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FIXED POINTS FOR RECIPROCALLY CONTINUOUS MAPPINGS AND POINTWISE *R*-WEAKLY COMMUTING MAPPINGS

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Abstract

In this paper, we introduce (ψ, ϕ)-weak contraction condition involving cubic terms of distance function and prove some fixed point theorems for pairs of reciprocally continuous and pointwise R-weakly commuting mappings satisfying newly introduced contraction condition. We also provide an application and an example in support of our result.

1. Introduction and Preliminaries

Banach contraction principle [2] is known as the basic tool of fixed point theory, it ensures the existence of a unique fixed point for every contraction mapping $T(say)$ defined on a complete metric space E. The mapping T in Banach contraction principle is always uniformly continuous. For the last ten decades, authors are continuously trying to extend and generalize the Banach contraction principle in various directions.

In 1969, Boyd and Wong [3] introduced ϕ contraction condition of the form $d(Tu, Tv) \leq \phi(d(u, v))$ for all $u, v \in E$, where T is a self map on a complete metric space E and $\phi : [0, \infty) \rightarrow [0, \infty)$ is an upper semi continuous

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function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. In 1997, Alber and Guerre-Delabriere [1] generalized ϕ contraction to ϕ-weak contraction in Hilbert spaces, which was further extended and proved by Rhoades [16] in complete metric space.

A self map T on a complete metric space is said to be a ϕ-weak contraction if for each $u, v \in E$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0$ if and only if $t > 0$ such that

$$d(Tu, Tv) \leq d(u, v) - \phi(d(u, v)). \tag{1.1}$$

The function ϕ in the above inequality (1.1) is known as control function or altering distance function. The notion of control function was given by Khan et al. [12] as follows.

Definition 1.1 [12]**.** An altering distance is a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following

(i) ϕ is an increasing and continuous function,

(ii) $\phi(t) = 0$ if and only if $t = 0$.

In 2009, Zhang and Song [19] gave the notion of generalized ϕ-weak contraction by generalizing the concept of ϕ-weak contraction.

Definition 1.2 [19]**.** Two self mappings S and T on a metric space (E, d) are said to be generalized ϕ-weak contractions if there exists a mapping $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(t) = 0$ such that

$$d(Su, Tv) \leq M(u, v) - \phi(M(u, v)) \text{ for all } u, v \in E,$$

where $M(u, v) = \max\{d(u, v), d(u, Su), d(v, Tv), \frac{d(u, Tv) + d(v, Su)}{2}\}.$

Another direction of generalization of Banach contraction principle concerns the coincidence points and common fixed points of pair of mappings satisfying contractive conditions. In 1976, G. Jungck [8] generalized Banach contraction principle by using the notion of commuting maps. The use of notion of commutative mappings in fixed point theory literature became a turning moment. Then, first attempt to relax commutative condition of mapping to weak commutative condition was initiated by Sessa [17].

In 1986, Jungck [9] further weakened the notion of commutativity/weak commutative to compatible mappings. In 1994, R. Pant [14] presented the concept of pointwise R-weak commuting mappings and showed a relation between compatible mappings and pointwise R-weak commuting mappings by proving that compatible mappings are pointwise R-weak commuting mappings but converse may not be true.

In 2013, Murthy and Prasad [13] introduced a weak contraction that involves cubic terms of distance function. In 2018, Jain et al. [6] generalized this result of Murthy and Prasad [13] for the pairs of compatible mappings. In 2019, Jain et al. [7] generalized this result of Murthy and Prasad [13] for the pairs of pointwise R-weakly commuting mappings.

In this paper, we introduce a generalized (ψ, ϕ)-weak contraction condition involving cubic terms of distance function and state and prove common fixed point theorem for pairs of pointwise R-weakly commuting mappings satisfying the newly defined contraction condition along with reciprocal continuity, which generalize the result of Murthy and Prasad [13] and Jain et al. [7].

Now we first recall some basic concepts which are useful for our work.

