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THE CORPORATE DOMINATION NUMBER OF THE CARTESIAN PRODUCT OF TWO CYCLES

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Abstract

Let $G = (V, E)$ be a graph. Let $C = V_1 \cup E_1 \subseteq (V \cup E)$. Take $P = \{u \in V(G[E_1])/|N(u) \cap N(w)| \leq 1$ for all $w(\neq u) \in V(G[E_1])\}$ where $V(G[E_1])$ denotes the vertex set of an edge induced sub graph $G[E_1]$ and $Q = \{v \in V_1/N(v) \cap N(w) = \varphi$ for all $w(\neq v) \in V_1\}$. A subset C is said to be a corporate dominating set if every vertex $v \notin P \cup Q$ is adjacent to exactly one vertex of $P \cup Q$. The corporate domination number of G, denoted by $\gamma_{cor}(G)$, is the minimum cardinality of elements in C. Let $C_m \square C_n$ denote the Cartesian product of C_m and C_n, the cycle of length m and n where $m, n \geq 3$. In this paper, we determine the exact value of the corporate domination number for the Cartesian product of two cycles.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E of size m. Throughout this paper, all graphs are finite, simple, and undirected.

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The open neighborhood of a vertex v is $N(v) = \{u/uv \in E(G)\}$ and the closed neighborhood of a vertex v is $N[v] = N(v) \cup \{v\}$. For graph theoretic terminology, we refer Chartrand, Lesinak [1] and Harray [4].

Let $x, y \in V \cup E$ are said to be associated if they are adjacent of incident in G. Consider the two sets $X, Y \in (V, E, V \cup E)$, a subset $D \subseteq X$ dominates Y if every element of $Y \setminus (D \cap Y)$ is associated with an element of D. The minimum cardinality among all the subsets of X is denoted by $\gamma_{X,Y}(G)$. This concept is suitable for the following fundamental domination numbers such as $\gamma_{V,V}(G)$, $\gamma_{V,E}(G)$, $\gamma_{V,VUE}(G)$, $\gamma_{E,E}(G)$, $\gamma_{E,V}(G)$, $\gamma_{E, VUE}(G)$, $\gamma_{VUE,V}(G)$, $\gamma_{VUE,E}(G)$, $\gamma_{VUE,VUE}(G)$. Various domination parameters have been focused to dominate the vertices, edges, mixing the vertices and edges. The detailed study of numerous domination parameters was established in [5, 6].

A dominating set $S \subseteq V$ of a graph G is said to be perfect if each vertex in $V - S$ is dominated by exactly one vertex of S. The minimum cardinality of S, denoted by $\gamma_p(G)$ is the perfect domination number of G. Fellows and Hoover [3] have addressed some formulation of perfect domination where the vertices are required to have at most one or exactly one neighborhood.

Let $S \subseteq V(G)$. Then S is said to be independent if no two vertices in S are adjacent. If a dominating set S is both perfect and independent, then S is called an efficient dominating set. The graph G is an efficient open domination graph if there exists an efficient open dominating set S, for which $\bigcup_{v \in D} N(v) = V(G)$ and $N(u) \cap N(v) = \phi$ for every $u, v(u \neq v) \in S$. Moreover, the efficient open domination graphs among graph products were characterized in [2].

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is a graph with $V(G_1 \square G_2) = V(G_1) \square V(G_2)$ and $((x_1, y_1), (x_2, y_2)) \in E(G_1 \square G_2)$ if and only if either $x_1 = y_1$ and x_2 adjacent to y_2 in G_2 or $x_2 = y_2$ and x_1 adjacent y_1 in G_1. A detailed study of the dominating set and its algorithm of Cartesian product of paths and cycles have been discussed by Polana Palvic, Janez Zerovnik [9]. We have determined the exact value of the corporate domination number for some classes of graphs such as cycle, path, wheel, complete graph in [7] and also established the corporate domination number of the Cartesian product of path and cycle in [8]. In this paper, we find the corporate domination number of the Cartesian product of two cycles and finally we conclude the paper along with related work for further research.

