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UNION AND INTERSECTION OF BIPOLAR *L*-FUZZY SUB ℓ-HX GROUPS

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Abstract

In this paper, we define the concept of union and intersection of any two bipolar L-fuzzy subsets of sub ℓ-group and sub ℓ-HX group. We introduce the concepts of union and Intersection of any two bipolar L-fuzzy sub ℓ-groups and sub ℓ-HX groups with suitable examples.

1. Introduction

L. A. Zadeh [11] introduced the conception of fuzzy set. Rosenfeld [9] developed the conception of fuzzy subgroups. In fuzzy sets, the membership degree range is [0, 1]. J. A Goguen [2] replaced the valuation set [0, 1]

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through complete lattice and introduce the L-fuzzy set. In Bipolar-valued fuzzy sets, the membership degree range is enlarged from the interval [0, 1] to $[-1, 1]$. The membership degree (0, 1] indicates that components somewhat satisfy the property and also the membership degree $[-1, 0)$ indicates that components somewhat satisfy the implicit counter-property. Li Hongxing [3] introduced the idea of *HX* group and also the authors Luo Chengzhong, Mi Honghai, Li Hongxing [4] introduced the idea of fuzzy *HX* group. G. S. V. Satya Saibaba [10] introduced the idea of Fuzzy lattice ordered groups. During this paper we tend to discuss some properties of bipolar L-Fuzzy sub ℓ- HX group using the idea of union and intersection.

2. Preliminaries

In this section, we provide some basic definitions. Throughout this paper $G = (G, *, \leq)$ could be a lattice ordered group or a ℓ- group, e is that the identity of G and mn we tend to mean $m * n$.

Definition 2.1. A bipolar L-fuzzy subset α of G is said to be bipolar L-fuzzy sub ℓ- group of G if for any $m, n \in G$

(i) $\alpha^{+}(mn) \geq \alpha^{+}(m) \wedge \alpha^{+}(n)$

(ii) $\alpha^{-}(mn) \geq \alpha^{-}(m) \vee \alpha^{-}(n)$

(iii) $\alpha^{-}(m^{-1}) = \alpha^{+}(m)$

(iv) $\alpha^{-}(m^{-1}) = \alpha^{-}(m)$

(v) $\alpha^{+}(m \vee n) \geq \alpha^{+}(m) \wedge \alpha^{+}(n)$

(vi) $\alpha^{-}(m \vee n) \leq \alpha^{-}(m) \vee \alpha^{-}(n)$

(vii) $\alpha^{+}(m \wedge n) = \alpha^{+}(m) \wedge \alpha^{+}(n)$

(viii) $\alpha^{-}(m \wedge n) \leq \alpha^{-}(m) \vee \alpha^{-}(n)$.

Definition 2.2. Let α be a bipolar L-fuzzy subset defined on G. Let $\vartheta \subset 2^{G} - \{\phi\}$ be a ℓ- HX group on G. A bipolar L-fuzzy set ρ^{α} defined on ϑ is said to be a bipolar L-fuzzy sub ℓ- HX group on ϑ if for all $P, Q \in \vartheta$.

(i) $(\rho^{\alpha})^{+}(PQ) \geq (\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q)$

(ii) $(\rho^{\alpha})^{-}(PQ) \leq (\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q)$

(iii) $(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(P^{-1})$

(iv) $(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(P^{-1})$

(v) $(\rho^{\alpha})^{+}(P \vee Q) \geq (\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q)$

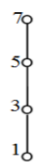
(vi) $(\rho^{\alpha})^{-}(P \vee Q) \geq (\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q)$

(vii) $(\rho^{\alpha})^{+}(P \wedge Q) \geq (\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q)$

(viii) $(\rho^{\alpha})^{-}(P \wedge Q) \leq (\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q)$.

Where $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)/$ for all $m \in P \subseteq G\}$ and $(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)/$ for all $m \in P \subseteq G\}$.

Example 2.3. Let $(G, \cdot_8) = (\{1, 3, 5, 7\}, \cdot_8\}$ be a group where G is the non-negative integer relatively prime to 8. Consider the partial order relation "Less than or equal to" on G. It gives the following Hasse Diagram.



So, we can easily verify both GLB and LUB exist for each pair of G. Clearly the poset $(G, \leq)$ is a Lattice.

Hence $(G, \cdot_8, \leq)$ is a Lattice ordered group.

Let $\vartheta \subset 2^{G} - \{\phi\}$ be a non-empty set. Let us consider $\vartheta = \{P, Q\} = \{\{1, 3\}, \{5, 7\}\}$

$\cdot_8$	P	Q
P	P	Q
Q	Q	P

Clearly $(\vartheta, \cdot_8)$ is a HX-group.

Consider the partial order relation $P \subseteq Q$ iff $p \leq q$ for all $p \in P$ and $q \in Q$ on ϑ. It gives the following Hasse Diagram.

So, we can easily verify both GLB and LUB exist for each pair of ϑ.

$\wedge$	P	Q
P	P	P
Q	Q	Q

$\vee$	P	Q
P	P	Q
Q	Q	Q

Clearly the poset $(\vartheta, \subseteq)$ is a Lattice.

Hence, $(\vartheta, ._{8}, \subseteq)$ is a Lattice ordered HX-group.

Definition 2.4 [10]. Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ are bipolar L-fuzzy subsets of sub ℓ-group of G. The intersection of α and β is $(\alpha \cap \beta) = ((\alpha \cap \beta)^{+}, (\alpha \cap \beta)^{-})$ defined as

(i) $(\alpha \cap \beta)^{+}(m) = (\alpha)^{+}(m) \wedge (\beta)^{+}(m)$

(ii) $(\alpha \cap \beta)^{-}(m) = (\alpha)^{-}(m) \vee (\beta)^{-}(m)$, for all $m \in G$

3. Some Properties and Examples of Bipolar L-Fuzzy sub ℓ-group using Union and Intersection

In this section, we discuss some of the properties and examples of bipolar L-fuzzy sub ℓ-group using union and intersection.

Theorem 3.1. *If* α *and* β *be any two bipolar L-fuzzy sub* ℓ*-group of G then* $\alpha \cap \beta$ *is also a bipolar L-fuzzy sub* ℓ*-group of G.*

Proof. Let $\alpha = (\alpha^+, \alpha^-)$ and $\beta = (\beta^+, \beta^-)$ are bipolar L-fuzzy sub ℓ-group of G. for $m, n \in G$, we have

(i) $(\alpha \cap \beta)^+(mn^{-1}) = \alpha^+(mn^{-1}) \wedge \beta^+(mn^{-1})$

$$\geq (\alpha^+(m) \wedge \alpha^+(n)) \wedge (\beta^+(m) \wedge \beta^+(n))$$

$$= (\alpha^+(m) \wedge \beta^+(m)) \wedge (\alpha^+(m) \wedge \beta^+(n))$$

$$= (\alpha \cap \beta)^+(m) \wedge (\alpha \cap \beta)^+(n)$$

$(\alpha \cap \beta)^+(mn^{-1}) \leq (\alpha \cap \beta)^+(m) \wedge (\alpha \cap \beta)^+(n)$