Definition 1.3. Let (E, d) be a metric space. Two mappings $S, T : E \to E$ are said to be compatible [9] if and only if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z,$ for some $z \in E.$

Definition 1.4. Let (E, d) be a metric space. Two mappings $S, T : E \to E$ are said to be non-compatible [9] if there exists a sequence $\{u_n\}$ in E such that

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = t,$$

for some $t \in E,$ but $\lim_{n\to\infty} d(STu_n, TSu_n)$ is either non zero or non-existent.

Definition 1.5. Let S and T be two self mappings on a metric space (E, d). Then S and T are said to be pointwise R-weakly commuting mappings on E, if given $u \in E$, there exists $R > 0$ such that

$$d(STu, TSu) \leq Rd(Su, Tu).$$

Remark 1.1. It is obvious that the notion of pointwise R-weak commutativity is equivalent to the notion of commutativity at coincidence points.

Remark 1.2. Since compatible mappings commute at their coincidence points, therefore, compatible mappings are pointwise R-weakly commuting mappings but converse may not be true.

Example 1.1. Let $E = [0, 10]$ and d be a usual metric. Let $S, T : E \to E$ be two mappings defined by

$$Su = \begin{cases} u, & u = 0; \\ 8, & u \in (0, 2.5]; \\ u - 2.5, & u \in (2.5, 10]. \end{cases} \qquad Tu = \begin{cases} u, & u = 0; \\ 0, & u \in (2.5, 10]; \\ 4, & u \in (0, 2.5]. \end{cases}$$

The mappings S and T are pointwise R-weakly commuting mappings, since they commute at their coincidence points. Let us consider the sequence $\{u_n\}$ defined by $u_n = 2.5 + \frac{1}{n}$, $n \geq 1$. Then $Su_n \to 0, Tu_n \to 0$ $STu_n \to 0$ and $TSu_n \to 4$. as $n \to \infty$. So, S and T are non-compatible. Moreover, mappings S and T are discontinuous at $u = 0$. Hence, mappings S and T are pointwise R-weakly commuting but not compatible.

In 1999, Pant [15] introduced the notion of reciprocally continuity as follows.

Definition 1.6 [15]**.** Let S and T be self maps on a metric space (E, d). S and T are said to be reciprocally continuous, if

$$\lim_{n\to\infty} STu_n = Sz \text{ and } \lim_{n\to\infty} TSu_n = Tz,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} TSu_n = z$, for some $z \in E$.

Remark 1.3. It is clear that a pair of continuous self maps is reciprocally continuous, but the converse may not true.

Example 1.2. Let $E = [0, 10]$ and d be a usual metric. Let $S, T : E \to E$ be two mappings defined by

$$Su = \begin{cases} 0, & u = 0; \\ 2, & u \in (0, 10]. \end{cases} \qquad Tu = \begin{cases} 0, & u = 0; \\ 4, & u \in (0, 10]. \end{cases}$$

Clearly, mappings S and T are reciprocally continuous but not continuous.

Remark 1.4. Compatibility and reciprocal continuous are independent of each other (see [18]).

2. Fixed Point Theorem

In this paper, we will prove fixed point theorems for pairs of pointwise R-weakly commuting mappings by using the control function $\psi \in \Psi$, where Ψ is a collection of all functions $\psi : [0, \infty)^4 \to [0, \infty)$ satisfying the following conditions:

(ψ_1) ψ is non decreasing and upper semi continuous in each coordinate variables,

(ψ_2) $\Delta(t) = \max\{\psi(t, t, 0, 0), \psi(0, 0, 0, t), \psi(0, 0, t, 0), \psi(t, t, t, t)\} \leq t$, for each $t > 0$.

Let Φ be a collection of all the functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions.

(ϕ_1) ϕ is a continuous function,

(ϕ_2) $\phi(t) > t$ for each $t > 0$ and $\phi(0) = 0$.

