Advances and Applications in Mathematical Sciences
Volume 21, Issue 12, October 2022, Pages 6783-6791


# UPPER TOTAL SPLIT DOUBLE GEODETIC NUMBER OF A GRAPH

**T. JEBARAJ and D. SAJITHA**

[1]Assistant Professor
[2]Reg. no. 20213082092001
Department of Mathematics
Malankara Catholic College, Mariagiri
Kanyakumari District, 629153
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli-627012, Tamilnadu, India
E-mail: jebaraj.math@gmail.com
sajithajomrv@gmail.com

## Abstract

A set $S_1 = \{a_1, a_2, a_3, \ldots, a_t\} \subset G$, $X \subseteq S_1$, $A$, $B$ and $C \subset S_1$ is said to be a minimal total split double geodetic set if only the improper subset $X$ is the total split double geodetic set of $G$. Then $|S_1|$ is maximum, it is the upper total split dg-number $dgt_s^+(G)$ of $G$. The intend of this treatise is to deliver the perception about upper total split double geodetic number of a graph, yet we substantiate for non-negative integers $r_1, d_1$ and $k_1 \geq 4$ with $r_1 \leq d_1 \leq 2r_1$ in an unparted graph (connected) $G$, $Rad(G) = r_1$, $Diam(G) = d_1$, $dgt_s^+(G) = k_1$. As well, we derive $4 \leq m_1 \leq n_1$ such that $dgt_s = m_1$ and $dgt_s^+(G) = n_1$.

## 1. Introduction

In 2012, A. P. Santhakumaran et al., introduced the "Double geodetic number of a graph" [4] which is developed from the notion of geodetic number. In 2014, Venkanagouda M. Goudar and others established the Split geodetic concepts [6]. Moving from [4], [6], [12], [5] and [14], we formed the

2020 Mathematics Subject Classification: 05C12.
Keywords: double geodetic set, minimum double geodetic set, upper total split dg-set, minimal dg-set.
Received Please provide.

upper total split dg-set concept, which is based on the notion of minimal dg-set. The below in formations needed to prove the results.

**Theorem 1.1 [14].** *Let* $G_1$ *be an unparted graph with* $|V(G)| = n$, *such that* $G_1$ *has a total split dg-set* $S_1$ *then* $2 \leq dg_s(G_1) \leq dgt_s(G_1) \leq n - 2$.

**Corollary 1.2.** *If* $dg_t(G) = n$ *or* $n - 1$ *and* $|V(G)| = n$ *then* $G$ *has no total split dg-set.*

**Theorem 1.3 [15]**

*For the Gear graph*

$$G_n, (n \geq 3), dgt_s(G_n) = \begin{cases} 4 & \text{if } n = 3, 4 \\ \dfrac{3(n+1)}{2} & \text{if } n \text{ is odd and } n \geq 5 \\ \dfrac{3n+2}{2} & \text{if } n \text{ is even and } n \geq 6 \end{cases}$$

## 2. The Upper Total Split Double Geodetic Number of a Graph

**Definition 2.1.** A set $S_1 = \{a_1, a_2, a_3, \ldots, a_t\} \subset G$, $X \subseteq S_1$, $A$, $B$ and $C \subset S_1$ is said to be a minimal total split double geodetic set if only the improper subset $X$ is the total split double geodetic set of $G$. Then $|S_1|$ is maximum, it is the upper total split dg-number $dgt_s^+(G)$ of $G$.

**Example 2.2.** In Figure $A$, $S_1 = \{v_1, v_4, v_7, v_8\}$ forms a total split dg-set of $G$, $dgt_s(G) = 4$. Also $S_2 = \{v_3, v_7, v_8, v_6, v_2, v_5\}$ make a total split dg-set of $G$, as well only the improper subset of $S_2$ forms total split dg-set of $G$. Consequently, $S_2$ is an upper total split dg-set of $G$. Further we easily checked that, none of the proper subsets of $S_2$ forms a total split dg-set and so $dgt_s^+(G) = 6$.

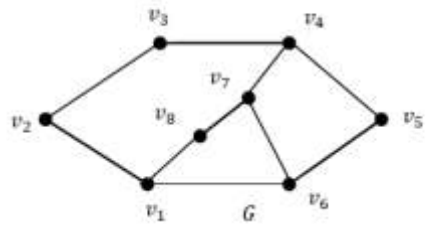



**Figure A**

**Remark 2.3.** In a graph $G$, every minimal total split dg-set is not a minimum total split dg-set but the contrary is true. In Figure $A$, $S_2 = \{v_3, v_7, v_8, v_6, v_2, v_5\}$ forms a minimal total split dg-set as well as it is not a least total split dg-set of $G$.

