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A THEOREM ON GENERALIZATION ABSOLUTE MATRIX SUMMABILITY OF FACTORED INFINITE SERIES

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Abstract

Sulaiman [16] has obtained a theorem dealing with weighted mean summability factors of infinite series. In this paper, we have generalized this theorem to $|A, p_n, \delta|_k$ summability method for infinite series.

1. Introduction

Let $\sum a_n$ be a given infinite series with (s_n) as the sequence of its n-th partial sums. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation $(s_n) \to (T_n)$ with

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (T_n) of the weighted arithmetic mean or simply the $(\overline{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [12]).

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The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty.$$

$|\bar{N}, p_n|_1$ is the same as $|\bar{N}, p_n|$.

2. Known Results

Given the series $\sum a_n$, if a series $\sum a_n \varepsilon_n$ is summable in some sense while, in general, $\sum a_n$ is itself not so summable, then (ε_n) is said to be a summability factor of the series $\sum a_n$. If the summability in question is absolute, the factors are naturally called absolute summability factors. Given any sequences $(x_n), (y_n)$, it is customary to write $y_n = O(x_n)$, if there exist κ and $N, |y_n| \leq k|x_n|$, for every $n > N$ (see [12]). In [2]-[10], [15],[18] some results have been studied dealing with absolute summability of factored infinite series, and also Sulaiman [16] has obtained the following theorem for $|\bar{N}, p_n|_k$ summability method.

Theorem 2.1 *Let* t_n *be the* $n-th$ *Cesàro mean of first order of the sequence* $\{na_n\}$. *Let* $\{\varepsilon_n\}$ *be a sequence of constants. If*

$$\sum_{n=1}^{m} n^{-1} |\varepsilon_n|^k |t_n|^k = O(1) \quad m \to \infty, \tag{1}$$

$$\sum_{n=1}^{m} p_n P_n^{-1} |\varepsilon_n|^k |t_n|^k = O(1) \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} n^{k-1} |\Delta\varepsilon_n|^k |t_n|^k = O(1) \quad m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} n^{-1} P_n = O(P_m) \quad m \to \infty, \tag{4}$$

holds, then the series $\sum a_n \varepsilon_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

3. Main Results

Given a normal matrix $A = (a_{nv})$, i.e., a lower triangular matrix of nonzero diagonal entries. A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \ \ n = 0, 1, \cdots$$

We use two lower semimatrices

$\overline{A} = (\overline{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \ \ n, v = 0, 1, \cdots$$

and

$$\hat{a}_{00} = \overline{a}_{00} = a_{00}, \ \hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1, v}, \ \ n = 1, 2, \cdots$$

It may be noted that $\overline{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively.

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v \sum_{v=0}^{n} \overline{a}_{nv} a_v \tag{5}$$

and

$$A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{6}$$

The series $\sum a_n$ is said to be summable $|A, p_n, \delta|_k$, $k \geq 1$ and $\delta \geq 0$, (see [11]) if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

In the special case, if we take $\delta = 0$, then we have $|A, p_n|_k$ summability method (see [17]). Also, if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\overline{N}, p_n|_k$ summability method.

Sulaiman obtained Theorem 2.1 dealing with $|\overline{N}, p_n|_k$ summability method by using weighted mean matrix. We have studied this theorem for $|A, p_n|_k$ summability method in (see [14]). In this paper we extend Theorem 2.1 to the $|A, p_n, \delta|_k$ summability of the series $\sum a_n \varepsilon_n$. $|A, p_n, \delta|_k$ summability method is more general than $|A, p_n|_k$ and $|\overline{N}, p_n|_k$ summability methods. When we extend Theorem 2.1 to the following theorem, we use normal matrices instead of weighted mean matrices.

Theorem 3.1. *Let* $A = (a_{nv})$ *be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \quad n = 0, 1, \cdots,$$

$$a_{n-1, v} \geq a_{nv}, \quad \text{for } n \geq v + 1,$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{7}$$

$$\sum_{v=1}^{n-1} \frac{|\hat{a}_{n, v+1}|}{v} = O(a_{nn}), \quad \sum_{v=1}^{n-1} \frac{|\hat{a}_{n, v}|}{v} = O(a_{nn}), \tag{8}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n, v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \ m \to \infty, \tag{9}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left(a_{vv} \left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \ m \to \infty, \tag{10}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} n^{-1} |\varepsilon_n|^k |t_n|^k = O(1) \ as \ m \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} a_{nn} |\varepsilon_n|^k |t_n|^k = O(1) \ as \ m \to \infty, \tag{12}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} n^{k-1} |\Delta\varepsilon_n|^k |t_n|^k = O(1) \ as \ m \to \infty. \tag{13}$$

Let t_n *be the* $n-th$ *Cesàro mean of first order of the sequence* $\{na_n\}$, *that is*

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v,$$

and $\{\varepsilon_n\}$ *be a sequence of constants. Then the series* $\sum a_n\varepsilon_n$ *is summable* $|A, p_n, \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < \frac{1}{k}$.

In general, in the proof of theorems dealing with absolute matrix summability, we use Minkowski, Hölder's inequalities and Abel transform.