Let (E, d) be a metric space and f, g, S and T be four self mappings on E satisfying the following conditions:

(C_1) $S(E) \subset g(E)$ and $T(E) \subset f(E)$,

(C_2) for $\psi \in \Psi$, $\phi \in \Psi$, real number $p > 0$ and for all $u, v \in E$,

$$[1 + pd(fu, gv)]\, d^2(Su, Tv) \le$$

$$p\psi(d^2(fu, Su)\, d(gv, Tv),\ d(fu, Su)d^2(gv, Tv),$$
$$d(fu, Su)d(fu, Tv)d(gv, Su),$$
$$d(fu, Tu)d(gv, Su)d(gv, Tv))$$
$$+\ m(fu, gv) - \phi(m(fu, gv)),$$

where

$$m(fu, gv) = \max\{d^2(fu, gv),\ d(fu, Su)d(gv, Tv),\ d(fu, Tv)d(gv, Su),$$
$$\frac{1}{2}[d(fu, Su)d(fu, Tv) + d(gv, Su)d(gv, Tv)]\}.$$

Then for arbitrary point $u_0 \in E$, by (C_1), one can find u_1 such that $Su_0 = gu_1 = v_0$. For this u_1, one can find $u_0 \in E$ such that $Tu_1 = fu_2 = v_1$.

Continuing in this fashion, one can construct sequences such that

$$v_{2n} = Su_{2n} = gu_{2n+1} \text{ and } v_{2n+1} = Tu_{2n+1} = fu_{2n+2}, \tag{2.1}$$

for each $n = 0, 1, 2, 3 \ldots$

Before proving the main results, first we shall prove following two Lemmas which are useful for our work.

Lemma 2.1. $\lim_{n \to \infty} d(v_n, v_{n+1}) = 0$, *where* $\{v_n\}$ *is the sequence in E defined by equation* (2.1).

Proof. For simplicity, let us denote

$$\gamma_n = d(v_n, v_{n+1}, n = 0, 1, 2, 3, \ldots \tag{2.2}$$

First, we prove that $\{\gamma_n\}$ is non-increasing sequence, i.e., $\gamma_{n+1} \le \gamma_n$ for $n = 1, 2, 3, \ldots$

Case I. If n is even. By taking $u = u_{2n}$ and $v = u_{2n+1}$ in (C_2) and using equation (2.1) and (2.2), we get

$$[1 + p\gamma_{2n-1}]\gamma_{2n}^2 \leq p\psi(\gamma_{2n-1}^2 \leq \gamma_{2n}, \gamma_{2n-1}\gamma_{2n}^2, 0, 0)$$
$$+ m(v_{2n-1}, v_{2n}) - \phi(m(v_{2n-1}, v_{2n})), \tag{2.3}$$

where $m(v_{2n-1}, v_{2n}) = \max\{\gamma_{2n-1}^2, \gamma_{2n-1}, \gamma_{2n}, 0, \frac{1}{2}[\gamma_{2n-1}d(v_{2n-1}, v_{2n+1}) + 0]\}$.

Using triangular inequality, we get

$$d(v_{2n-1}, v_{2n+1}) \leq d(v_{2n-1}, v_{2n}) + d(v_{2n}, v_{2n+1}) = \gamma_{2n-1} + \gamma_{2n}.$$

Hence,

$$m(v_{2n-1}, v_{2n}) \leq \max\{\gamma_{2n-1}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}[\gamma_{2n-1} + \gamma_{2n})]\}. \tag{2.4}$$

Now we claim that $\{\gamma_{2n}\}$ is non-increasing. Suppose it is not possible, i.e., $\gamma_{2n-1} < \gamma_{2n}$, then by using the inequality (2.4) with the property of ϕ and ψ, inequality (2.3) reduces to

$$[1 + p\gamma_{2n-1}]\gamma_{2n}^2 \leq p\gamma_{2n-1}\gamma_{2n}^2 + \gamma_{2n-1}\gamma_{2n} - \phi(\psi_{2n-1}\gamma_{2n}),$$

i.e., $\gamma_{2n}^2 < \gamma_{2n}^2$, a contradiction. Therefore, $\gamma_{2n} \leq \gamma_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\gamma_{2n+1} \leq \gamma_{2n}$. It follows that the sequence $\{\gamma_n\}$ is non-increasing.

