Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 883-905
© 2021 Mili Publications, India

INTUITIONISTIC TRIANGULAR FUZZY THREE-DIMENSIONAL NUMBERS AND ITS APPLICATION TO MULTI-CRITERIA DECISION-MAKING PROBLEM

THANGARAJ BEAULA and R. SELVAKUMARI

PG and Research
Department of Mathematics
T. B. M. L. College, Affiliated to Bharathidasan
University Tiruchirappalli
Porayar, Tamil Nadu, India

Department of Mathematics
D. S. E. College, Affiliated to Bharathidasan
University Tiruchirappalli
Perambalur, Tamil Nadu, India

Abstract

Intuitionistic triangular fuzzy three dimensional number (ITrFTD-number) is a special type of intuitionistic fuzzy number ona real number set. In the ITrFTD-number, criteria behind is the chance of the same or the different membership and non-membership values. In this paper, ITrFTD-numbers are defined based on multiple criteria decision making problems in which the inputs are considered as ITrFTD-numbers. Operational using *t*-norm, *t*-conorm is defined aggregation operators on ITrFTD-numbers are developed. The ranking order is defined correspondingly to the similarity with respect to the positive ideal solution.

1. Introduction

Atanassov [1] (1986) first introduced the theory of intuitionistic fuzzy set in order to study uncertainty. Many researches have been undergone on operations on intuitionistic fuzzy sets [2, 10], multi-criteria decision-making method on intuitionistic fuzzy numbers [11, 12], intuitionistic fuzzy

---

2020 Mathematics Subject Classification: 03E72.

Keywords: Intuitionistic triangular fuzzy three-dimensional numbers, Multi-criteria decision-making problem.

Received October 25, 2021; Accepted November 10, 2021

information aggregation [14, 5], new distance between triangular intuitionistic fuzzy numbers [4] distance and similarity measures of intuitionistic fuzzy multi set [6]. Yager [13] indicated that fuzzy multiset occurs more than once with the same or different membership values. Shinoj and John [7, 8], Shinoj and Sunil [9] developed intuitionistic fuzzy multi set and its applications as a brand-new research area. Ejegwa [3] presented mathematical techniques to transform intuitionistic fuzzy multisets to fuzzy sets.

In this paper intuitionistic triangular fuzzy three-dimensional number (ITrFTD) is introduced and aggregation operators are formulated using weighted harmonic mean and weighted arithmetic mean. Using the aggregation operator on ITrFTD-numbers an algorithm to solve multi-criteria decision making problem is proposed further illustrated by a numerical example.

2. Preliminaries

In this section some basic concepts related to fuzzy set, multi-fuzzy set, intuitionistic fuzzy set, intuitionistic fuzzy multi set and intuitionistic fuzzy numbers are defined.

Definition 1. Let X be a universal set. A fuzzy set A over X is defined as $\widetilde{A} = \{\mu_{\widetilde{A}}(x) / x \in X\}$ where $\mu_{\widetilde{A}}$ is the membership function defined from X to [0, 1]. For each $x \in X$, the value of $\mu_{\widetilde{A}}(x)$ represents the degree of x belonging to the fuzzy set.

Definition 2. Fuzzy intersection t-norms is a binary function t that maps $[0, 1] \times [0, 1]$ to $[0, 1]$ satisfying the following conditions

Conditions:

(i) $t(0, 0) = 0$ and $t(a, 1) = t(1, a) = a$, for all $a \in [0, 1]$ (boundary condition)

(ii) If $a_1 \leq a_2$ and $b_1 \leq b_2$ then $t(a_1, b_1) \leq t(a_2, b_2)$ (monotonicity)

(iii) $t(a, b) = t(b, a)$ (commutative)

(iv) $t(a, t(b, c)) = t(t(a, b), c)$, for all $a, b, c \in X$ (associative)

Definition 3. s-norms is a binary function s that maps $[0, 1] \times [0, 1]$ to $[0, 1]$ satisfying associative, monotonic and commutative with properties.

Conditions:

(i) $s(1, 1) = 1$ and $s(a, 0) = s(0, a) = a$, for all $a \in [0, 1]$ (boundary condition)

(ii) If $a_1 \leq a_2$ and $b_1 \leq b_2$ then $s(a_1, b_1) \leq s(a_2, b_2)$ (monotonicity)

(iii) $s(a, b) = s(b, a)$ (commutative)

(iv) $s(a, s(b, c)) = s(s(a, b), c)$, for all $a, b, c \in X$ (associative)

Definition 2.4. Let X be a non-empty set, a three dimensional fuzzy set $\widetilde{A}$ on X is defined as $\widetilde{A} = \{(x, t_1, t_2, t_3) : x \in X\}$ where $t_i \in [0, 1]$ for $i = 1, 2, 3$; such that $t_1 \geq t_2 \geq t_3$.

