, t) \geq \mathcal{V}\left(\mathcal{Q}\mathfrak{z}, \mathcal{Q}\mathfrak{z}, \ldots, \mathcal{Q}\mathfrak{z}, \mathcal{Q}z, \frac{t}{\rho}\right)$$

$$= \mathcal{V}\left(\mathfrak{z}, \mathfrak{z}, \ldots, \mathfrak{z}\ z, \frac{t}{\rho}\right)$$

for $0 \leq \rho < 1$, so this is a contraction. Therefore, $z = \mathfrak{z}$. Hence, the common fixed point of $\mathcal{P}$ and $\mathcal{Q}$ is unique.

Example 3.2. Let $\mathcal{X} = [-1, 1]$ with $\mathcal{V}(v_1, v_2, \ldots, v_{n-1}, v_n, t) = \dfrac{t}{t + \mathcal{A}(v_1, v_2, \ldots, v_{n-1}, v_n)}$. Clearly, $(\mathcal{X}, \mathcal{V}, *)$ is a complete $\mathcal{V} - FMS$. Let $\mathcal{P}$ and $\mathcal{Q}$ be maps from $\mathcal{X}$ into itself defined as $\mathcal{P}(v) = \dfrac{v}{6}$ and $\mathcal{Q}(v) = \dfrac{v}{6}$ for all $v \in \mathcal{X}$. Then,

$$\mathcal{P}(\mathcal{X}) = \left[-\frac{1}{6}, \frac{1}{6}\right] \subset \left[-\frac{1}{4}, \frac{1}{4}\right] = \mathcal{Q}(\mathcal{X})$$

and the maps $\mathcal{P}$, $\mathcal{Q}$ are continuous. Also,

$$\mathcal{V}(\mathcal{P}v_1, \mathcal{P}v_2, \ldots, \mathcal{P}v_{n-1}, \mathcal{P}v_n, t) \geq \mathcal{V}(\mathcal{Q}v_1, \mathcal{Q}v_2, \ldots, \mathcal{Q}v_{n-1}, \mathcal{Q}v_n, t)$$

satisfies for all $v_1, v_2, v_{n-1}, v_n \in \mathcal{X}$ and $\dfrac{4}{6} \leq \rho < 1$. Also, the maps $\mathcal{P}$ and $\mathcal{Q}$ are w-compatible since

$$\lim_{h \to \infty} \mathcal{V}(\mathcal{QP}v_h, \mathcal{QP}v_h, \ldots, \mathcal{QP}v_h, \mathcal{PQ}v_h, t) = 1$$

where $\{v_h\} = \dfrac{1}{h}$ is a sequence for $h = 1, 2, \ldots$ in $\mathcal{X}$ such that

$$\lim_{h \to \infty} \mathcal{Q}v_h = \lim_{h \to \infty} \frac{1}{6h} = 0$$

and

$$\lim_{h \to \infty} \mathcal{P}v_h = \lim_{h \to \infty} \frac{1}{6h} = 0$$

for $0 \in \mathcal{X}$. Thus, 0 is the unique common fixed point of $\mathcal{P}$ and $\mathcal{Q}$.

Conclusion

In this paper, verified the existence of unique common fixed point for w-compatible maps in $\mathcal{V} - FMS$ with suitable example.

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