**Theorem 2.4.** *If $G$ has a total split dg-set $S$ and $|V(G)| = n$ then* $3 \le dgt_s(G) \le dgt_s^+(G) \le n-2$.

**Proof.** Any total split dg-set requires at least 3 vertices, consequently $dgt_s(G) \ge 3$. More over $dgt_s(G) \le dgt_s^+(G)$, since the cardinality is maximum in upper total split dg-set and only the improper subset of $S \subset G$ forms a total split dg-set. Also by Theorem [1.1] and corollary [1.2], we get $dgt_s^+(G) \le n-2$.

**Corollary 2.5.** *If $G$ has a total split dg-set $S$ with $|V(G)| = n$ and* $dgt_s(G) = n-2$ *then* $dgt_s^+(G) = n-2$.

**Proof.** From [2.4] and remark [2.3], we acquire $dgt_s^+(G) = n-2$.

**Theorem 2.6.** *In a ladder graph* $L_n = G, dgt_s^+(G) = \begin{cases} 4 & \text{if } n = 3 \\ 5 & \text{if } n = 4 \\ 6 & \text{if } n \ge 5 \end{cases}$

**Proof.** Let $V(L_n) = \{a_1, a_2, a_3, \ldots, a_{2n}\}$, $n \ge 3$.

**Case (i)** $n = 3$.

Let $M = \{a_1, a_2, a_6\}$ be the split double geodetic set of $L_n$. Since it covers all the pair of vertices $\alpha, \beta$ in $L_n$ as well $\alpha, \beta \in I[M]$ and $\langle V - M \rangle$ is disconnected. Further the induced subgraph $\langle M \rangle$ has one isolated vertex, consequently $M$ doesn't form a total split dg-set of $L_n$. Add a vertex $a_5$ to $M$ which is an appropriate neighborhood of $a_6$. Therefore $M' = \{a_1, a_2, a_6, a_5\}$ and only the improper subset of $M'$ form a total split dg-set of $L_n$. Hence $M'$ is an upper total split double geodetic set of $L_n$, $dgt_s^+(G) = 4$.

**Case (ii)** $n = 4$.

Let $N = \{a_1, a_8, a_i\}$ be the split double geodetic set of $L_n$. Since it covers all the pair of vertices $x, y$ in $L_n$, as well $x, y \in I[N]$. Moreover $\langle V - N \rangle$ is detached. But the induced subgraph $\langle N \rangle$ has isolated vertices. Add the appropriate neighborhood of each vertex in $N$, to form a total split double geodetic set of $L_n$. Clearly $N' = \{a_1, a_8, a_i, a_2, a_j\}$ is the minimal total split dg-set of $L_n$. Obviously $N'$ is an upper total split dg-set of $L_n$, $dgt_s^+(G) = 5$.

**Case (iii)** $n \geq 5$.

Let $L = \{a_1, a_{2n}, a_i, a_j\}$ be the split dg-set of $L_n; a_i, a_j$ are the cut vertices of $L_n$, which are adjacent with each other. Clearly $\langle V - N \rangle$ is disconnected and the induced subgraph $\langle L \rangle$ has isolated vertices. Add the appropriate neighborhood of $a_1$ and $a_{2n}$, to form a total split dg-set of $L_n$. Obviously $L' = \{a_1, a_2, a_{2n}, a_i, a_j, a_{2n-1}\}$ is the minimal total split dg-set of $L_n$. Hence $dgt_s^+(G) = 6$.

**Theorem 2.7.** *For non-negative integers $r_1, d_1, r_1 \leq d_1 \leq 2r_1$ and $k_1 \geq 4$ in G then*

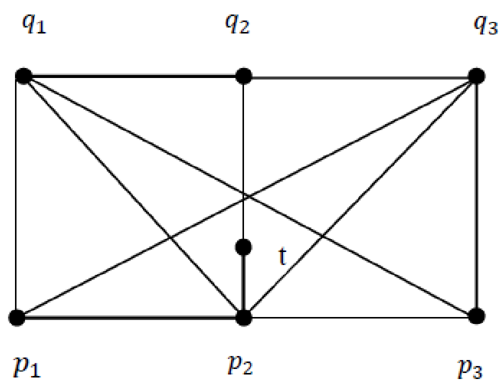
$$Rad(G) = r_1,\ diam(G) = d_1 \text{ and } dgt_s^+(G) = k_1.$$

**Proof.** If $r_1 = 2$, obviously know $d_1 = 2, 3$ and 4.

**Case (i)** $r_1 = 2$. Then $d_1 = 2$.

Assume that $2r_1 + 1 = k_1$. Consider two paths $P$ and $Q$, its vertices are

$q_1, q_2, q_3$ and $p_1, p_2, p_3$ respectively. Take one more new vertex $t$ and connect with $q_2$ and $p_2$. Also $q_1$ and $q_3$ are connected with $p_1$, $p_2$ and $p_3$. Now we get Figure $B$, which is named as $G$ then $G$ has radius 2 and diameter 2.