2. Corporate Domination Number of a Graph

We here start with the definition of corporate domination number with example and state related results on corporate domination.

Definition 2.1. Let $C = V_1 \cup E_1 (\subseteq V \cup E)$. Then C will be in one of the following forms. (i) $V_1 \neq \phi$ and $E_1 = \phi$. (ii) $V_1 = \phi$ and $E_1 \neq \phi$. (iii) $V_1 \neq \phi$ and $E_1 \neq \phi$. Take $P = \{u \in V(G[E_1])/|N(u) \cap N(w)| \leq 1$ for all $w(\neq u) \in V(G[E_1])\}$ where $V(G[E_1])$ denote the vertex set of an edge induced subgraph $G[E_1]$ and $Q = \{v \in V_1 / N(v) \cap N(w) = \varphi$ for all $w(\neq v) \in V_1\}$. A subset C is said to be a corporate dominating set if every vertex $v \in V - P \cup Q$ is adjacent to exactly one vertex of $P \cup Q$. The corporate domination number, denoted by $\gamma_{cor}(G)$ equals the minimum cardinality of a corporate dominating set of G.

Example 2.2. For a graph G which is given in Figure 1, $\gamma_{cor}(G) = 2$. Here $E_1 = \{v_2v_3\}$, $V(G[E_1]) = \{v_2, v_3\}$ and $C = \{v_2v_3, v_6\}$

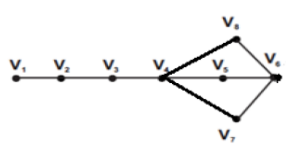

Figure 1. G with $\gamma_{cor}(G) = 2$.

Proposition 2.3. *Let G be a graph. Then* $\gamma_{cor}(G) = 1$ *if and only if one of the following holds.*

(i) *There exists a full degree vertex in G.*

(ii) *There exists an edge* uv *in G such that* uv *does not lie on any triangle and* $d(u) + d(v) = n$.

Remark 2.4. Corporate domination need not exist for all graphs.

Proposition 2.5. *Let C be a corporate dominating set with* $C = V_1$. *Then*

(i) *every corporate dominating set is the dominating set as well as the perfect dominating set.*

(ii) *both dominating and perfect dominating sets need not be the corporate dominating set.*

Example 2.6. For a cycle C_9, the corporate dominating set $C = \{v_2, v_5, v_8\}$ which is also the dominating set as well as the perfect dominating set.

Proposition 2.7.

(i) *For any cycle* C_m *and Path* $P_n (m \geq 3, n \geq 2)$, $\gamma_{cor}(C_m) = \left\lceil \frac{m}{4} \right\rceil$ *and* $\gamma_{cor}(P_n) = \left\lceil \frac{n}{4} \right\rceil$

(ii) *For any complete graph* $K_n (n \geq 3)$, $\gamma_{cor}(K_n) = 1$.

(iii) *For any star graph* $K_{1,n} (n \geq 2)$, $\gamma_{cor}(K_{1,n}) = 1$.

(iv) *For any wheel graph* $W_n (n > 3)$, $\gamma_{cor}(W_n) = 1$.

3. Main Results

In this section, we determine the exact value of the corporate domination number for the Cartesian product of two cycles.

Theorem 3.1. *Let* C_m *and* $C_n (m, n \geq 3)$ *where* $m \equiv 0 (\bmod 3)$, $n \equiv 0 (\bmod 3)$ *and* $m \neq n$ *be any two cycles. Then*

$$\gamma_{cor}(C_m \square C_n) = \begin{cases} \left(\frac{m}{3}\right)\left\lceil \frac{n}{2} \right\rceil & \text{if } m, n \text{ are odd and } m < n \\ \left\lceil \frac{m}{2} \right\rceil\left(\frac{n}{3}\right) & \text{if } m, n \text{ are odd and } m > n \\ \frac{mn}{6} & \text{otherwise.} \end{cases}$$

Proof. Let C_m and $C_n(m, n \geq 3)$ and $m \neq n$ be any two cycles with $m \equiv 0(\bmod 3)$ and $n \equiv 0(\bmod 3)$. Consider the following cases.

