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APPROXIMATING FIXED POINTS OF GENERALIZED α-NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract

In this paper, we transform M iteration process in the setting of CAT(0) space to approximate fixed point of generalized α-nonexpansive mappings and then, we also obtain some convergence results in CAT(0) space. Our results are the improvement, extension and generalization of many recent results in the literature of fixed point theory in CAT(0) space.

1. Introduction

Ullah and Arshad [7] introduced a new three step iteration process known as "M iteration process", described as.

$$x_0 \in K$$

$$Z_n = (1 - \alpha_n)x_n + \alpha_n Tx_n$$

$$y_n = Tz_n$$

$$x_{n+1} = Ty_n$$

where $n \geq 0$, $\{\alpha_n\}$ is a real sequence in $[0, 1]$ and K is a nonempty subset of a Banach space.

We transform the M iteration process in the setting of CAT(0) space described as.

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$$x_0 \in K$$

$$Z_n = (1-\alpha_n)x_n \oplus \alpha_n T x_n$$

$$y_n = Tz_n$$

$$x_{n+1} = T y_n,$$

where $n \geq 0$, $\{\alpha_n\}$ is a real sequence in [0, 1] and K is a nonempty subset of a CAT(0) space E.

In 2008, Suzuki [6] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). Let K be a nonempty subset of a CAT(0) space E, a mapping $T : K \to K$ is said to satisfy condition (C) if

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) < d(x, y) \text{ for all } x, y \in K.$$

It is obvious that, every mapping satisfying condition(C) with a fixed point is quasi-nonexpansive mapping. Recently, Aoyoma and Kohsaka [1] introduced the class of α-nonexpansive mappings, a mapping $T : K \to K$ is said to be α-nonexpansive if there exists an $\alpha \in [0, 1]$ such that for all $x, y \in K$,

$$d(Tx, Ty)^2 \leq ad(Tx, y)^2 + \alpha d(x, Ty)^2 + (1-2\alpha)d(x, y).$$

Very recently, Pant and Shukla [4] introduced the class of generalized α-non expansive mapping which contains the mapping satisfying condition (C), a mapping $T : K \to K$ is said to be generalized α-nonexpansive if there exists an $\alpha \in [0, 1]$ such that for all $x, y \in K$,

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(1-2\alpha) d(x, y).$$

Our results are generalizations of recent well-known results of Pant and Shukla [4], Piri et al. [5] and many others.

2. Preliminaries

For details as regards CAT(0) spaces please see [2]. Some results are recalled here for CAT(0) space E.

Lemma 2.1. *Let K be a nonempty subset of a $CAT(0)$ space E and $T: K \to K$ a generalized α-nonexpansive mapping. Then, $F(T)$ is closed.*

Proof. Let $\{Z_n\}$ be a sequence in $F(T)$ such that $\{Z_n\}$ converges to a point $z \in K$.

Since $\frac{1}{2} d(z_n, Tz_n) = 0 = d(z_n, z)$ by definition of generalized α-nonexpansive mapping and continuity of metric on E, we have

$$\begin{aligned}\lim_{n\to\infty} d(z_n, Tz) &= \lim_{n\to\infty} d(Tz_n, Tz) \\ &\leq \lim_{n\to\infty}[\alpha d(Tz_n, Z) + \alpha d(z_n, Tz) + (1 - 2\alpha) d(z_n, z)] \\ &= \lim_{n\to\infty}[\alpha d(Tz, Z_n) + (1-\alpha) d(z_n, z)] \\ &\leq \alpha \lim_{n\to\infty} d(Tz, z_n) + (1-\alpha) \lim_{n\to\infty} d(z_n, z).\end{aligned}$$

Since $(1-\alpha) > 0$, the above inequality reduces to

$$\lim_{n\to\infty} d(z_n, Tz) \leq \lim_{n\to\infty} d(z_n, z).$$

And $T(z) = z$. Therefore $F(T)$ is closed.

Lemma 2.2. *Let K be a nonempty subset of a $CAT(0)$ space E and $T: K \to K$ a generalized α-nonexpansive mapping with a fixed point $p \in K$. Then T is quasi-nonexpansive.*

Proof. Let $p \in F(T)$ and $x \in K$.

Since $\frac{1}{2} d(p, Tp) = 0 \leq d(x, p)$,

$$d(Tx, p) = d(Tx, Tp) \leq \alpha d(Tx, p) + \alpha d(Tp, x) + (1-2\alpha) d(x, p).$$

This implies that

$$(1-\alpha) d(Tx, p) \leq (1-\alpha) d(x, p).$$

Since $(1-\alpha) > 0$, we get $d(Tx, y) \leq d(x, y)$.