$G$

**Figure B**

Let $S' = \{p_1, p_2, p_3, t, q_2\}$ be a total split dg-set of $G$. Since $G[S']$ incited by $S'$ has no detached vertices and $\langle V - S' \rangle$ is detached. More over, only the improper subset of $S'$ is a total split dg-set of $G$ and hence $S'$ is an upper total split dg-set of $G$. From [2.3], conspicuously know $S'$ is not a least total split dg-set of $G$. Thus $dgt_s^+(G) = 5 = 2r_1 + 1 = k_1$.

**Case (ii)** $r_1 = 2$. Then $d_1 = 3$.

Let $G = G_3$ (Gear graph with $n = 3$) and $n + 1 = k_1$ then by Theorem [1.3], $dgt_s^+(G) = k_1$.

**Case (iii)** $r_1 = 2$. Then $d_1 = 4$.

Set up a graph $G$ as follows. Let $u_1, u_2, u_3$ be the vertices of a path $P_3$. Take $r$ new vertices $w_1, w_2, \ldots, w_r$ and join each $w_i (1 \le i \le r)$ to $u_1$ and $u_3$ and obtain the graph $H$. Also add some $K_2$ and join one end of each $K_2$ to $u_3$ which is denoted by $v_1, v_2, \ldots, v_{k_1 - l} (3 < l < k_1)$ and the other ends of $K_2$ is denoted by $x_1, x_2, \ldots\ldots, x_{l-3}$ respectively and get the graph $G$ in Figure C.

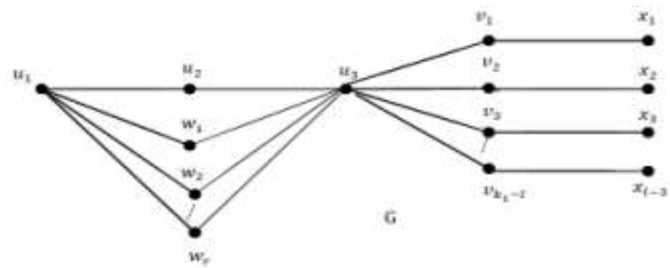


**Figure C**

Then $G$ has radius $r_1 = 2$ and diameter $d_1 = 4$. Obviously $S = \{u_1, u_3, x_1, x_2, \ldots\ldots, x_{l-3}\}$ is a least split dg-set of $G$, yet the subgraph $G[S]$ incited by $S$ has detached vertices with degree zero. Therefore to make a total split dg-set of $G$ from $S$, the appropriate neighborhood of each vertex is to be added. Clearly $S' = \{u_1, u_3, w_1, x_1, x_2, \ldots\ldots, x_{l-3}, v_1, v_2, \ldots\ldots, v_{k_1-l}\}$ and only the improper subset of $S'$ forms a total split dg-set of $G$. Thus $dgt_s^+(G) = k_1$.