Case 1. Let m and n be odd

Subcase 1.1. Let $m < n$. Then for $2 \leq i \leq m-1$ and $i \equiv 2(\bmod 3)$, let $C = \{v_i v_{m+i}, v_{2m+i} v_{3m+i}, \ldots, v_{mn-3m+i} v_{mn-2m+i}, v_{mn-m+i}\}$. Here $P = \{v_i, v_{m+i}, \ldots, v_{mn-3m+i}, v_{mn-2m+i}\}$ and $Q = \{v_{mn-m+i}\}$. Clearly, $|Q| = \frac{m}{3}$.

Since for any $u \in (P \cup Q)^c$, $N(u) \cap (P \cup Q) = \{w\}$ where $w \in P \cup Q$, C is the corporate dominating set.

Since every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c$, $|P \cup Q| = \frac{mn}{3}$. Therefore, $|P| = \frac{m(n-1)}{3}$, as $|Q| = \frac{m}{3}$. Hence C contains $\frac{m(n-1)}{6}$ edges and $\frac{m}{3}$ vertices.

Thus, $|C| = \frac{m(n-1)}{6} + \frac{m}{3} = \frac{m(n-1)+2m}{6} = \frac{m(n-1+2)}{6} = \frac{m(n+1)}{6}$ $= \frac{m}{3}\left\lceil \frac{n}{2} \right\rceil$.

To prove C is minimum, let C' be any other corporate dominating set and P', Q' be the sets corresponding to C' such that every vertex not in $P' \cup Q'$ is adjacent to exactly one vertex in $P' \cup Q'$. Further the set C' will be in one of the following forms. (i) $C' = V_1'$ (ii) $C' = E_1'$ (iii) $C' = V_1' \cup E_1'$.

If (i) holds, then $P' = \varphi$ and $Q' \neq \varphi$. Since for any $u \in V_1'$, $N(u) \cap N(w) \neq \varphi$ for some $w \in V_1'$, which is a contradiction. If (ii) holds, then $P' \neq \varphi$ and $Q' = \varphi$. Let $|P'| > |P|$ with $|P'| = \frac{mn}{3}$ and $|Q'| = 0$. Then $|C'| \leq \frac{mn}{3}$ and hence $|C'| \geq |C|$. If (iii) holds, then $P' \neq \varphi$ and $Q' \neq \varphi$. We observe that $|P' \cup Q'| \geq \frac{mn}{3} (= |P \cup Q|)$. If it is less than $\frac{mn}{3}$, then there exists at least one vertex $v_i \in (P' \cup Q')^c$ which is adjacent to none of the vertices in $P' \cup Q'$, which is a contradiction.

(a) Let $|P'| < |P|$ and $|Q'| > |Q|$ with $2 \le |P'| \le \frac{mn - m - 3}{3}$ and $\frac{m+3}{3} \le |Q'| \le \frac{2m}{3}$. Then C' contains at most $\frac{mn - m - 3}{3}$ edges and $\frac{2m}{3}$ vertices and hence $|C'| \le \frac{mn - m - 6}{3} + \frac{2m}{3} = \frac{mn + m - 6}{3}$. Thus $|C'| \ge |C|$.

(b) Suppose $|P'| \ge |P|$ and $|Q'| \le |Q|$ with $\frac{mn - m}{3} \le |P'| \le \frac{mn}{3} - 1$ and $1 \le |Q'| \le \frac{m}{3}$ Then it follows that C' has at most $\frac{mn - 6}{3}$ edges and $\frac{m}{3}$ vertices. Therefore, $|C'| \ge |C|$.