Now we prove that $\lim_{n\to\infty} \gamma_n = 0$. Suppose $\lim_{n\to\infty} \gamma_n \neq 0$, i.e.,

$$\lim_{n\to\infty} \gamma_n = t, \text{ for some } t > 0. \tag{2.5}$$

Taking $n \to \infty$ in inequality (2.3) and using the inequality (2.4), the equation (2.5) with the property of ϕ, ψ, we have

$$[1 + pt]t^2 \leq pt^3 + t^3 - \phi(t^2).$$

This implies that $\phi(t^2) \leq 0$, a contradiction to the definition of ϕ. Therefore,

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} d(v_n, v_{n-1}) = 0. \tag{2.6}$$

Lemma 2.2. *The sequence* $\{v_n\}$, *defined by equation* (2.1), *is a Cauchy sequence in E.*

Proof. Let us assume that $\{v_n\}$ is not a Cauchy sequence, so there exists an $\epsilon > 0$, for which, one can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$d(v_{m(k)}, v_{n(k)}) \geq \epsilon \text{ and } d(v_{m(k)}, v_{n(k)-1}) \geq \epsilon, \tag{2.7}$$

for all positive integers k, $n(k) > m(k) > k$.

Now

$$\epsilon \leq d(v_{m(k)}, v_{n(k)}) \geq d(v_{m(k)}, v_{n(k)-1}) + d(v_{n(k)-1}, v_{m(k)}).$$

Letting $k \to \infty$, we get

$$\lim_{n\to\infty} d(v_{m(k)}, v_{n(k)}) = \epsilon. \tag{2.8}$$

Now from the triangular inequality, we have,

$$| d(v_{n(k)}, v_{m(k)+1}) - d(v_{m(k)}, v_{n(k)}) | \leq d(v_{m(k)}, v_{m(k)+1}).$$

Taking limit as $k \to \infty$ and using equations (2.6) and (2.8) in the above inequality, we have

$$\lim_{n\to\infty} d(v_{n(k)}, v_{m(k)+1}) = \epsilon. \tag{2.9}$$

Again from the triangular inequality, we have

$$| d(v_{m(k)}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)}) | \leq d(v_{n(k)}, v_{n(k)+1}).$$

Taking limit as $k \to \infty$ and using equations (2.6) and (2.8), we have

$$\lim_{n\to\infty} d(v_{m(k)}, v_{n(k)+1}) = \epsilon. \tag{2.10}$$

Similarly, on using triangular inequality, we have

$$| d(v_{m(k)+1}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)}) | \leq d(v_{m(k)}, v_{m(k)+1}) + d(v_{n(k)}, v_{n(k)+1}).$$

Taking limit as $k \to \infty$ and using equations (2.6) and (2.8) in the above inequality, we have

$$\lim_{n\to\infty} d(v_{n(k)+}, v_{m(k)+1}) = \epsilon. \tag{2.11}$$

On taking $u = u_{m(k)}$ and $v = u_{n(k)}$ in (C_2) and using equation (2.1), letting $k \to \infty$ and using equations (2.6)-(2.11) with the property of ϕ and ψ, we obtain

$$[1 + p\epsilon]\epsilon^2 \leq p\psi(0, 0, 0, 0) + \epsilon^2 - \phi(\epsilon^2) < \epsilon^2,$$

which is a contradiction. Thus, the sequence $\{v_n\}$ is a Cauchy sequence in E.

Now, we state and prove fixed point theorem for pointwise R-weakly commuting mappings along with reciprocal continuity as follows:

Theorem 2.1. *Let (f, S) and c be pairs of pointwise R-weakly commuting self mappings of a complete metric space (E, d) satisfying conditions (C_1) and (C_2). If (f, S) or (g, T) is a pair of compatible and reciprocally continuous mappings, then f, g, S and T have a unique common fixed point in E.*

Proof. By Lemma 2.2, the sequence $\{v_n\}$ defined by (2.1), is a Cauchy sequence in E. Since (E, d) is a complete metric space. Therefore, $\{v_n\}$ converges to a point $z \in E$, as $k \to \infty$. Consequently, the sub sequences $\{Su_{2n}\}, \{fu_{2n+2}\}, \{Tu_{2n+1}\}$ and $\{gu_{2n+1}\}$ also converges to the same point z. Suppose that (f, S) is pair of compatible and reciprocally continuous mappings. Thus by reciprocal continuity of f and S, we have

$$\lim_{n\to\infty} fSu_{2n} = fz, \lim_{n\to\infty} Sfu_{2n} = Sz.$$

Also, by compatibility of f and S, $fz = Sz$.