(ii) $(\alpha \cap \beta)^-(mn^{-1}) = \alpha^-(mn^{-1}) \vee \beta^-(mn^{-1})$

$$\leq (\alpha^-(m) \vee \alpha^+(n)) \vee (\beta^-(m) \wedge \beta^-(n))$$

$$= (\alpha^-(m) \vee \beta^-(m)) \vee (\alpha^-(n) \vee \beta^-(n))$$

$$= (\alpha \cap \beta)^-(m) \vee (\alpha \cap \beta)^-(n)$$

$(\alpha \cap \beta)^-(mn^{-1}) \leq (\alpha \cap \beta)^-(m) \vee (\alpha \cap \beta)^-(n)$

(iii) $(\alpha \cap \beta)^+(m \vee n) = \alpha^+(m \vee n) \wedge \beta^+(m \vee n)$

$$\geq (\alpha^+(m) \wedge \alpha^+(n)) \wedge (\beta^+(m) \wedge \beta^+(n))$$

$$= (\alpha^+(m) \wedge \beta^+(m)) \wedge (\alpha^+(m) \wedge \beta^+(n))$$

$$= (\alpha \cap \beta)^+(m) \wedge (\alpha \cap \beta)^+(n)$$

$(\alpha \cap \beta)^+(m \vee n) \geq (\alpha \cap \beta)^+(m) \wedge (\alpha \cap \beta)^+(n)$

(iv) $(\alpha \cap \beta)^-(m \vee n) = \alpha^-(m \vee n) \wedge \beta^-(m \vee n)$

$$\leq (\alpha^-(m) \vee \alpha^-(n)) \vee (\beta^-(m) \vee \beta^-(n))$$

$$= (\alpha^-(m) \vee \beta^-(m)) \vee (\alpha^-(n) \vee \beta^-(n))$$

$$= (\alpha \cap \beta)^{-}(m) \vee (\alpha \cap \beta)^{-}(n)$$

$$(\alpha \cap \beta)^{-}(m \vee n) \leq (\alpha \cap \beta)^{-}(m) \vee (\alpha \cap \beta)^{-}(n)$$

(v) $(\alpha \cap \beta)^{+}(m \wedge n) = \alpha^{+}(m \wedge n) \wedge \beta^{+}(m \wedge n)$

$$\leq (\alpha^{+}(m) \wedge \alpha^{+}(n)) \wedge (\beta^{+}(m) \wedge \beta^{+}(n))$$

$$= (\alpha^{+}(m) \wedge \beta^{+}(m)) \wedge (\alpha^{+}(m) \wedge \beta^{+}(n))$$

$$= (\alpha \cap \beta)^{+}(m) \wedge (\alpha \cap \beta)^{+}(n)$$

$$(\alpha \cap \beta)^{+}(m \wedge n) \leq (\alpha \cap \beta)^{+}(m) \wedge (\alpha \cap \beta)^{+}(n)$$

(vi) $(\alpha \cap \beta)^{-}(m \wedge n) = \alpha^{-}(m \wedge n) \wedge \beta^{-}(m \wedge n)$

$$\geq (\alpha^{-}(m) \vee \alpha^{-}(n)) \vee (\beta^{-}(m) \vee \beta^{-}(n))$$

$$= (\alpha^{-}(m) \vee \beta^{-}(m)) \vee (\alpha^{-}(n) \vee \beta^{-}(n))$$

$$= (\alpha \cap \beta)^{-}(m) \vee (\alpha \cap \beta)^{-}(n)$$

$$(\alpha \cap \beta)^{-}(m \wedge n) \leq (\alpha \cap \beta)^{-}(m) \vee (\alpha \cap \beta)^{-}(n).$$

Hence, $\alpha \cap \beta$ is a bipolar L-fuzzy sub ℓ-group of G.

Example 3.2. The above theorem can be explained in this example. Let $(G, {}_{.15}, \leq) = (\{1, 4, 7, 13\}, {}_{.15}, \leq)$ be a ℓ-group where G is the non-negative integer relatively prime to 15. Let $\alpha = \{\langle m, \alpha^{+}(m), \alpha^{-}(m)\rangle : m \in G\}$ and $\beta = \{\langle m, \beta^{+}(m), \beta^{-}(m)\rangle : m \in G\}$ are bipolar L-fuzzy subsets of G.

The mappings $\alpha^{+} : G \to L, \alpha^{-} : G \to L, \beta^{+} : G \to L$ and $\beta^{-} : G \to L$ are defined as,

$\alpha^{+}(1) = 0.7$	$\alpha^{-}(1) = -0.8$	$\beta^{+}(1) = 0.8$	$\beta^{-}(1) = -0.9$
$\alpha^{+}(4) = 0.6$	$\alpha^{-}(4) = -0.5$	$\beta^{+}(4) = 0.7$	$\beta^{-}(4) = -0.6$

$\alpha^{+}(7) = 0.5$	$\alpha^{-}(7) = -0.4$	$\beta^{+}(7) = 0.6$	$\beta^{-}(7) = -0.3$
$\alpha^{+}(13) = 0.5$	$\alpha^{-}(13) = -0.4$	$\beta^{+}(13) = 0.6$	$\beta^{-}(13) = -0.3$

Clearly, α and β is a bipolar *L*-fuzzy sub ℓ-group of *G*.

The intersection of α and β is calculated as,

$(\alpha \cap \beta)^{+}(1) = 0.8$	$(\alpha \cap \beta)^{-}(1) = 0.9$
$(\alpha \cap \beta)^{+}(4) = 0.7$	$(\alpha \cap \beta)^{-}(4) = -0.8$
$(\alpha \cap \beta)^{+}(7) = 0.6$	$(\alpha \cap \beta)^{-}(7) = -0.7$
$(\alpha \cap \beta)^{+}(13) = 0.6$	$(\alpha \cap \beta)^{-}(13) = -0.7$

Hence, $\alpha \cap \beta$ is a bipolar *L*-fuzzy sub ℓ-group of *G*.

Definition 3.3 [10]**.** Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ are bipolar *L*-fuzzy subsets of sub ℓ-group of *G*. The union of α and β is $(\alpha \cup \beta) = ((\alpha \cup \beta)^{+}, (\alpha \cup \beta)^{-})$ defined as

(i) $(\alpha \cup \beta)^{+}(m) = (\alpha)^{+}(m) \vee (\beta)^{+}(m)$

(ii) $(\alpha \cup \beta)^{-}(m) = (\alpha)^{-}(m) \wedge (\beta)^{-}(m)$, for all $m \in G$.

Remark 3.4. Union of two bipolar *L*-fuzzy sub ℓ-group of a ℓ-group *G* need not be a bipolar *L*-fuzzy sub ℓ-group which can be explained by the following example.

Example 3.5. Let $(G, \cdot_{15}, \leq) = (\{1, 4, 7, 13\}, \cdot_{15}, \leq)$ be a ℓ-group where *G* is a non-negative integer relatively prime to 15. Let $\alpha = \{\langle m, \alpha^{+}(m), \alpha^{-}(m)\rangle : m \in G\}$ and $\beta = \{\langle m, \beta^{+}(m), \beta^{-}(m)\rangle : m \in G\}$ are bipolar *L*-fuzzy subsets of *G*.