Subcase 1.2. Let $m > n$. Then for $1 \le t \le n - 2$ and $t \equiv 2 (\operatorname{mod} 3)$, let $C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \dots, v_{(t+1)m-2}v_{(t+1)m-1}, v_{(t+1)m}\}$. Here $P = \{v_{tm+1}, v_{tm+2}, \dots, v_{(t+1)m-2}, v_{(t+1)m-1}\}$ and $Q = \{v_{(t+1)m}\}$. Clearly, $|Q| = \frac{n}{3}$. Proceed as in sub case 1.1, we get C is the corporate dominating set and $|P \cup Q| = \frac{mn}{3}$. Hence $|P| = \frac{m(n-1)}{3}$, as $|Q| = \frac{m}{3}$. Thus C contains $\frac{m(n-1)}{6}$ edges and $\frac{n}{3}$ vertices.

Therefore $|C| = \left\lceil \frac{m}{2} \right\rceil \left(\frac{n}{3} \right)$.

We shall prove that C is minimum. Proceed the similar argument of sub case 1.1 by replacing m by n and n by m, $|C'| \ge |C|$.

Case 2. Let either (m or n) or both (m and n) be even.

Subcase 2.1. Let m be even. Then n may be even or odd.

For $1 \le t \le n - 2$ and $t \equiv 1 (\operatorname{mod} 3)$, let $C = \{v_{tm+1}v_{tm+2}, \dots, v_{(t+1)m-1} v_{(t+1)m}\}$.

Here $P = \{v_{tm+1}, v_{tm+2}, \dots, v_{(t+1)m-1}, v_{(t+1)m}\}$ and $Q = \varphi$.

Since every vertex not in $P \cup Q$ is adjacent to exactly one vertex in $P \cup Q$, C is the corporate dominating set.

As every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c$, $|P| = |P \cup Q| = \frac{mn}{3}$.

Hence C contains $\frac{mn}{6}$ edges.

To prove C is minimum, let C' be any other corporate dominating set and P', Q' be the sets corresponding to C' such that $|N(u) \cap (P' \cup Q')| = 1$ for all $(P' \cup Q')^c$. Further, the set C' will be in one of the following forms.

(i) $C' = V_1'$ (ii) $C' = E_1'$ (iii) $C' = V_1' \cup E_1'$

If (i) holds, then $P' = \varphi$ and $Q' \neq \varphi$. Since for any $u \in V_1'$, $N(u) \cap N(w) \neq \varphi$ for some $w \in V_1'$, which is a contradiction.

If (ii) holds, then $P' \neq \varphi$ and $Q' = \varphi$. Let $|P'| \geq P$ with $|P'| = \frac{mn}{3}$. Then $|C'| \leq \frac{mn}{3}$ and hence $|C'| \geq |C|$.

If (iii) holds, then $P' \neq \varphi$ and $Q' \neq \varphi$.

If $|P' \cup Q'| > |P \cup Q|$, then $|N(w) \cap (P' \cup Q')| > 1$ for some $w \in (P' \cup Q')^c$, which is a contradiction.

Suppose $|P' \cup Q'| < |P \cup Q|$. Then there exists at least one vertex $u \in (P' \cup Q')^c$, which is adjacent to none of the vertices in $P' \cup Q'$, which is a contradiction. Hence $|P' \cup Q'| = |P \cup Q| = \frac{mn}{3}$.

Let $|P'| < |P|$ and $|Q'| > |Q|$ with $(m-2)\left(\frac{n}{3}\right) \leq |P'| \leq \frac{mn}{3} - 1$ and $1 \leq |Q'| \leq \frac{2n}{3}$.

Then C' has at most $\frac{mn}{3} - 2$ edges and $\frac{2n}{3}$ vertices. Therefore $|C'| \leq (m-2)\left(\frac{n}{3}\right) - 2$ and hence $|C'| \geq |C|$.

Subcase 2.2. Let n be even. Then m may be even or odd.

For $2 \leq i \leq m-1$ and $i \equiv 2 (\bmod 3)$, let $C = \{v_i v_{m+i}, v_{2m+i} v_{3m+i}, \ldots, v_{mn-2m+i} v_{mn-m+i}\}$.