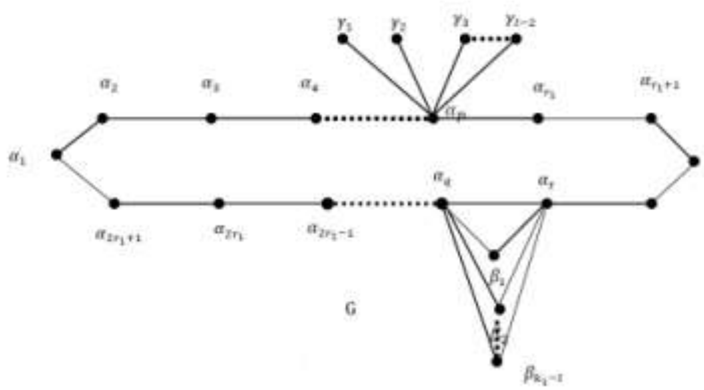


**Figure D**

**Case (iv)** $r_1 \geq 3$.

If $k_1 \geq 4$. Put up a graph $G$ as follows. Consider $\alpha_1, \alpha_2, \ldots\ldots, \alpha_{r_1}, \alpha_{r_1+1}, \ldots \alpha_{2r_1}, \alpha_{2r_1+1}$ be the vertices of odd cycle $C_{2r_1+1}$. Take $K_1 - l$ new vertices $\beta_1, \beta_2, \ldots, \beta_{k_1-l}$ and append each $\beta_i (1 \leq i \leq K_1 - l)$ to $\alpha_q$ and $\alpha_t$, where $\alpha_q, \alpha_t$, are the two adjacent vertices of cycle $C_{2r_1+1}$. Further more add on $l-2$ new vertices $\gamma_1, \gamma_2, \ldots, \gamma_{l-2}$ and connect each $\gamma_i (1 \leq i \leq l-2)$ to $\alpha_p$ such that $d(\alpha_p, \beta_i) = r_1 + 1$ and acquire Figure D. Then $G$ has radius $r_1$ and diameter $d_1$. Also $S = \{\beta_1, \beta_2, \ldots, \beta_{k_1-1}, \gamma_1, \gamma_2, \ldots, \gamma_{l-2}, \alpha_p, \alpha_t\}$ forms a total split dg-set and only the improper subset of $S$ make a total split dg-set of $G$. Consequently $S$ is the minimal total split dg-set of $G$, $dgt_s^+(G) = k_1$.

**Case (v)** $r_1 = d_1$.

Let $V(C_{2r_1}) : u_1, u_2, \ldots, u_{r_1+1}, \ldots, u_{2r_1}$. Take $k_1 - l - 1$ new vertices $v_1, v_2, \ldots, v_{k_1-l-1}$ to $C_{2r_1}$ and append each $v_i (1 \leq i \leq k_1 - l - 1)$ to $u_2$ and $u_{2r_1}$. Also add $l-3$ new vertices $w_1, w_2, \ldots, w_{l-3}$ and connect each $w_j (1 \leq j \leq l-3)$ to $u_{r_1}$ and $u_{r_1+2}$. Now attain the graph $G$ of Figure E. From figure E, We clear that eccentricity of each vertex of $G$ is $r_1$ so that Rad $(G) = Diam(G) = r_1$. The weak extreme vertices of $G$ are $E = \{u_1, u_{r_1+1}, v_1, v_2, \ldots, v_{k_1-l-1}, w_1, w_2, \ldots, w_{l-3}\}$. Also $E$ is the dg-set of $G$ and $\langle V - E \rangle$ is a detached graph. Consequently define another set $E' = E \cup \{u_{r_1}, u_{2r_1}\}$, where $u_{r_1}$ and $u_{2r_1}$ are the neighbors of some elements of $E$. Clearly $E'$ forms a total split dg-set of $G$, and only the improper subset of $E'$ forms a total split dg-set of $G$. Hence $E'$ is an upper total split dg-set of $G$. It is easily verified that, no other subsets with more than $k_1$ elements forms a minimal total split dg-set of $G$, $dgt_s^+(G) = k_1$.

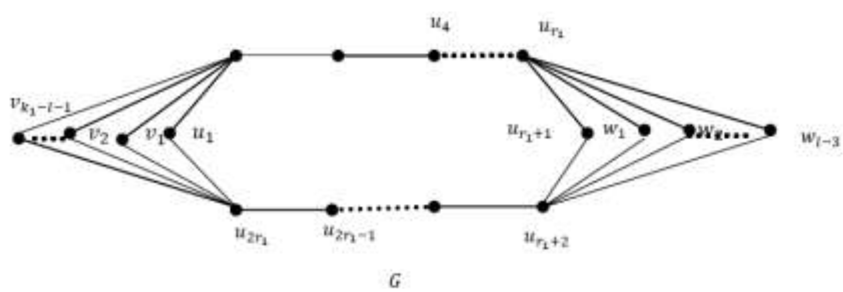



**Figure E**

**Theorem 2.8.** *For any non-negative integers* $4 \leq m_1 \leq n_1$ *in* $G$, *such that* $dgt_s(G) = m_1$ *and* $dgt_s^+(G) = n_1$.

**Proof.**

**Case (i)** If $m_1 = n_1$.

Let $G = C_{2n}, n \geq 3,$ then $dgt_s(G) = dgt_s^+(G) = m_1$.