The mappings $\alpha^{+} : G \to L$, $\alpha^{-} : G \to L$, $\beta^{+} : G \to L$ and $\beta^{-} : G \to L$ are defined as,

$\alpha^{+}(1) = 0.8$	$\alpha^{-}(1) = -0.7$	$\beta^{+}(1) = 0.9$	$\beta^{-}(1) = -0.6$
$\alpha^{+}(4) = 0.6$	$\alpha^{-}(4) = -0.4$	$\beta^{+}(4) = 0.7$	$\beta^{-}(4) = -0.5$
$\alpha^{+}(7) = 0.3$	$\alpha^{-}(7) = -0.2$	$\beta^{+}(7) = 0.6$	$\beta^{-}(7) = -0.4$
$\alpha^{+}(13) = 0.3$	$\alpha^{-}(13) = -0.2$	$\beta^{+}(13) = 0.6$	$\beta^{-}(13) = -0.4$

Clearly, α and β is a bipolar L-fuzzy sub ℓ-group of G.

The union of α and β is calculated as,

$(\alpha \cup \beta)^{+}(1) = 0.5$	$(\alpha \cup \beta)^{-}(1) = -0.4$
$(\alpha \cup \beta)^{+}(4) = 0.8$	$(\alpha \cup \beta)^{-}(4) = -0.6$
$(\alpha \cup \beta)^{+}(7) = 0.6$	$(\alpha \cup \beta)^{-}(7) = -0.5$
$(\alpha \cup \beta)^{+}(13) = 0.6$	$(\alpha \cup \beta)^{-}(13) = -0.5$

Now, $(\alpha \cup \beta)^{+}(7.13) = (\alpha \cup \beta)^{+}(1) \geq (\alpha \cup \beta)^{+}(7) \wedge (\alpha \cup \beta)^{+}(13)$, $0.5 \geq (0.6 \wedge 0.6)$ $0.5 \geq 0.6$. And $(\alpha \cup \beta)^{-}(7.13) = (\alpha \cup \beta)^{-}(1) \leq (\alpha \cup \beta)^{-}(7) \vee (\alpha \cup \beta)^{-}(13)$, $-0.4 \leq (-0.5 \vee -0.5)$, $-0.4 \leq -0.5$.

This is not true.

Hence, $\alpha \cup \beta$ is not a bipolar L-fuzzy sub ℓ-group of G.

Theorem 3.6. *If α, β be any two bipolar L-fuzzy sub ℓ-group of G then $\alpha \cup \beta$ is a bipolar L-fuzzy sub ℓ-group of G if and only if $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof. It is obvious.

4. Some Properties and Examples of Bipolar *L*-fuzzy sub ℓ-HX group using the Concept of Union and Intersection

In this section, we discuss some properties and examples of bipolar L-fuzzy sub ℓ-HX group using the concept of union and intersection.

Theorem 4.1. *Let G be a ℓ-group. If α is a bipolar L-fuzzy sub ℓ-group of G then the bipolar L-fuzzy set ρ^{α} is a bipolar L-fuzzy sub ℓ-HX group of ϑ.*

Proof. Let α be a bipolar L-fuzzy sub ℓ-group of G and ρ^{α} be a bipolar L-fuzzy subset on G for any $P, Q \in \vartheta \subset G$

(i) $(\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q) = (\vee \{\alpha^{+}(m)/ \text{ for all } m \in P \subset G\}) \wedge (\vee \{\alpha^{+}(n)/ \text{ for all } n \in Q \subset G\})$

$= \alpha^{+}(m_0) \wedge \alpha^{+}(n_0)$, some $m_0 \in P$, $n_0 \in Q$ and $P, Q \subset G$

$\leq \alpha^{+}(m_0 n_0)$, α is a bipolar L-fuzzy sub ℓ-group on G

$= \vee\{\alpha^{+}(mn)/ \text{ for all } m \in P, n \in Q \text{ and } P, Q \subset G\}$

$= (\rho^{\alpha})^{+}(PQ)$

$(\rho^{\alpha})^{+}(PQ) \geq (\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q)$

(ii) $(\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q) = (\wedge \{\alpha^{-}(m)/ \text{ for all } m \in P \subset G\}) \vee (\wedge \{\alpha^{-}(n)/ \text{ for all } n \in Q \subset G\})$

$= \alpha^{-}(m_0) \vee \alpha^{-}(n_0)$, some $m_0 \in P$, $n_0 \in Q$ and $P, Q \subset G$

$\geq \alpha^{-}(m_0 n_0)$, α is a bipolar L-fuzzy sub ℓ-group on G

$= \wedge \{\alpha^{-}(mn)/ \text{ for all } mP, n \in Q \text{ and } P, Q \subset G\}$

$= (\rho^{\alpha})^{-}(PQ)$

$(\rho^{\alpha})^{-}(PQ) \leq (\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q)$

(iii) $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)/ \text{ for all } m \in P \subset G\}$

$= \vee\{\alpha^{+}(m^{-1})/ \text{ for all } m^{-1} \in P \subset G\}$

$= \vee\{\alpha^{+}(m^{-1})/ \text{ for all } m^{-1} \in P^{-1} \subset G\}$

$(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(P^{-1})$

(iv) $(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)/$ for all $m \in P \subset G\}$

$= \wedge\{\alpha^{-}(m^{-1})/$ for all $m^{-1} \in P \subset G\}$

$= \wedge\{\alpha^{-}(m^{-1})/$ for all $m^{-1} \in P^{-1} \subset G\}$

$(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(P^{-1})$

(v) $(\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q) = (\vee \{\alpha^{+}(m)/$ for all $m \in P \subset G\}) \wedge (\vee \{\alpha^{+}(n)/$ for all $n \in Q \subset G\})$

$= \alpha^{+}(m_0) \wedge \alpha^{+}(n_0)$, some $m_0 \in P$, $n_0 \in Q$ and $P, Q \in G$

$\leq \alpha^{+}(m_0 \wedge n_0)$, α is a bipolar L-fuzzy sub ℓ-group on G

$= \vee\{\alpha^{+}(m \wedge n)/$ for all $m \in P$, $n \in Q$ and $P, Q \subset G\}$

$= (\rho^{\alpha})^{+}(P \wedge Q)$

$(\rho^{\alpha})^{+}(P \wedge Q) \geq (\rho^{\alpha})^{+}(P) \wedge (\lambda\rho^{\alpha})^{+}(Q)$

(vi) $(\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q) = (\wedge \{\alpha^{-}(m)/$ for all $m \in P \subset G\}) \vee (\wedge \{\alpha^{-}(n)/$ for all $n \in Q \subset G\})$

$= \alpha^{-}(m_0) \vee \alpha^{-}(n_0)$, some $m_0 \in P$, $n_0 \in Q$ and $P, Q \subset G$