Here $P = \{v_i, v_{m+i}, v_{2m+i}, \ldots, v_{mn-m+i}\}$ and $Q = \varphi$. Similarly, as in the sub case 2.1, we can prove that C is the corporate dominating set and $|P| = |P \cup Q| = n\left(\frac{m}{3}\right)$. Hence $|C| = \frac{mn}{6}$.

Now we claim that C is minimum. Replacing m by n and n by m in the sub case 2.1, C is the minimum corporate dominating set.

Corollary 3.2. *Let C_m and $C_n (m, n \geq 3, m \neq 4)$ be any two cycles with $m \not\equiv 0 (\bmod 3)$ and $n \equiv 0 (\bmod 3)$. Then*

$$\gamma_{cor}(C_m \square C_n) = \left\lceil \frac{m}{2} \right\rceil \left(\frac{n}{3}\right).$$

Proof. Let C_m and C_n be any two cycles with $m \not\equiv 0 (\bmod 3)$, $n \equiv 0 (\bmod 3)$ and $m \neq 4$. Consider the following cases.

Case 1. Let m be odd. Then for $1 \leq t \leq n-2$ and $t \equiv 1 (\bmod 3)$, let $C = \{v_{tm+1} v_{tm+2}, v_{tm+3} v_{tm+4}, \ldots, v_{(t+1)m-2} v_{(t+1)m-1}, v_{(t+1)m}\}$. Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, \ldots, v_{(t+1)m-1}\}$, $Q = \{v_{(t+1)m}\}$. Clearly, $|Q| = \frac{n}{3}$. Since for any $u \in (P \cup Q)^c$, $N(u) \cap (P \cup Q) = \{w\}$ where $w \in P \cup Q$, C is the corporate dominating set. Since every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c$, $|P \cup Q| = m\left(\frac{n}{3}\right)$. As $|Q| = \frac{n}{3}$, $|P| = (m-1)\left(\frac{n}{3}\right)$. Hence C contains $\frac{(m-1)n}{6}$ edges and $\frac{n}{3}$ vertices. Thus $|C| = \left\lceil \frac{m}{2} \right\rceil \left(\frac{n}{3}\right)$.

We claim that C is minimum. Let C' is any other corporate dominating set and P', Q' be the sets corresponding to C' such that $|N(w) \cap (P' \cup Q')| = 1$ for all $u \in (P' \cup Q')^c$. Further, the set C' will be in one of the following forms.

(i) $C' = V_1'$ (ii) $C' = E_1'$ (iii) $C' = V_1' \cup E_1'$.

If $C' = V_1'$ holds, then $P' = \varphi$ and $Q' \neq \varphi$. Since for any $u \in V_1'$, $N(u) \cap N(w) \neq \varphi$ for some $w \in V_1'$, which is a contradiction.

If $C' = E_1'$ holds, then $P' \neq \varphi$ and $Q' = \varphi$. Let $|P'| \geq |P|$ with $|P'| = \frac{mn}{3}$. Then $|C'| \leq \frac{mn}{3}$ and hence $|C'| \geq |C|$.

If $C' = V_1' \cup E_1'$ holds, then $P' \neq \varphi$ and $Q' \neq \varphi$. As in the proof of sub case 2.1.in Theorem 3.1, $|P' \cup Q'| = |P \cup Q|$.

(a) Let $|P'| < |P|$ and $|Q'| > |Q|$ with $2 \leq |P'| \leq \frac{mn-n-3}{3}$ and $\frac{n+3}{3} \leq |Q'| \leq 2\left(\frac{n}{3}\right)$. Then C' contains at most $\frac{mn-n-6}{3}$ edges and $2\left(\frac{n}{3}\right)$ vertices and hence $|C'| \geq |C|$.