**Case (ii)** If $m_1 < n_1$.

Take $n_1 - 1 = m_1$. Consider a series connection of '$k$' number of cycle $C_4$ and append $m_1 - 4$ new vertices to $C_{k+1}$. Also consider $\{l_1, l_2, \ldots, l_{n_1 - m_1 - 1}\}$ new vertices and connect each $l_i(1 \leq i \leq n_1 - m_1 - 1)$ to $x_1$ and $y_1$, then acquire $G$ in Figure F.

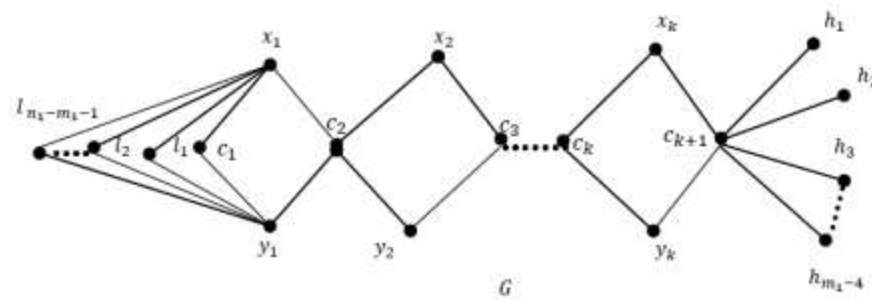



**Figure F**

It follows that $T = \{l_1, l_2, \ldots, l_{n_1 - m_1 - 1}, h_1, h_2, \ldots, h_{m_1 - 4}, C_1\}$ is the set of weak extreme vertices of given $G$. Therefore to make a total split dg-set of $G$, add the appropriate neighborhood of each isolated vertex in $T$. Hence $T' = \{l_1, l_2, \ldots, l_{n_1 - m_1 - 1}, h_1, h_2, \ldots, h_{m_1 - 4}, c_1, c_2, c_{k+1}, x_1\}$ is a least total split dg-set of $G$, and hence $dgt_s(G) = n_1 - 1 = m_1$.

Further $T'' = \{l_1, l_2, \ldots, l_{n_1 - m_1 - 1}, h_1, h_2, \ldots, h_{m_1 - 4}, c_1, c_j(\neq c_2), c_{k+1}, x_1, x_i\}$ and none of the proper subsets of $T''$ forms a total split dg-set of $G$, thus $dgt_s^+(G) = n_1$.

## References


[1] B. Bresar, S. Klavzar and A. T. Horvat, On the geodetic number and related metric sets in Cartesian product graphs, Discrete mathematics 308 (2008), 5555-5561.

[2] G. Chartrand, F. Harary and P. Zhang, On the geodetic number of a graph, Networks 39 (2002), 1-6.

[3] F. Harary, Graph Theory, Addision – Wesley, 1969.

[4] A. P. Santhakumaran and T. Jebaraj, Double geodetic number of a Graph, Discussions Mathematicae Graph Theory 32 (2012), 109-119.

[5] P. A. Ostrand, Graphs with specified radius and diameter, Discrete Math. 4 (1973), 71-75.

[6] Venkanagouda M. Goudar, K. S. Ashalatha and Venkatesha, Split geodetic number of a graph, Advances and Applications in Discrete mathematics 13(1) (2014), 9-22.

[7] F. Harary, E. Loukakis and C. Tsouros, The geodetic number of a graph, Math. Comput. Modeling 17 (1993), 89-95.

[8] R. Muntean and P. Zhang, On geodomonation in graphs, Congr. Numer. 143 (2000), 161-174.

[9] Z. Mihalic and N. Trinajstic, A graph theoretical approach to structure-property relationships, J. Chem. Educ. 69 (1999), 701-712.

[10] G. Chartrand and P. Zhang, Introduction to Graph Theory, Tata McGraw Hill Pub. Co. Ltd., 2006.

[11] M. Atici, Graph operations and geodetic numbers, Congressus Numerantium 141 (1999), 95-110.

[12] A. P. Santhakumaran and T. Jebaraj, The total double geodetic number of a graph, Proyecciones (Antofagasta, Online) 39(1) (2020).

[13] A. P. Santhakumaran, T. Jebaraj and S. V. Ullas Chandran, The linear geodetic number of a graph, Discrete Mathematics, Algorithms and Applications 3(3) (2011), 357-368. DOI: 10.1142/s1793830911001279.

[14] T. Jebaraj and D. Sajitha, Split double geodetic number of a graph, (communicated).

[15] T. Jebaraj and D. Sajitha, Total Split double geodetic number of a graph, Stochastic Modeling and Applications 25(3) (July-December, Special Issue 2021 Part-3).