Proof of Theorem 3.1. Let (Q_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n\varepsilon_n$. Then, by equations (5) and (6), we have

$$Q_n(s) - Q_{n-1}(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v\varepsilon_v = \sum_{v=1}^{n} v^{-1}\hat{a}_{nv}va_v\varepsilon_v.$$

Applying Abel's transformation (see [13]) to

$$\sum_{v=1}^{n} v^{-1}\hat{a}_{nv}va_v\varepsilon_v,$$

we have that

$$\sum_{v=1}^{n} v^{-1}\hat{a}_{nv}va_v\varepsilon_v = \sum_{v=1}^{n-1} \Delta_v\left(\hat{a}_{nv}\varepsilon_v v^{-1}\right)\sum_{r=1}^{v} ra_r + \frac{\hat{a}_{nn}\varepsilon_n}{n}\sum_{r=1}^{n} ra_r$$

$$= \sum_{v=1}^{n-1} (v+1)t_v\left(v^{-1}(v+1)^{-1}\hat{a}_{nv}\varepsilon_v + (v+1)^{-1}\Delta_v(\hat{a}_{nv})\varepsilon_v + (v+1)^{-1}\hat{a}_{n,v+1}\Delta_{\varepsilon_v}\right)$$

$$+ \frac{n+1}{n}a_{nn}\varepsilon_n t_n$$

$$= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\varepsilon_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,\, v+1}\Delta\varepsilon_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,\, v}\varepsilon_v\frac{t_v}{v} + a_{nn}\varepsilon_n t_n\frac{n+1}{n}$$

$$= Q_{n,\,1} + Q_{n,\,2} + Q_{n,\,3} + Q_{n,\,4}.$$

To complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|Q_{n,\,r}\right|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.$$

First, by applying Hölder's inequality (see [13]) with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, and by using the conditions (7), (10) and (12), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|Q_{n,\,1}| \le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\varepsilon_v||t_v|\right\}^k$$

$$\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\varepsilon_v|^k|t_v|^k \times \left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\varepsilon_v|^k|t_v|^k\right\}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\varepsilon_v|^k|t_v|^k\right\}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\varepsilon_v|^k|t_v|^k\right\}$$

$$= O(1)\sum_{v=1}^{m}|\varepsilon_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k} a_{vv}|\varepsilon_v|^k|t_v|^k = O(1) \text{ as } m \to \infty.$$

Then by using the conditions (7)-(9) and (13), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|Q_{n,\,2}|^k \le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,\,v+1}||\Delta\varepsilon_v||t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,\,v+1}|}{v}\,v\,|\Delta\varepsilon_v||t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,\,v+1}||\,\Delta\varepsilon_v\,|^k|\,t_v\,|^k v^{k-1}\times\left\{\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,\,v+1}|}{v}\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}(a_{nn})^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,\,v+1}||\,\Delta\varepsilon_v\,|^k|\,t_v\,|^k v^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,\,v+1}||\,\Delta\varepsilon_v\,|^k|\,t_v\,|^k v^{k-1}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\varepsilon_v|^k|\,t_v\,|^k v^{k-1}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,\,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\,\Delta\varepsilon_v\,|^k|\,t_v\,|^k v^{k-1} = O(1)\ \text{ as } m\to\infty,$$

and also by virtue of hypotheses of Theorem 3.1, and the conditions (7)-(9) and (11), we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|Q_{n,\,3}| \le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,\,v}\varepsilon_v\,\frac{t_v}{v}\right|^k$$

$$\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,\,v}||\,\varepsilon_v\,|\frac{|\,t_v\,|}{v}\right\}^k$$

$$\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,\,v}||\,\varepsilon_v\,|^k\,\frac{|\,t_v\,|^k}{v}\times\left\{\sum_{v=1}^{n-1}\ \frac{|\hat{a}_{n,\,v}|}{v}\ |\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}(a_{nn})^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,\,v}||\,\varepsilon_v\,|^k\ \frac{|\,t_v\,|^k}{v}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,\,v}||\,\varepsilon_v\,|^k\ \frac{|\,t_v\,|^k}{v}$$

$$= O(1)\sum_{v=1}^{m}|\,\varepsilon_v\,|^k\,\frac{|\,t_v\,|^k}{v}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,\,v}|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\,\varepsilon_v\,|^k\,\frac{|\,t_v\,|^k}{v} = O(1)\ \text{ as } m\to\infty.$$

Finally, by using the conditions (7) and (12), we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|Q_{n,\,4}|^k = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}(a_{nn})^{k-1}(a_{nn})\,|\,\varepsilon_n\,|^k|\,t_n\,|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}a_{nn}|\,\varepsilon_n\,|^k|\,t_n\,|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}a_{nn}|\,\varepsilon_n\,|^k|\,t_n\,|^k = O(1)\ \text{ as } m\to\infty.$$

This completes the proof of Theorem 3.1.

4. Conclusions

If we apply Theorem 3.1 to the weighted mean, then $A = (a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$, $v = 0, 1, \cdots, n;\ n = 0, 1, 2, \cdots,$ when $0 \le v \le n$, where $P_n = p_0 + p_1 + p_2 + \cdots + p_n$, it is clear that $\hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}$, and by taking $a_{nv} = \frac{p_v}{P_n}$, the condition (8) reduces to the condition (4) as follows:

$$\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,\,v+1}|}{v} = \sum_{v=1}^{n-1}\frac{1}{v}P_v\Big(\frac{p_n}{P_{n-1}P_n}\Big) = \Big(\frac{p_n}{P_{n-1}P_n}\Big)\sum_{v=1}^{n-1}\frac{P_v}{v} = O(a_{nn}).$$

It should be noted that if we take $\delta = 0$ and $a_{nv} = \dfrac{p_v}{P_n}$, the conditions (11), (12) and (13) reduce to the conditions (1), (2) and (3) respectively.

1. If we take $\delta = 0$ and $a_{nv} = \dfrac{p_v}{P_n}$ in Theorem 3.1, then we have Theorem 2.1 dealing with $|\overline{N}, p_n|_k$ summability method.

2. If we take $\delta = 0$ in Theorem 3.1, we have a result for $|A, p_n|_k$ summability method.

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