(b) Suppose $|P'| \geq |P|$ and $|Q'| \leq |Q|$ with $\frac{mn-n}{3} \leq |P'| \leq \frac{mn-3}{3}$ and $1 \leq |Q'| \leq \frac{n}{3}$. Then $|C'| \leq \frac{mn-n-6}{3}$. Thus $|C'| \geq |C|$.

Case 2. Let m be even. Then for $1 \leq t \leq n-2$ and $t \equiv 1 (\operatorname{mod} 3)$, let $C = \{v_{tm+1}v_{tm+2}, v_{tm+3}v_{tm+4}, \ldots, v_{(t+1)m-1}v_{(t+1)m}\}$. Here $P = \{v_{tm+1}, v_{tm+2}, v_{tm+3}, \ldots, v_{(t+1)m-1}, v_{(t+1)m}\}$ and $Q = \varphi$. Since for any $u \in (P \cup Q)^c$, $N(u) \cap (P \cup Q) = \{w\}$, where $w \in P \cup Q$, C is the corporate dominating set.

As every vertex in $P \cup Q$ is adjacent to exactly two vertices in $(P \cup Q)^c$, $|P \cup Q| = m\left(\frac{n}{3}\right)$. Hence C has $\frac{mn}{6}$ edges.

To prove C is minimum, let C' be any other corporate dominating set and P', Q' be the sets corresponding to C' such that $|N(w) \cap (P' \cup Q')| = 1$ for all $u \in (P' \cup Q')^c$. Further, C' will be in one of the following forms. (i) $C' = V_1'$ (ii) $C' = E_1'$ (iii) $C' = V_1' \cup E_1'$. Proceed as in case 1, we can prove that C is minimum if $C' = V_1'$ and $C' = E_1'$. If $C' = V_1' \cup E_1'$ holds, then $P' \neq \varphi$ and $Q' \neq \varphi$. As $|P' \cup Q'| = |P \cup Q|$, let $|P'| < |P|$ and $|Q'| > |Q|$ with $\frac{(m-2)n}{3} \leq |P'| \leq \frac{mn-3}{3}$ and $1 \leq |Q'| \leq \frac{2n}{3}$. Then $|C'| = \frac{mn+2n-6}{3}$. Hence $|C'| > |C|$.

Illustration 3.3.

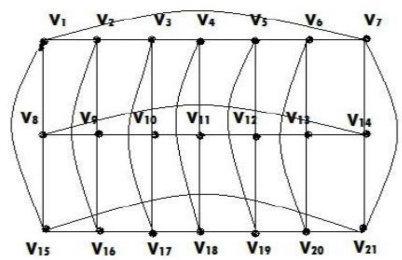

Figure 2. $C_7 \square C_3$.

In Figure 2, let $C = \{v_8v_9, v_{10}v_{11}, v_{12}v_{13}, v_{14}\}$. Since for any $u \in (P \cup Q)^c$, $N(u) \cap (P \cup Q) = \{w\}$ where $w \in P \cup Q$, C is the corporate dominating set and $\gamma_{cor}(C_7 \square C_3) = 4$.

Corollary 3.4. *Let C_m and C_n, $(m, n \geq 3, n \neq 4)$ be any two cycles with $m \equiv 0 (\text{mod } 3)$ and $n \not\equiv 0 (\text{mod } 3)$.*

Then $\gamma_{cor}(C_m \square C_m) = \left(\frac{m}{3}\right)\left\lceil \frac{n}{2} \right\rceil$

Proof. Similar to Corollary 3.2, by replacing m by n and n by m.

Corollary 3.5. *Let $C_m (m \geq 3)$ be any cycle with $m \equiv 0 (\text{mod } 3)$. Then*

$$\gamma_{cor}(C_m \square C_n) = \begin{cases} \frac{m(m+1)}{6} & \text{if } m \text{ is odd} \\ \frac{m^2}{6} & \text{if } m \text{ is even} \end{cases}$$

Proof. Similar to Theorem 3.1.