$\geq \alpha^{-}(m_0 \wedge n_0)$, α is a bipolar L-fuzzy sub ℓ-group on G

$= \wedge \{\alpha^{-}(m \wedge n)/$ for all $m \in P$, $n \in Q$ and $P, Q \subset G\}$

$= (\rho^{\alpha})^{-}(P \wedge Q)$

$(\rho^{\alpha})^{-}(P \wedge Q) \leq (\rho^{\alpha})^{-}(P) \vee (\rho^{\alpha})^{-}(Q)$

(vii) $(\rho^{\alpha})^{+}(P) \wedge (\rho^{\alpha})^{+}(Q) = (\vee \{\alpha^{+}(m)/$ for all $m \in P \subset G\}) \wedge (\vee \{\alpha^{+}(n)/$ for all $n \in Q \subset G\})$

$= \alpha^+(m_0) \wedge \alpha^+(n_0)$, some $m_0 \in P$, $n_0 \in Q$ and $P, Q \subset G$

$\leq \alpha^+(m_0 \vee n_0)$, μ is a bipolar L-fuzzy sub ℓ-group on G

$= \vee\{\alpha^+(m \vee n)/$ for all mP, $n \in Q$ and $P, Q \subset G\}$

$= (\rho^\alpha)^+(P \vee Q)$

$(\rho^\alpha)^+(P \vee Q) \geq (\rho^\alpha)^+(P) \wedge (\rho^\alpha)^+(Q)$

(viii) $(\rho^\alpha)^-(P) \vee (\rho^\alpha)^-(Q) = (\wedge\{\alpha^-(m)/$ for all $m \in P \subset G\}) \vee (\wedge\{\alpha^-(n)/$ for all $n \in Q \subset G\})$

$= \alpha^-(m_0) \vee \mu^-(n_0)$, some $m \in P$, $n_0 \in Q$ and $P, Q \subset G$

$\geq \alpha^-(m_0 \vee n_0)$, α is a bipolar L-fuzzy sub ℓ-group on G

$= \wedge\{\alpha^-(m \vee n)/$ for all $m \in P$, $n \in Q$ and $P, Q \subset G\}$

$= (\rho^\alpha)^-(P \vee Q)$

$(\rho^\alpha)^-(P \vee Q) \leq (\rho^\alpha)^-(P) \vee (\rho^\alpha)^-(Q)$.

Hence, ρ^α is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

Remark 4.2. Let G be a ℓ-group. Let $\alpha = (\alpha^+, \alpha^-)$ is a bipolar L-fuzzy subset of G. If $\rho^\alpha = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be a bipolar L-fuzzy sub ℓ-HX group of ϑ with $|M| \geq 2$ for all $M \in \vartheta$. then $\alpha = (\alpha^+, \alpha^-)$ need not be a bipolar L-fuzzy sub ℓ-group of G, which can be explained by the following example.

Example 4.3. Let $(G, ._{12}, \leq) = (\{1, 5, 7, 11\}, ._{12}, \leq)$ be a ℓ-group where G is the non-negative integer relatively prime to 12. Let $\alpha = \{\langle m, \alpha^+(m), \alpha^-(m)\rangle : m \in G\}$ be the bipolar L-fuzzy subset of G. The mappings $\alpha^+ : G \to L$, $\alpha^- : G \to L$ are defined as,

$\alpha^{+}(1) = 0.9$	$\alpha^{-}(1) = -0.6$
$\alpha^{+}(5) = 0.8$	$\alpha^{-}(3) = -0.4$
$\alpha^{+}(7) = 0.7$	$\alpha^{-}(5) = -0.3$
$\alpha^{+}(11) = 0.5$	$\alpha^{-}(7) = -0.2$

Assume $(\vartheta, ._{12}, \subseteq) = (\{P, Q\}, ._{12}, \subseteq) = (\{\{1, 5\}, \{7, 11\}\}, ._{12}, \subseteq)$ be a ℓ-HX group.

Let $\rho^{\alpha} = \{(m, (\rho^{\alpha})^{+}(m), (\rho^{\alpha})^{-}(m))\}/$ for all $m \in \vartheta\}$ is a bipolar L-fuzzy subset of ϑ and the mappings $(\rho^{\alpha})^{+} : \vartheta \to L$, $(\rho^{\alpha})^{-} : \vartheta \to L$ are calculated as,

$(\rho^{\alpha})^{+}(P) = 0.9$	$(\rho^{\alpha})^{-}(P) = -0.6$
$(\rho^{\alpha})^{+}(Q) = 0.8$	$(\rho^{\alpha})^{-}(Q) = -0.5$

Clearly, ρ^{α} is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

But, α is not a bipolar L-fuzzy sub ℓ-group of G.

Because, $\alpha^{+}(5.7) = \alpha^{+}(11) \geq \alpha^{+}(5) \wedge \alpha^{+}(7)$, $0.5 \geq (0.8) \wedge (0.7)$, $0.5 \geq 0.7$.

And $\alpha^{-}(5.7) = \alpha^{-}(11) \leq \alpha^{-}(5) \vee \alpha^{-}(7)$, $-0.2 \leq (-0.4) \vee (-0.3)$, $-0.2 \leq -0.3$.

This is not true.

Hence, α is not a bipolar L-fuzzy sub ℓ-group of G.

Definition 4.4 [6]**.** Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ are bipolar L-fuzzy subsets of G. Let $\vartheta \subset 2^{G} - \{\phi\}$ be a ℓ-HX group of G. Let $\rho^{\alpha} = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ and $\omega^{\beta} = ((\omega^{\beta})^{+}, (\omega^{\beta})^{-})$ are bipolar L-fuzzy subsets of ϑ. The intersection of ρ^{α} and ω^{β} is $(\rho^{\alpha} \cap \omega^{\beta}) = ((\rho^{\alpha} \cap \omega^{\beta})^{+}, (\rho^{\alpha} \cap \omega^{\beta})^{-})$ defined as

(i) $(\rho^{\alpha} \cap \omega^{\beta})^{+}(P) = (\rho^{\alpha})^{+}(P) \wedge (\omega^{\beta})^{+}(P)$

(ii) $(\rho^{\alpha} \cap \omega^{\beta})^{-}(P) = (\rho^{\alpha})^{-}(P) \vee (\omega^{\beta})^{-}(P)$

Where $(\rho^{\alpha})^{+}(P) = \vee \{\alpha^{+}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\rho^{\alpha})^{-}(P) = \wedge \{\alpha^{-}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\omega^{\beta})^{+}(P) = \vee \{\beta^{+}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\omega^{\beta})^{-}(P) = \wedge \{\beta^{-}(m)/ \text{ for all } m \in P \subseteq G\}$.

Theorem 4.5 [7]. *Let ρ^{α} and ω^{β} be any two bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group ϑ then $\rho^{\alpha} \cap \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group ϑ.*

Remark 4.6. Let G be a ℓ-group. Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ are bipolar L-fuzzy sub ℓ-group of G and $\alpha \cap \beta$ is also a bipolar L-fuzzy sub ℓ-group of G then $\Omega^{\alpha \cap \beta} = ((\Omega^{\alpha \cap \beta})^{+}, (\Omega^{\alpha \cap \beta})^{-})$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ induced by $\alpha \cap \beta$ of sub ℓ-group of G (By theorem 4.1).