Since $S(E) \subset g(E)$, there exists a point $t \in E$ such that $Sz = gt$, i.e., $fz = Sz = gt$.

We claim that $Sz = Tt$. Suppose not, then by taking $u = z, v = t$ in (C_2), we get

$$[1 + pd(fz,\ gt)]d^2(Sz,\ Tt) \le p\psi(0,\ 0,\ 0,\ 0) + m(fz,\ gt) - \phi(m(fz,\ gt)),$$

where

$$m(fz,\ gt) = \max\{d^2(fz,\ gt),\ d(fz,\ Sz)d(gt,\ Tt),\ d(fz,\ Tt)d(gt,\ Sz),$$

$$\frac{1}{2}[(fz,\ Sz)d(fz,\ Tt) + d(gt,\ Sz)d(gt,\ Tz)]\}$$

$$= 0.$$

On solving the above inequality using the value of $m(fz,\ gt)$ along with the property of ϕ and ψ, we get $d^2(Sz,\ Tt) < 0$, a contradiction. Hence $Sz = Tt$, i.e., $fz = Sz = gt = Tt$. By point wise R-weak commutativity of the pair $(f,\ S)$, there exists $0 < R$ such that

$$d(fSz,\ Sfz) \le Rd(fz,\ Sz) = 0,$$

this implies that $fSz = Sfz$ and $ffz = fSz = Sfz = SSz$.

Also, by pointwise R-weak commutativity of the pair $(g,\ T)$, we have $TTt = Tgt = gTt = ggt$.

Now, we show that Sz is a fixed point of f and S. For this taking $u = Sz$ and $v = t$ in (C_2), we have

$$[1 + pd(fSz,\ gt)]d^2(SSz,\ Tt) \le p\psi(0,\ 0,\ 0,\ 0) + m(fSz,\ gt) - \phi(m(fDz,\ gt)),$$

where

$$m(fSz,\ gt) = \max\{d^2(fSz,\ gt),\ d(fSz,\ SSz)d(gt,\ Tt),\ d(fSz,\ Tt)d(gt,\ SSz),$$

$$\frac{1}{2}[(fSz,\ SSz)d(fSz,\ Tt) + d(gt,\ SSz)d(gt,\ Tz)]\}$$

$$= d^2(fSz,\ Sz).$$

On solving it, we get $d^2(fSz,\ Sz) = 0$ which implies that $fSz = Sz$.

So, $Sz = fSz = SSz$. Thus, Sz is a common fixed point of f and S.

Similarly one can prove that $Tt(= Sz)$ is a common fixed point of g and T.

Thus, Sz is a common fixed point of f, g, T and S. Uniqueness follows easily. This completes the proof.

Example 2.1. Let $E = [0, 10]$ and d be a usual metric. Let $f, g, S, T : E \to E$ be four mappings defined by

$$fu = \begin{cases} 0, & u = 0; \\ 8, & u \in (0, 2.5] \\ u - 2.5, & u \in (2.5, 10]. \end{cases} \qquad Su = \begin{cases} 4, & u \in (0, 2.5]; \\ 0, & u \in (2.5, 10] \cup \{0\}. \end{cases}$$

$$gu = \begin{cases} u, & u = 0; \\ 4, & u \in (2.5, 10]. \end{cases} \qquad Tu = \begin{cases} u, & u = 0; \\ 2, & u \in (0, 10]. \end{cases}$$

Let $0 < p$ be a real number and $\phi : [0, \infty) \to [0, \infty)$ be a function defined by $\phi(t) = \frac{3}{2}t$, for $t \geq 0$ and $\psi : [0, \infty)^4 \to [0, \infty)$ be a function defined by

$$\psi(w_1, w_2, w_3, w_4) = \max\left\{\frac{w_1 + w_2}{2}, w_3, w_4\right\},$$

$w_i \geq 0, 1 \leq i \leq 4$.