Lemma 2.3 *Let K be a nonempty subset of a $CAT(0)$ space E and $T: K \to K$ a generalized α-nonexpansive mapping. Then, for all $x, y \in K$:*

i. $d(Tx, T^2x) \leq d(x, Tx)$.

ii. *Either* $\frac{1}{2}d(x, Tx) \leq d(x, y)$ or $\frac{1}{2}d(Tx, T^2x) \leq d(Tx, y)$.

iii. *Either* $d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(x, Ty) + (1 - 2\alpha)d(x, y)$ or $d(T^2x, Ty) \leq \alpha d(Tx, Ty) + \alpha d(T^2x, y) + (1 - 2\alpha)d(Tx, y)$.

Proof. Since

$$\frac{1}{2}d(x, Tx) \leq d(x, Tx).$$

By using the definition of generalized α-nonexpansive mapping, we have

$$d(Tx, T^2x) \leq d(x, Tx).$$

To prove (ii), arguing by contradiction, we suppose that

$$\frac{1}{2}d(x, Tx) > d(x, y) \text{ and } \frac{1}{2}d(Tx, T^2x) > d(Tx, y)$$

By (i), and the triangle inequality, we get

$$\begin{aligned} d(x, Tx) &\leq d(x, y) + d(Tx, y) \\ &\leq \frac{1}{2}d(x, Tx) + \frac{1}{2}d(Tx, T^2x) \\ &\leq d(x, Tx), \end{aligned}$$

which is a contradiction. Thus (ii) holds. The condition (iii) directly follows from (ii).

Lemma 2.4. *Let K be a nonempty subset of a $CAT(0)$ space E and $T : K \to K$ a generalized α-nonexpansive mapping. Then, for all $x, y \in K$*

$$d(x, Ty) \leq \frac{3 + \alpha}{1 - \alpha}d(x, Tx) + d(x, y).$$

Proof. From Lemma 2.3, we have for all $x, y \in K$, either

$$d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(x, Ty) + (1 - 2\alpha)d(x, y)$$

or

$$d(T^2x,\ Ty) \leq \alpha d(Tx,\ Ty) + \alpha d(T^2x,\ y) + (1 - 2\alpha)d(Tx,\ y).$$

In the first case, we have

$$\begin{aligned} d(x,\ Ty) &\leq d(x,\ Tx) + d(Tx,\ Ty) \\ &\leq d(x,\ Tx) + \alpha d(Tx,\ y) + \alpha d(Ty,\ x) + (1 - 2\alpha)d(x,\ y) \\ &\leq d(x,\ Tx) + \alpha d(Tx,\ x) + \alpha d(x,\ y) + \alpha d(Ty,\ x) + (1 - 2a)d(x,\ y). \end{aligned}$$

This implies that

$$d(x,\ Ty) \leq \frac{(1+\alpha)}{(1-\alpha)} d(Tx,\ x) + d(x,\ y)$$

In the other case, we have

$$\begin{aligned} d(x,\ Ty) &\leq d(x,\ Tx) + d(Tx,\ T^2x) + d(T^2x,\ Ty) \\ &\leq 2d(x,\ Tx) + \alpha d(Tx,\ Ty) + \alpha d(T^2x,\ y) + (1 - 2\alpha)d(Tx,\ y) \\ &\leq 2d(x,\ Tx) + \alpha d(Tx,\ x) + \alpha d(Ty,\ x) + \alpha d(T^2x,\ Tx) + \alpha d(Tx,\ y) \\ &+ (1 - 2\alpha)d(Tx,\ y) \\ &\leq (2 + \alpha)d(x,\ T) + \alpha d(Ty,\ x) + \alpha d(x,\ Tx) + (1 - \alpha)d(Tx,\ y) \\ &\leq (2+\alpha)d(x, Tx) + \alpha d(Ty, x) + \alpha d(x, Tx) + (1-\alpha)d(Tx, y) + (1-\alpha)d(x, y). \end{aligned}$$

This implies that

$$d(x,\ Ty) \leq \frac{(3+\alpha)}{(1-\alpha)} d(x,\ Tx) + d(x,\ y).$$

Therefore in the both the cases, we get the desired result.