Theorem 4.7. *If $\rho^{\alpha}, \omega^{\beta}, \Omega^{\alpha \cap \beta}$ are bipolar L-fuzzy sub ℓ-HX group of ϑ induced by bipolar L-fuzzy sub ℓ-groups α, β and $\alpha \cap \beta$ of G respectively then $\Omega^{\alpha \cap \beta} = \rho^{\alpha} \cap \omega^{\beta}$.*

Proof. Let $\rho^{\alpha} = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ and $\omega^{\beta} = ((\omega^{\beta})^{+}, (\omega^{\beta})^{-})$ are bipolar L-fuzzy sub ℓ-HX group of ϑ then $\rho^{\alpha} \cap \omega^{\beta}$ are bipolar L-fuzzy sub ℓ-HX group of ϑ and $\Omega^{\alpha \cap \beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ induced by bipolar L-fuzzy subset $\alpha \cap \beta$ of sub ℓ-group of G.

(i) $(\Omega^{\alpha \cap \beta})^{+}(P) = \vee \{(\alpha \cap \beta)^{+}(m)/m \in P \subseteq G\}$

$$= \vee \{\{\alpha^{+}(m) \wedge \beta^{+}(m)\}/m \in P \subseteq G\}$$

$$= (\vee \{\alpha^{+}(m)/m \in P \subseteq G\}) \wedge (\vee \{\beta^{+}(m)\}/m \in P \subseteq G\})$$

$$= (\rho^{\alpha})^{+}(P) \wedge (\omega^{\beta})^{+}(P)$$

$$= (\rho^{\alpha} \cap \omega^{\beta})^{+}(P)$$

(ii) $(\Omega^{\alpha\cap\beta})^{-}(P) = \wedge\{(\alpha\cap\beta)^{-}(m)/m \in P \subseteq G\}$

$$= \wedge\{\{\alpha^{-}(m) \vee \beta^{-}(m)\}/m \in P \subseteq G\}$$

$$= (\wedge\{\alpha^{-}(m)/m \in P \subseteq G\}) \vee (\wedge\{\beta^{-}(m)\}/m \in P \subseteq G\})$$

$$= (\rho^{\alpha})^{-}(P) \vee (\omega^{\beta})^{-}(P)$$

$$= (\rho^{\alpha} \cap \omega^{\beta})^{-}(P).$$

Hence, $\Omega^{\alpha\cap\beta} = \rho^{\alpha} \cap \omega^{\beta}$.

Example 4.8. The above theorem can be illustrated in this example.

Let $(G, ._8, \leq) = (\{1, 3, 5, 7\}, ._8, \leq)$ be a ℓ-group where G is the non-negative integer relatively prime to 8. Let $\alpha = \{\langle m, \alpha^{+}(m), \alpha^{-}(m)\rangle : m \in G\}$ and $\beta = \{\langle m, \beta^{+}(m), \beta^{-}(m)\rangle : m \in G\}$ are bipolar L-fuzzy subsets of G.

(i) The mappings $\alpha^{+} : G \to L$, $\alpha^{-} : G \to L$, $\beta^{+} : G \to L$ and $\beta^{-} : G \to L$ are defined as,

$\alpha^{+}(1) = 0.8$	$\alpha^{-}(1) = -0.7$	$\beta^{+}(1) = 0.9$	$\beta^{-}(1) = -0.7$
$\alpha^{+}(3) = 0.6$	$\alpha^{-}(3) = -0.5$	$\beta^{+}(3) = 0.7$	$\beta^{-}(3) = -0.6$
$\alpha^{+}(5) = 0.4$	$\alpha^{-}(5) = -0.3$	$\beta^{+}(5) = 0.6$	$\beta^{-}(5) = -0.5$
$\alpha^{+}(7) = 0.4$	$\alpha^{-}(7) = -0.3$	$\beta^{+}(7) = 0.6$	$\beta^{-}(7) = -0.5$

Clearly, α and β is a bipolar L-fuzzy sub ℓ-group of G.

(ii) The intersection of α and β is calculated as, clearly, $\alpha \cap \beta$ is a bipolar L-fuzzy sub ℓ-group of G.

$(\alpha \cap \beta)^{+}(1) = .8$	$(\alpha \cap \beta)^{-}(1) = -0.9$
$(\alpha \cap \beta)^{+}(3) = 0.7$	$(\alpha \cap \beta)^{-}(2) = -0.6$

$(\alpha \cap \beta)^{+}(5) = 0.6$	$(\alpha \cap \beta)^{-}(3) = -0.4$
$(\alpha \cap \beta)^{+}(7) = 0.6$	$(\alpha \cap \beta)^{-}(4) = -0.4$

(iii) Let $\vartheta = \{P, Q\} = \{\{1, 3\}, \{5, 7\}\}$ be a ℓ-HX group of G.

.8	P	Q
P	P	Q
Q	Q	P

$\wedge$	P	Q
P	P	P
Q	P	Q

$\vee$	P	Q
P	P	Q
Q	Q	Q

Define $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\omega^{\beta})^{+}(P) = \vee\{\beta^{+}(m)/ \text{ for all } m \in P \subseteq G\}$ and

$(\omega^{\beta})^{-}(P) = \wedge\{\beta^{-}(m)/ \text{ for all } m \in P \subseteq G\}$.

Now

$(\rho^{\alpha})^{+}(P) = .8$	$(\rho^{\alpha})^{-}(P) = -0.7$	$(\omega^{\beta})^{+}(P) = 0.9$	$(\omega^{\beta})^{-}(P) = -0.7$
$(\rho^{\alpha})^{+}(Q) = 0.4$	$(\rho^{\alpha})^{-}(Q) = -0.3$	$(\omega^{\beta})^{+}(Q) = 0.6$	$(\omega^{\beta})^{-}(Q) = -0.5$

Clearly, ρ^{α} and ω^{β} is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

(iv) The intersection of ρ^{α} and ω^{β} is calculated as,

$(\rho^{\alpha} \cap \omega^{\beta})^{+}(P) = 0.8$	$(\rho^{\alpha} \cap \omega^{\beta})^{-}(P) = -0.7$
$(\rho^{\alpha} \cap \omega^{\beta})^{+}(Q) = 0.4$	$(\rho^{\alpha} \cap \omega^{\beta})^{-}(Q) = -0.3$

Clearly, $\rho^{\alpha} \cap \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

(v) Let $\Omega^{\alpha \cap \beta} = ((\Omega^{\alpha \cap \beta})^{+}, (\Omega^{\alpha \cap \beta})^{-})$ be a bipolar L-fuzzy set of ϑ and the mappings $(\Omega^{\alpha \cap \beta})^{+} : \vartheta \to L$, $(\Omega^{\alpha \cap \beta})^{-} : \vartheta \to L$ are defined as

$$(\Omega^{\alpha \cap \beta})^{+}(P) = \vee\{(\alpha \cap \beta)^{+}(P)/m \in P \subseteq G\}$$

$$(\Omega^{\alpha \cap \beta})^{-}(P) = \wedge\{(\alpha \cap \beta)^{-}(P)/m \in P \subseteq G\}.$$