It is clear that (g, T) is the pair of reciprocally continuous mappings. But neither g nor T is continuous, not continuous even at the common fixed point $u = 0$. Also, f and S commute at their coincidence points, so f and S are pointwise R-weakly commuting mappings, but f and S are non-compatible, let us consider the sequence $\{v_n\}$ defined by $u_n = 2.5 + \frac{1}{n}, n \geq 1$. Then $fu_n \to 0, Su_n \to 0, fSu_n \to 0$ and $Sfu_n \to 4$. Hence, f and S are non-compatible. One can easily verified that all the conditions of the Theorem 2.1 are satisfied and f, g, T and S have a unique common fixed point $u = 0$. Moreover, all the mappings involved in this example are discontinuous at $u = 0$.

Remark 2.1. If we consider the $\psi : [0, \infty)^4 \to [0, \infty)$ defined by

$$\psi(t_1, t_2, t_3, t_4) = \max\left\{\frac{1}{2}[t_1 + t_2], t_3, t_4\right\},$$

in Theorem 2.1, then we conclude that our result generalizes the results of Jain et al. [7] for pointwise R-weakly commuting mappings.

Corollary 2.1. *Let* (E, d) *be a complete metric space. Suppose that* $S, T : E \to E$ *are two pointwise R-weakly commuting mappings satisfying the following conditions*

$(C_{1^*})\, T(E) \subset S(E),$

(C_{2^*}) *for all* $u, v \in E,$ *real number* $p > 0,\ \psi \in \Psi,\ \phi \in \Phi,$

$$[1 + pd(Su, Sv)]d^2(Tu, Tv) \leq p\psi(d^2(Su, Tu)d(Sv, Tv),\ d(Su, Tu)d^2(Sv, Tv),$$

$$d(Su, Tu)d(Su, Tv)d(Sv, Tu),\ d(Su, Tv)d(Sv, Tu)d(Sv, Tv))$$

$$+ m(Su, Sv) - \phi(m(Su, Sv)),$$

where

$$m(Su, Sv) = \max\{d^2(Su, Sv),\ d(Su, Tu)d(Sv, Tv),\ d(Su, Tv)d(Sv, Tu),$$

$$\frac{1}{2}[d(Su, Tu)d(Su, Tv) + d(Sv, Tu)d(Sv, Tv)]\},$$

If S and T are compatible and reciprocally continuous mappings, then S and T have a unique common fixed point in E.

Proof. On substituting $f = g = S$ and $S = T$ in Theorem 2.1 one can deduce this corollary, which generalize the result of Murthy and Parsad [13] and Jain et al. [5].

3. Application

In 2001, Branciari [4] obtained Banach contraction principle for mapping satisfying an integral type contraction condition. On the similar lines, we analyze our results for mappings satisfying a generalized (ψ, ϕ)-weak contraction condition of integral type.

Theorem 3.1. *Let* (f, S) *and* (g, T) *be pairs of pointwise R-weakly commuting mappings from a complete metric space* (E, d) *to itself satisfying* *the conditions* (C_1) *and* (C_4) *for* $u, v \in E$,

$$\int_o^{M(u,\, v)} \gamma(t) dt \leq \int_o^{N(u,\, v)} \gamma(t)\, dt,$$

where

$$M(u, v) = [1 + pd(fu, gv)] d^2(Su, Tv),$$

$$N(u, v) = p\psi(d^2(fu, Su) d(gv, Tv), d(fu, Su) d^2(gv, Tv), d(fu, Tv) d(gv, Su),$$
$$d(fu, Tv) d(gv, Su) d(gv, Tv)) + m(fu, gv) - \phi(m(fu, gv)),$$

where

$$m(fu, gv) = \max\{d^2(fu, gv), d(fu, Su) d(gv, Tv), d(fu, Tv) d(gv, Su),$$
$$\frac{1}{2}[d(fu, Su) d(fu, Tv) + d(gv, Su) d(gv, Tv)]\},$$

$\psi \in \Psi$, $\phi \in \Phi$, $p > 0$ is a real number and $\gamma : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0, \infty)$ such that for each $\epsilon > 0$, $\int_0^{\epsilon} \gamma(t) dt > 0$. If pairs (f, S) or (g, T) is compatible mappings and reciprocally continuous mappings, then f, g, S and T have a unique common fixed point.

Proof. On putting $\gamma(t) = c$ (some non zero constant), it reduces to Theorem 2.1.

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