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DISTANCE DEGREE SEQUENCE OF SOME GRAPHS

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Abstract

The distance degree sequence of graphs were helpful in the study of differentiating molecules with the same molecular formula, that is, isomers in Chemistry. This was done by representing the molecules as graphs. Let G be a graph with vertex set V and Edge set E. Let u be a vertex in the graph G. The distance degree sequence of vertex u is denoted by $DDS(u)=(d_0(u), d_1(u), \ldots, d_{e(u)}(u))$. Here $d_i(u)$ is the number of vertices at distance i from u. Also $e(u)$ is the eccentricity of the vertex u in G. In this paper, we determine the distance degree sequence of rooted product of graphs, the distance degree sequence of strong product of graphs.

1. Introduction

A sequence for a graph is represented as a list of numbers that is an invariant rather than a single number. It is much easier to handle sequences compared to a single invariant as sequences contain or carry more information about a graph than a single invariant [1], [2], [3], [4], [5]. There are many sequences representing a graph available in literature viz, degree sequence, eccentric sequence, distance degree sequence, status sequence, path degree sequence, etc. First sequence introduced onto a graph was its degree sequence. Havel and Hakimi independently worked on the realizability of a degree sequence by a graph [9], [10]. The eccentric

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sequences were the first distance related sequences introduced for undirected graphs.

Major contributions are due to Lesniak, Ostrand, Behzad and Simpson, Nandakumar [11], [12] [15]. Next distance based sequences were the path degree sequences and distance degree sequences studied by Randic. Initially these sequences were defined to model a chemical structure by its molecular graph and study them to distinguish isomers. Bloom et al. continued the study of distance degree sequences by defining two classes of graphs namely distance degree regular graphs (DDR) and distance degree injective (DDI) graphs [6],[7], [8], [13]. Many researchers have contributed to the study of these graphs as they have peculiar properties and are extremes in terms of their structures. DDI graphs are highly irregular whereas DDR graphs are highly regular [14], [15], [16].

2. Preliminaries

We will now define the terminologies required in this paper.

Let $G(V, E)$ denote a graph with set of vertices V, and the set of edges E. The distance $d(u, v)$ from a vertex u of G to a vertex v is the length of a shortest u to v path. The degree of the vertex u is the number of vertices at distance one. The sequence of numbers of vertices having $0, 1, 2, 3, \ldots$ is called the degree sequence, which is the list of degrees of vertices of G arranged in non-decreasing order.

The eccentricity $e(v)$ of v is the distance of a farthest vertex from v.

The minimum of the eccentricities is the radius, rad (G) and the maximum is the diameter, diam(G) of G.

A graph is said to be self centered if all the vertices have the same eccentricity.

The eccentric sequence of a connected graph G is a list of the eccentricities of its vertices arranged in non-decreasing order.

The distance degree sequence of a vertex is a generalization of its degree sequence. The distance degree sequence (dds) of a vertex v in a graph G is a list of the number of vertices at distance $1, 2, \ldots, e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus the sequence $(d_{i0}, d_{i1}, d_{i2}, \ldots, d_{ij} \ldots)$ is the distance degree sequence of a vertex v_i in G where d_{ij} denotes the number of vertices at distance j from v_i.

The rooted product of a graph G and a rooted graph H is obtained by taking $|V(G)|$ copies of H, and for every i^{th} vertex of G, identify v_i with the root node of the i^{th} copy of H.

The strong product $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if:

- $u = v$ and u' is adjacent to v', or
- u', v' and u is adjacent to v, or
- u is adjacent to v and u' is adjacent to v'.

3. Main Results

Theorem 3.1. *The Distance Degree Sequence of the rooted product of path p_m and cycle C_3 is given as follows*

(i) $\{(2, 2, 3, 4, 4, 4, \ldots, 4)^2, (2, 3, 3, 4, \ldots, n)^2,$

$(2, 3, 5, \ldots, (n+1))^2, (1, 2, 2, 3, \ldots, 4)^2, (2, 2, 2, \ldots, n)^2, (2, 2, 3, \ldots, (n+1))^2,$

$(1, 1, 2, 2, \ldots, 4)^4, \ldots, (1, 2, 2, \ldots, n)^4, (1, 2, 2, \ldots (n+1))^4\}$ *when n is even*

(ii) $\{(2, 2, 3, 4, 4, \ldots, 4)^2, (2, 2, \ldots, (n+1))^2, (3, 4, \ldots, (n+1)), (1, 2, 2, \ldots, 4)^2,$

$(2, 2, \ldots, (n+1))^2, (2, 3, 4, \ldots, (n+1)), (1, 1, 2, \ldots, 4)^4, (1, 2, 2, \ldots, (n+1))^4,$

$(1, 2, 2, \cdots (n+1))^2\}$ *when n is odd*

Proof. We prove the result for even n. The distance degree sequence of the vertex 1 and vertex m for path P_m are computed first where the eccentricity of these vertices is 4. Then the distance degree sequence of vertex 2 and vertex $m-1$ is computed. The eccentricity of these vertices is $n+1$. Continuing this way we determine the distance degree sequence of the path P_m. We note that there is a pair of middle vertices which have the same distance degree sequence. Now we consider the cycle C_3. By the definition of Rooted product of graphs we have m copies of C_3. We determine the distance degree sequence of each vertex of every m copies of C_3. Combining the distance degree sequence of every vertex of the rooted product of the above said graphs we obtain the desired result when n is even.

We now prove the result for odd n. The eccentricity of the first and the last vertices (i.e. vertex 1 and vertex m) in the path P_m is 4 and thus we obtain the distance degree sequence of these vertices. Similarly the distance degree sequence of all the vertices are determined. Here we note that there is a single middle vertex for which the distance degree sequence is computed. Now we consider the cycle C_3. We determine the distance degree sequence of each vertex of every m copies of C_3. Combining the distance degree sequence of every vertex of the rooted product of the above said graphs we obtain the desired result when n is odd.

4. Result

The Distance Degree Sequence of the strong product of two path graphs (Kings Graph) is given as follows. The Kings graph is a strong product of two path graphs.

For a nXm king's graph the total number of vertices is nm and the number of edges is $4nm-3(n+m)+2$. For a square nXn king's graph, the total number of vertices is n^2 and the total number of edges is $(2n-2)(2n+1)$.

We find the distance degree sequence of 4×5 Kings graph.

Here $n=4$ and $m=5$.

The distance degree sequence of the first vertex is $3, 4, 5, 7$. Since this is the edge vertex of the kings graph, the remaining three edge vertices will also have the same distance degree sequence. The vertex next to the edge vertex will have the distance degree sequence 5, 6, 8. The remaining vertices next to the edge vertices will have the same distance degree sequence as the former vertex. Continuing this way we obtain the distance degree sequence of the 4×5 Kings graph as

$$\{(3, 4, 5, 7)^4, (5, 6, 8)^4, (5, 5, 8)^4 (4, 4, 5, 6)^4, (4, 7, 8)^4, (8, 11)^2\}.$$

Similar results can be determined to the kings graph for any value of m and n.

5. Conclusion

In this paper we have determined the distance degree sequence of the rooted product of graphs and the distance degree sequence of strong product of graphs. Similar results can be extended to various types of graphs and their properties can be determined. To find the distance degree sequence of some classes of graphs itself is very challenging which paves way for many open problems. Such problems can be considered in the future and various results and properties can be computed.

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