Definition 5. Let X be a nonempty set. An intuitionistic fuzzy set (IFS) $\widetilde{A}$ is of the form $\widetilde{A} = \{(x, t, s) : x \in X\}$, where the function $t, s \in [0, 1]$ defines the degree of membership and the degree of nonmember ship of the element $x \in X$ to the set $\widetilde{A}$ with $0 \leq t + s \leq 1$.

Definition 6. An intuitionistic triangular fuzzy number, $\widetilde{A}$ is defined by its membership function $\eta_{\widetilde{A}}$ and non-membership function $\gamma_{\widetilde{A}}$ as

$$t(x) = \begin{cases} \frac{(x-a)}{(b-a)} \eta_{\widetilde{A}}, & a \leq x \leq b \\ \frac{(c-x)}{(c-b)} \eta_{\widetilde{A}}, & b < x \leq c \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

and

$$s(x) = \begin{cases} \dfrac{(b-x)+\gamma_{\tilde{A}}(x-a_1)}{(b-a_1)}, & a_1 \le x \le b \\ \dfrac{(x-b)+\gamma_{\tilde{A}}(c_1-x)}{(c_1-b)}, & b < x \le c_1 \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

where $0 \le t \le 1$ and $0 \le s \le 1$ with $t + s \le 1$

$\tilde{A} = \langle([a, b, c]; t), ([a_1, b, c_1]; s)\rangle$ and represented as generally denoted as $\tilde{A} = \langle[a, b, c]; t, s\rangle$.

Definition 7. Let $\tilde{A} = \langle[a_1, b_1, c_1]; t_1, s_1\rangle$ and $\tilde{B} = \langle[a_2, b_2, c_2]; t_2', s_2'\rangle$ be two intuitionitic triangular fuzzy numbers and $k \ge 0$ then

(i) $\tilde{A} + \tilde{B} = ((a_1 + a_2, b_1 + b_2, c_1 + c_2), 1 - \dfrac{(1-t_1)(1-t_2')}{\max(1-t_1, 1-t_2', k)},$

$\dfrac{s_1 s_2'}{\max(s_1, s_2', k)})$

(ii) $\tilde{A} \cdot \tilde{B} = ((a_1a_2, b_1b_2, c_1c_2), \dfrac{t_1t_2'}{\max(t_1, t_2', k)}, 1 - \dfrac{(1-s_1)(1-s_2')}{\max(1-s_1, 1-s_2', k)})$

(iii) $k\tilde{A} = ([ka_1, kb_1, kc_1], 1 - \dfrac{1}{kt_1}, \dfrac{1}{ks_1})$

(iv) $\tilde{A}^k = ([a_1^k, b_1^k, c_1^k], 1 - \dfrac{1}{t_1^k}, \dfrac{1}{s_1^k})$

Definition 8. Let $\tilde{A} = \langle[a_1, b_1, c_1]; t_1\rangle$, $\tilde{B} = \langle[a_2, b_2, c_2]; t_2\rangle$, $0 \le a_1 \le b_1 \le c_1 \le 1$, $0 \le a_2 \le b_2 \le c_2 \le 1$ then the degree of similarity $S(\tilde{A}, \tilde{B})$ between the generalized triangular fuzzy numbers $P(A)$ and $P(B)$ is calculated as

$$S(\tilde{A}, \tilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$$

$$\left.\left(\frac{\min\{P(A), P(B)\} + \min\{t_1, t_2\}}{\max\{P(A), P(B)\} + \max\{t_1, t_2\}}\right)\right]$$

where $S(\widetilde{A}, \widetilde{B}) \in [0, 1]$; $P(\widetilde{A})$ and $P(\widetilde{B})$ are defined as follows:

$$P(\widetilde{A}) = \sqrt{(a_1 - b_1)^2 + (t_1)^2} + \sqrt{(c_1 - b_1)^2 + (t_1)^2} + (c_1 - a_1),$$

$$P(\widetilde{B}) = \sqrt{(a_2 - b_2)^2 + (t_2)^2} + \sqrt{(c_2 - b_2)^2 + (t_2)^2} + (c_2 - a_2)$$

$P(\widetilde{A})$ and $P(\widetilde{B})$ denote the perimeters of the generalized triangular fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ respectively.