Now

$(\Omega^{\alpha \cap \beta})^{+}(P) = 0.8$	$(\Omega^{\alpha \cap \beta})^{-}(P) = -0.7$
$(\Omega^{\alpha \cap \beta})^{+}(Q) = 0.4$	$(\Omega^{\alpha \cap \beta})^{-}(Q) = -0.3$

Clearly $\Omega^{\alpha \cap \beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

(vi) Let $\Omega^{\alpha \cap \beta} = ((\Omega^{\alpha \cap \beta})^{+}, (\Omega^{\alpha \cap \beta})^{-})$ be a bipolar L-fuzzy sub ℓ-HX group of ϑ induced by $\alpha \cap \beta$ of sub ℓ-group of G. where

$$(\Omega^{\alpha \cap \beta})^{+}(P) = \vee\{(\alpha \cap \beta)^{+}(P)/m \in P \subseteq G\}$$

$$(\Omega^{\alpha \cap \beta})^{-}(P) = \wedge\{(\alpha \cap \beta)^{-}(P)/m \in P \subseteq G\}.$$

$(\Omega^{\alpha \cap \beta})^{+}(P) = 0.8$	$(\Omega^{\alpha \cap \beta})^{-}(P) = -0.7$
$(\Omega^{\alpha \cap \beta})^{+}(Q) = 0.4$	$(\Omega^{\alpha \cap \beta})^{-}(Q) = -0.3$

Clearly, $\Omega^{\alpha \cap \beta} = \rho^{\alpha} \cap \sigma^{\beta}$.

Hence, by the conditions (i), (ii), (iii), (iv), (v) and (vi), $\Omega^{\alpha \cap \beta} = \rho^{\alpha} \cap \sigma^{\beta}$ is proved.

Definition 4.9 [6]. Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ are bipolar L-fuzzy subsets of G. Let $\vartheta \subset 2^{G} - \{\phi\}$ be a ℓ-HX group of G. Let $\rho^{\alpha} = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ and $\omega^{\beta} = ((\omega^{\beta})^{+}, (\omega^{\beta})^{-})$ are bipolar L-fuzzy subsets of ϑ. The union of ρ^{α} and ω^{β} is $(\rho^{\alpha} \cup \omega^{\beta}) = ((\rho^{\alpha} \cup \omega^{\beta})^{+}, (\rho^{\alpha} \cup \omega^{\beta})^{-})$ defined as,

(i) $(\rho^{\alpha} \cup \omega^{\beta})^{+}(P) = (\rho^{\alpha})^{+}(P) \vee (\omega^{\beta})^{+}(P)$

(ii) $(\rho^{\alpha} \cup \omega^{\beta})^{-}(P) = (\rho^{\alpha})^{-}(P) \wedge (\omega^{\beta})^{-}(P)$

Where $(\rho^{\alpha})^{+}(P) = \vee\{\mu^{+}(m) / \text{ for all } m \in P \subseteq G\}$

$(\rho^{\alpha})^{-}(P) = \wedge\{\mu^{-}(m) / \text{ for all } m \in P \subseteq G\}$

$(\omega^{\beta})^{+}(P) = \vee\{\beta^{+}(m) / \text{ for all } m \in P \subseteq G\}$

$(\omega^{\beta})^{-}(P) = \wedge\{\beta^{-}(m) / \text{ for all } m \in P \subseteq G\}$.

Theorem 4.10 [6]. *Let ρ^{α} and ω^{β} be any two bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group ϑ then $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group ϑ.*

Example 4.11. If ρ^{α} and ω^{β} be any two bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group ϑ then $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group ϑ, which is described in this example.

Let $(G, ._{15}, \leq) = (\{1, 4, 7, 13\}, ._{15}, \leq)$ be a ℓ-group where G is the non-negative integer relatively prime to 15. Let $\alpha = \{\langle m, \alpha^{+}(m), \alpha^{-}(m)\rangle : m \in G\}$ and $\beta = \{\langle m, \beta^{+}(m), \beta^{-}(m)\rangle : m \in G\}$ are bipolar L-fuzzy subsets of sub ℓ-group of G and the mappings $\alpha^{+} : G \to L, \alpha^{-} : G \to L, \beta^{+} : G \to L$ and $\beta^{-} : G \to L$ are defined as,

$\alpha^{+}(1) = 0.5$	$\alpha^{-}(1) = -0.6$	$\beta^{+}(1) = 0.4$	$\beta^{-}(1) = -0.8$
$\alpha^{+}(4) = 0.4$	$\alpha^{-}(4) = -0.5$	$\beta^{+}(4) = 0.3$	$\beta^{-}(4) = -0.5$

$\alpha^{+}(7) = 0.3$	$\alpha^{-}(7) = -0.4$	$\beta^{+}(7) = 0.2$	$\beta^{-}(7) = -0.3$
$\alpha^{+}(13) = 0.3$	$\alpha^{-}(13) = -0.4$	$\beta^{+}(13) = 0.2$	$\beta^{-}(13) = -0.3$

Clearly, α and β are bipolar L-fuzzy sub ℓ-group of G.

The union of α and β is calculated as,

$(\alpha \cup \beta)^{+}(1) = 0.9$	$(\alpha \cup \beta)^{-}(1) = -0.8$
$(\alpha \cup \beta)^{+}(4) = 0.8$	$(\alpha \cup \beta)^{-}(4) = -0.7$
$(\alpha \cup \beta)^{+}(7) = 0.7$	$(\alpha \cup \beta)^{-}(7) = -0.6$
$(\alpha \cup \beta)^{+}(13) = 0.7$	$(\alpha \cup \beta)^{-}(13) = -0.6$

Clearly, $\alpha \cup \beta$ is a bipolar L-fuzzy sub ℓ-group of G.

Case i. $|M| = 1$ for all $M \in \vartheta$. Let $\vartheta = \{P, Q, R, S\} = \{\{1\}, \{4\}, \{7\}, \{13\}\}$ be a ℓ-HX group of G with usual multiplication and $|M| = 1$ for all $M \in \vartheta$. Let ρ^{α} and ω^{β} are bipolar L-fuzzy subsets of ϑ and the mappings $(\rho^{\alpha})^{+} : \vartheta \to L$, $(\rho^{\alpha})^{-} : \vartheta \to L$ are calculated as,

$(\rho^{\alpha})^{+}(P) = 0.6$	$(\rho^{\alpha})^{-}(P) = -0.7$	$(\omega^{\beta})^{+}(P) = 0.8$	$(\omega^{\beta})^{-}(P) = -0.7$
$(\rho^{\alpha})^{+}(Q) = 0.5$	$(\rho^{\alpha})^{-}(Q) = -0.6$	$(\omega^{\beta})^{+}(Q) = 0.6$	$(\omega^{\beta})^{-}(Q) = -0.5$
$(\rho^{\alpha})^{+}(R) = 0.4$	$(\rho^{\alpha})^{-}(R) = -0.5$	$(\omega^{\beta})^{+}(R) = 0.4$	$(\omega^{\beta})^{-}(R) = -0.3$
$(\rho^{\alpha})^{+}(S) = 0.4$	$(\rho^{\alpha})^{-}(S) = -0.5$	$(\omega^{\beta})^{+}(S) = 0.4$	$(\omega^{\beta})^{-}(S) = -0.3$