Theorem 3.6. *Let* C_m *and* $C_n (m, n \geq 3)$ *be any two cycles. Then* $\gamma_{cor}(C_m \square C_n)$ *does not exist if and only if* $m \not\equiv 0 (\bmod 3)$ *and* $n \not\equiv 0 (\bmod 3)$.

Proof. Let $C_m \square C_n$ be any graph and let $m \not\equiv 0 (\bmod 3)$ and $n \not\equiv 0 (\bmod 3)$. Suppose $\gamma_{cor}(C_m \square C_n)$ exist. Then there exist a corporate dominating set say C, which will be in one of the following forms. (i) $C = V_1$ (ii) $C = E_1$ (iii) $C = V_1 \cup E_1$.

If $C = V_1$ holds, then $P = \varphi$ and $Q \neq \varphi$. Since for any $u \in Q$, $N(u) \cap N(u) \neq \varphi$ for some $u \in Q$, which is a contradiction.

If $C = E_1$ holds, then $P \neq \varphi$ and $Q = \varphi$. Since for any $u \in P$, $|N(u) \cap N(W)| > 1$ for some $w \in P$, which is a contradiction.

If $C = V_1 \cup E_1$ holds, then $P \neq \varphi$ and $Q \neq \varphi$. Let $|P'| \geq 2$ and $|Q| \geq 1$. Then there exists at least one vertex in $(P \cup C)^c$ is adjacent to more than one vertex in $P \cup C$, which is a contradiction, as for any $w \in P$ or $w \in Q$, $|N(u) \cup N(w)| > 1$ for some $w \in P$ or $w \in Q$. Hence $\gamma_{cor}(C_m \square C_n)$ does not exist.

Conversely, assume that $\gamma_{cor}(C_m \square C_n)$ does not exist. We have to prove that $m \not\equiv 0 (\bmod 3)$ and $n \not\equiv 0 (\bmod 3)$. W. l. g let us take either $m \not\equiv 0 (\bmod 3)$ or $n \not\equiv 0 (\bmod 3)$. By Corollary 3.2 or Corollary 3.4, $\gamma_{cor}(C_m \square C_n)$ exist, which is a contradiction.

Suppose $m \equiv 0 (\bmod 3)$ and $n \equiv 0 (\bmod 3)$. Then by Theorem 3.1, $\gamma_{cor}(C_m \square C_n)$ exist, which is a contradiction.

Corollary 3.7. *Let* C_4 *be a cycle and* $C_m (m \geq 3)$ *be any cycle. Then* $\gamma_{cor}(C_m \square C_4)$ *does not exist.*

Proof. Let $C_m \square C_4$ be any graph. Consider the following cases.

Case 1. Let $m \not\equiv 0 (\bmod 3)$ and $n = 4$. Since $m \not\equiv 0 (\bmod 3)$ and $n \not\equiv 0 (\bmod 3)$, by Theorem 3.6 $\gamma_{cor}(C_m \square C_4)$ does not exist.

Case 2. Let $m \equiv 0 (\bmod 3)$ and $n = 4$. Suppose $\gamma_{cor}(C_m \square C_4)$ exist. Then there exists a corporate dominating set C which will be in one of the following forms. (i) $C = V_1$ (ii) $C = E_1$ (iii) $C = V_1 \cup E_1$.

As for any $u \in P$, or $u \in Q$, $| N(u) \cap N(w) | > 1$ for some $w \in P$, or $w \in Q$, $\gamma_{cor}(C_m \square C_4)$ does not exist, which is a contradiction.

4. Conclusion and Future work

In this paper, we have found the exact value of corporate domination number for $C_m \square C_n$, the Cartesian product of two cycles C_m and C_n. Furthermore, we will list out some interesting problems for further research that we have planned during the course of our investigation.

(1) Find the corporate domination number of the Cartesian product of two paths P_m and P_n.

(2) Study of the domination chain connecting the parameters $\gamma_{cor}(G)$, $\Gamma_{cor}(G)$, $ir_{cor}(G)$, $IR_{cor}(G)$, $i_{cor}(G)$ which is one of the strongest focal points of research in domination theory.

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