Lemma 2.5 [3]**.** *Suppose that E is a complete $CAT(0)$ space and $x \in E$. $\{\alpha_n\}$ is a sequence in $[b,\ c]$ for some $b,\ c \in (0,\ 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in E such that, for some $r \geq 0$, we have*

$$\lim_{n\to\infty} \sup d(x_n,\ x) \leq r,\ \lim_{n\to\infty} \sup d(y_n,\ x) \leq r$$

and

$$\lim_{n\to\infty} \sup d(\alpha_n x_n \oplus (1-\alpha_n) y_n, x) \le r;$$

Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

3. Convergence Results for Generalized α-nonexpansive Mappings

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space E and $T : K \to K$ a generalized α-nonexpansive mapping. Suppose that the sequence $\{x_n\}$ be defined by M iteration process, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. By lemma 2.2, T is quasi-nonexpansive, so

$$\begin{aligned} d(z_n, p) &= d((1-\alpha_n)x_n \oplus \alpha_n Tx_n, p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(Tx_n, p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ d(z_n, p) &\le d(x_n, p). \end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned} d(y_n, p) &= d(Tz_n, p) \\ &\le d(z_n, p) \\ d(y_n, p) &\le d(x_n, p). \end{aligned} \tag{3.2}$$

From (3.2), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(Ty_n, p) \\ &\le d(y_n, p) \\ &\le d(x_n, p). \end{aligned}$$

This implies that $\{d(x_n, p)\}$ is bounded and non-increasing. Hence $\lim_{n\to\infty} d(x_n, p)$ exists.

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete*

CAT(0) space E and $T: K \to K$ a generalized α-nonexpansive mapping. Suppose that the sequence $\{x_n\}$ be defined by M iteration process. Then $F(T) \neq \phi$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. Suppose that $F(T) \neq \phi$ and $p \in F(T)$. Then by lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n, p) = k.$$

By the proof of lemma 3.1, we have

$$\limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = k. \tag{3.3}$$

By using lemma 2.2, we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = k. \tag{3.4}$$

Also, we have

$$d(z_n, p) \leq d(x_n, p)$$

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = k \tag{3.5}$$

and

$$\begin{aligned} d(x_{n+1}, p) = d(Ty_n, p) &\leq d(y_n, p) \\ &\leq d(z_n, p). \end{aligned}$$

Therefore,

$$r \leq \liminf_{n\to\infty} d(z_n, p). \tag{3.6}$$

By using (3.5) and (3.6), we get

$$r = \lim_{n\to\infty} d(z_n, p)$$

$$r = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n Tx_n, p). \tag{3.7}$$

Hence, by (3.3), (3.4), (3.7) and lemma (2.5), we obtain

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Let $p \in A(K, \{x_n\})$.

By lemma 2.4, we have

$$r(Tp, \{x_n\}) = \limsup_{n\to\infty} d(x_n, Tp)$$

$$\leq \left(\frac{3+\alpha}{1-\alpha}\right)\limsup_{n\to\infty} d(Tx_n, x_n) + \limsup_{n\to\infty} d(x_n, p)$$

$$= \limsup_{n\to\infty} d(x_n, p) = r(p, \{x_n\}).$$

Hence, we conclude that $Tp \in A(K, \{x_n\})$. Since E is CAT(0) space, $A(K, \{x_n\})$ is singleton. Thus, we have $Tp = p$.

Now we are in the position to prove strong convergence theorem.

Theorem 3.3. *Let K, E, T and $\{x_n\}$ be as in Theorem* 3.2. *Suppose that $F(T) \neq \phi$ and $\liminf_{n\to\infty} d(x_n F(T)) = 0$ where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists, for all $p \in F(T)$. So, $\lim_{n\to\infty} d(x_n, F(T))$ exists, thus

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\{z_j\}$ in $F(T)$ such that $d(x_{n_j}, z_j) \leq \frac{1}{2^j}$ for all $j \in N$.

By the proof of Lemma 3.1, $\{x_n\}$ is decreasing. So, $d(x_{n_{j+1}}, z_j) \leq d(x_{n_j}, z_j) \leq \frac{1}{2^j}$.

Therefore,

$$d(z_{j+1}, z_j) \leq d(z_{j+1}, x_{n_{j+1}}) + d(x_{n_{j+1}}, z_j)$$

$$\leq \frac{1}{2^{j+1}} + \frac{1}{2^j}$$

$$\leq \frac{1}{2^{j-1}} \to 0,\ j \to \infty.$$

Hence, we conclude that $\{z_j\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since, by Lemma 2.1, $F(T)$ is closed, then $p \in F(T)$. So, $\{x_{n_j}\}$ converges strongly to $p \in F(T)$ and since $\lim_{n\to\infty} d(x_n, p)$ exists, hence $\{x_n\}$ converges strongly to $p \in F(T)$.

4. Conclusions

The extension of the linear version of convergence results to nonlinear spaces has its own importance. Here we extend a linear version of convergence results to the fixed point of a generalized α-nonexpansive mapping for the already defined M iteration method [7] in the setting of Banach space to nonlinear CAT(0) spaces.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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