Clearly, ρ^{α} and ω^{β} is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

The union of ρ^{α} and ω^{β} is calculated as,

$(\rho^{\alpha} \cup \omega^{\beta})^{+}(P) = 0.5$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(P) = -0.6$
$(\rho^{\alpha} \cup \omega^{\beta})^{+}(Q) = 0.4$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(R) = -0.5$

$(\rho^{\alpha} \cup \omega^{\beta})^{+}(R) = 0.3$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(S) = -0.4$
$(\rho^{\alpha} \cup \omega^{\beta})^{+}(S) = 0.3$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(S) = -0.4$

Clearly, $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

So, α and β are bipolar L-fuzzy sub ℓ-group of G and $\alpha \cup \beta$ is a bipolar L-fuzzy sub ℓ-group of G then $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

Case ii. $|M| = 2$ for all $M \in \vartheta$. Let $\vartheta = \{P, Q\} = \{\{1, 3\}, \{5, 7\}\}$ be a ℓ-HX group of G with usual multiplication and $|M| = 2$ for all $M \in \vartheta$. Let ρ^{α} and ω^{β} are bipolar L-fuzzy subsets of ϑ and the mappings $(\rho^{\alpha})^{+} : \vartheta \to L$, $(\rho^{\alpha})^{-} : \vartheta \to L$ are calculated as,

.15	P	Q
P	P	Q
Q	Q	P

$\wedge$	P	Q
P	P	P
Q	P	Q

$\vee$	P	Q
P	P	Q
Q	Q	Q

Define $(\rho^{\alpha})^{+}(P) = \vee \{\alpha^{+}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\rho^{\alpha})^{-}(P) = \wedge \{\alpha^{-}(m)/ \text{ for all } m \in P \subseteq G\}$

$(\omega^{\beta})^{+}(P) = \vee \{\beta^{+}(m)/ \text{ for all } m \in P \subseteq G\}$ and

$(\omega^{\beta})^{-}(P) = \wedge\{\beta^{-}(m)/ \text{ for all } m \in P \subseteq G\}.$

Now

$(\rho^{\alpha})^{+}(P) = 0.4$	$(\rho^{\alpha})^{-}(P) = -0.5$	$(\omega^{\beta})^{+}(P) = 0.7$	$(\omega^{\beta})^{-}(P) = -0.9$
$(\rho^{\alpha})^{+}(Q) = 0.3$	$(\rho^{\alpha})^{-}(Q) = -0.2$	$(\omega^{\beta})^{+}(Q) = 0.6$	$(\omega^{\beta})^{-}(Q) = -0.8$

Clearly, ρ^{α} and ω^{β} is a bipolar L-fuzzy sub ℓ-HX group of ϑ. The union of ρ^{α} and ω^{β} is calculated as,

$(\rho^{\alpha} \cup \omega^{\beta})^{+}(P) = 0.8$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(P) = -0.7$
$(\rho^{\alpha} \cup \omega^{\beta})^{+}(Q) = 0.6$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(Q) = -0.5$

Clearly, $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

Hence, by cases (i) and (ii), $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

Remark 4.12. Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ are bipolar L-fuzzy sub ℓ-group of G and $\alpha \cup \beta$ is a bipolar L-fuzzy subsets of sub ℓ-group of G and need not be a bipolar L-fuzzy sub ℓ-group of G.

If $\eta^{\alpha\cup\beta}$ is a bipolar L-fuzzy subset induced by $\alpha \cup \beta$ then $\eta^{\alpha\cup\beta} = ((\eta^{\alpha\cup\beta})^{+}, (\eta^{\alpha\cup\beta})^{-})$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ with $|M| \geq 2$ for all $M \in \vartheta$.

Theorem 4.13. *Let* $\rho^{\alpha}, \omega^{\beta}, \Omega^{\alpha\cup\beta}$ *are bipolar L-fuzzy sub* ℓ*-HX group of* ϑ *induced by bipolar L-fuzzy subsets* α, β *and* $\alpha \cup \beta$ *of sub* ℓ*-group of G respectively then* $\Omega^{\alpha\cup\beta} = \rho^{\alpha} \cup \omega^{\beta}$.

Proof. Let $\rho^{\alpha} = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ and $\omega^{\beta} = ((\omega^{\beta})^{+}, (\omega^{\beta})^{-})$ are bipolar L-fuzzy sub ℓ-HX group of ϑ then $\rho^{\alpha} \cup \omega^{\beta}$ are bipolar L-fuzzy sub ℓ-HX group of ϑ and $\Omega^{\alpha\cup\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ induced by bipolar L-fuzzy subset $\alpha \cup \beta$ of sub ℓ-group of G.

(i) $(\Omega^{\alpha\cup\beta})^{+}(P) = \vee\{(\alpha\cup\beta)^{+}(m)/m \in P \subseteq G\}$

$$= \vee\{\{\alpha^{+}(m) \wedge \beta^{+}(m)\}/m \in P \subseteq G\}$$

$$= (\vee\{\alpha^{+}(m)/m \in P \subseteq G\}) \vee (\vee\{\beta^{+}(m)\}/m \in P \subseteq G\})$$

$$= (\rho^{\alpha})^{+}(P) \vee (\omega^{\beta})^{+}(P)$$

$$= (\rho^{\alpha} \cup \omega^{\beta})^{+}(P)$$

(ii) $(\Omega^{\alpha\cup\beta})^{-}(P) = \wedge\{(\alpha\cup\beta)^{-}(m)/m \in P \subseteq G\}$

$$= \wedge\{\{\alpha^{-}(m) \wedge \beta^{-}(m)\}/m \in P \subseteq G\}$$

$$= (\wedge\{\alpha^{-}(m)/m \in P \subseteq G\}) \wedge (\wedge\{\beta^{-}(m)\}/m \in P \subseteq G\})$$

$$= (\rho^{\alpha})^{-}(P) \vee (\omega^{\beta})^{-}(P)$$

$$= (\rho^{\alpha} \cup \omega^{\beta})^{-}(P).$$

Hence, $\Omega^{\alpha\cup\beta} = \rho^{\alpha} \cup \omega^{\beta}$.

Example 4.14. The above theorem can be illustrated in this example.

Let $(G, ._{8}, \leq) = (\{1, 3, 5, 7\}, ._{8}, \leq)$ be a ℓ-group where G is the non-negative integer relatively prime to 8. Let $\alpha = \{\langle m, \alpha^{+}(m), \alpha^{-}(m)\rangle : m \in G\}$ and $\beta = \{\langle m, \beta^{+}(m), \beta^{-}(m)\rangle : m \in G\}$ are bipolar L-fuzzy subsets of G.

(i) The mappings $\alpha^{+} : G \to L$, $\alpha^{-} : G \to L$, $\beta^{+} : G \to L$ and $\beta^{-} : G \to L$ are defined as,

$\alpha^{+}(1) = 0.7$	$\alpha^{-}(1) = -0.8$	$\beta^{+}(1) = 0.8$	$\beta^{-}(1) = -0.9$
$\alpha^{+}(3) = 0.6$	$\alpha^{-}(3) = -0.5$	$\beta^{+}(3) = 0.5$	$\beta^{-}(3) = -0.7$
$\alpha^{+}(5) = 0.4$	$\alpha^{-}(5) = -0.6$	$\beta^{+}(5) = 0.3$	$\beta^{-}(5) = -0.5$
$\alpha^{+}(7) = 0.4$	$\alpha^{-}(7) = -0.6$	$\beta^{+}(7) = 0.3$	$\beta^{-}(7) = -0.5$

Clearly, α and β is a bipolar L-fuzzy sub ℓ-group of G.

(ii) The union of α and β is calculated as,

$(\alpha \cup \beta)^{+}(1) = 0.5$	$(\alpha \cup \beta)^{-}(1) = -0.6$
$(\alpha \cup \beta)^{+}(3) = 0.4$	$(\alpha \cup \beta)^{-}(3) = -0.5$
$(\alpha \cup \beta)^{+}(5) = 0.3$	$(\alpha \cup \beta)^{-}(5) = -0.4$
$(\alpha \cup \beta)^{+}(7) = 0.2$	$(\alpha \cup \beta)^{-}(7) = -0.3$

Now, $(\alpha \cup \beta)^{+}(3.5) = (\alpha \cup \beta)^{+}(7) \geq (\alpha \cup \beta)^{+}(3) \wedge (\alpha \cup \beta)^{+}(5), 0.2 \geq (0.4 \wedge 0.3)$, $0.2 \geq 0.3$.

And $(\alpha \cup \beta)^{-}(3.5) = (\alpha \cup \beta)^{-}(7) \leq (\alpha \cup \beta)^{-}(3) \vee (\alpha \cup \beta)^{-}(5), -0.3 \leq (-0.5 \vee -0.4)$, $-0.3 \leq -0.4$.

This is not true.

Therefore, $\alpha \cup \beta$ is not a bipolar L-fuzzy sub ℓ- group of G.

(iii) Let $\vartheta = \{\{1, 3\}, \{5, 7\}\}$ be a ℓ- HX group of G, $P = \{1, 3\}$ and $Q = \{5, 7\}$.

.8	P	Q
P	P	Q
Q	Q	P

$\wedge$	P	Q
P	P	P
Q	P	Q

$\vee$	P	Q
P	P	Q
Q	Q	Q

Define $(\rho^{\alpha})^{+}(P) = \vee\{\mu^{+}(m)/$ for all $m \in P \subseteq G\}$

$(\rho^{\alpha})^{-}(P) = \wedge\{\mu^{-}(m)/$ for all $m \in P \subseteq G\}$

$(\omega^{\beta})^{+}(P) = \vee\{\beta^{+}(m)/$ for all $m \in P \subseteq G\}$ and

$(\omega^{\beta})^{-}(P) = \wedge\{\beta^{-}(m)/$ for all $m \in P \subseteq G\}$.

Now

$(\rho^{\alpha})^{+}(P) = 0.3$	$(\rho^{\alpha})^{-}(P) = -0.8$	$(\omega^{\beta})^{+}(P) = 0.8$	$(\omega^{\beta})^{-}(P) = -0.9$
$(\rho^{\alpha})^{+}(Q) = 0.4$	$(\rho^{\alpha})^{-}(Q) = -0.6$	$(\omega^{\beta})^{+}(Q) = 0.3$	$(\omega^{\beta})^{-}(Q) = -0.5$

Clearly, ρ^{α} and ω^{β} is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

(iv) The union of ρ^{α} and ω^{β} is calculated as,

$(\rho^{\alpha} \cup \omega^{\beta})^{+}(P) = 0.5$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(P) = -0.7$
$(\rho^{\alpha} \cup \omega^{\beta})^{+}(Q) = 0.8$	$(\rho^{\alpha} \cup \omega^{\beta})^{-}(Q) = -0.5$

Clearly, $\rho^{\alpha} \cup \omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

(v) Let $\Omega^{\alpha\cup\beta} = ((\Omega^{\alpha\cup\beta})^{+}, (\Omega^{\alpha\cup\beta})^{-})$ be a bipolar L-fuzzy set of ϑ and the mappings $(\Omega^{\alpha\cup\beta})^{+} : \vartheta \to L$, $(\Omega^{\alpha\cup\beta})^{-} : \vartheta \to L$ are defined as

$$(\Omega^{\alpha\cup\beta})^{+}(P) = \vee\{(\alpha \cup \beta)^{+}(P)/m \in P \subseteq G\}$$

$$(\Omega^{\alpha\cup\beta})^{-}(P) = \wedge\{(\alpha \cup \beta)^{-}(P)/m \in P \subseteq G\}$$

Now

$(\Omega^{\alpha\cup\beta})^{+}(P) = 0.5$	$(\Omega^{\alpha\cup\beta})^{-}(P) = -0.7$
$(\Omega^{\alpha\cup\beta})^{+}(Q) = 0.8$	$(\Omega^{\alpha\cup\beta})^{-}(Q) = -0.5$

Clearly, $\Omega^{\alpha\cup\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ϑ.

(vi) Let $\Omega^{\alpha\cup\beta} = ((\Omega^{\alpha\cup\beta})^{+}, (\Omega^{\alpha\cup\beta})^{-})$ be a bipolar L-fuzzy sub ℓ-HX group of ϑ induced by $\alpha \cup \beta$ of sub ℓ-group of G.

where $(\Omega^{\alpha\cup\beta})^{+}(P) = \vee\{(\alpha \cup \beta)^{+}(P)/m \in P \subseteq G\}$

$$(\Omega^{\alpha\cup\beta})^{-}(P) = \wedge\{(\alpha \cup \beta)^{-}(P)/m \in P \subseteq G\}$$

$(\Omega^{\alpha\cup\beta})^{+}(P) = 0.5$	$(\Omega^{\alpha\cup\beta})^{-}(P) = -0.7$
$(\Omega^{\alpha\cup\beta})^{+}(Q) = 0.8$	$(\Omega^{\alpha\cup\beta})^{-}(Q) = -0.5$

Therefore, $\Omega^{\alpha\cup\beta} = \rho^{\alpha} \cup \omega^{\beta}$.

Hence, by the conditions (i), (ii), (iii), (iv), (v) and (vi), $\Omega^{\alpha\cup\beta} = \rho^{\alpha} \cup \omega^{\beta}$ is proved.

5. Conclusion

In this paper, we have presented some properties and examples for union and intersection of bipolar L-fuzzy sub ℓ-groups and bipolar L-fuzzy sub ℓ-HX groups.

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