3. Intuitionistic Triangular Fuzzy Three Dimensional Number

Definition 1. Let X be a non empty set. An intuitionistic triangular fuzzy three dimensional set $\widetilde{A}$ (ITrFTD) on X is of the form $\widetilde{A} = \{(x : (t_1, t_2, t_3), (s_1, s_2, s_3)) : x \in X\}$ where the function $t_i, s_i \in [0, 1], i = 1, 2, 3$ defines the degree of membership and the degree of non membership with $0 \leq t_i + s_i \leq 1$ subject to $t_1 \geq t_2 \geq t_3$, $s_1 \leq s_2 \leq s_3$.

Definition 2. Let $t_i, s_i \in [0, 1], i = 1, 2, 3$ and $a, b, c \in \mathbb{R}$ with $a \leq b \leq c$. An intuitionistic triangular fuzzy three dimensional number (ITrFTD number) $\widetilde{A} = \langle([a, b, c]; (t_1, t_2, t_3), (s_1, s_2, s_3))\rangle$ is a special intuitionistic fuzzy multi set on the real number set $\mathbb{R}$, whose membership functions $\eta^i_{\widetilde{A}}$ and non-membership functions $\gamma^i_{\widetilde{A}}$ for $i = 1, 2, 3$ are defined as

$$t_i = \begin{cases} \dfrac{(x-a)}{(b-a)} \eta^i_{\widetilde{A}}, & a \leq x \leq b \\ \dfrac{(c-x)}{(c-b)} \eta^i_{\widetilde{A}}, & b < x \leq c \\ 0, & \text{otherwise} \end{cases}$$

and

$$s_i = \begin{cases} \dfrac{(b-x)+\gamma^i_{\tilde{A}}(x-a_1)}{(b-a_1)}, & a_1 \le x \le b \\ \dfrac{(x-b)+\gamma^i_{\tilde{A}}(c_1-x)}{(c_1-b)}, & b < x \le c_1 \\ 1, & \text{otherwise} \end{cases}$$

Example 3. Define ITrFTD number function with its membership and non membership function as

$$t_1 = \begin{cases} \dfrac{(x-1)}{3}(0.4), & 1 \le x \le 4 \\ \dfrac{(6-x)}{2}(0.4), & 4 < x \le 6 \\ 0, & \text{otherwise} \end{cases}$$

$$s_1 = \begin{cases} \dfrac{(4-x)+0.3(x-2)}{2}, & 2 \le x \le 4 \\ \dfrac{(x-5)+0.3(5-x)}{1}, & 4 < x \le 5 \\ 1, & \text{otherwise} \end{cases}$$

$$t_2 = \begin{cases} \dfrac{(x-1)}{3}(0.6), & 1 \le x \le 4 \\ \dfrac{(6-x)}{2}(0.6), & 4 < x \le 6 \\ 0, & \text{otherwise} \end{cases}$$

$$s_2 = \begin{cases} \dfrac{(4-x)+0.1(x-2)}{2}, & 2 \le x \le 4 \\ \dfrac{(x-5)+0.1(5-x)}{1}, & 4 < x \le 5 \\ 1, & \text{otherwise} \end{cases}$$

$$t_3 = \begin{cases} \dfrac{(x-1)}{3}(0.2), & 1 \le x \le 4 \\ \dfrac{(6-x)}{2}(0.2), & 4 < x \le 6 \\ 0, & \text{otherwise} \end{cases}$$

$$s_2 = \begin{cases} \dfrac{(6-x)+0.5(x-2)}{2}, & 2 \le x \le 4 \\ \dfrac{(x-5)+0.5(5-x)}{1}, & 4 < x \le 5 \\ 1, & \text{otherwise} \end{cases}$$

It is usually represented as $\langle [2, 4, 6]; (0.4, 0.6, 0.2), (0.3, 0.1, 0.5) \rangle$

Definition 4. The arithmetic operations on ITrFTD are defined as

Let $\widetilde{A} = \langle ([a_1, b_1, c_1]; (t_1, t_2, t_3), (s_1, s_2, s_3) \rangle$,

$\widetilde{B} = \langle ([a_2, b_2, c_2]; (t_1', t_2', t_3'), (s_1', s_2', s_3') \rangle$, then,

(i) $\widetilde{A} + \widetilde{B} = \langle [a_1 + a_2, b_1 + b_2, c_1 + c_2]; \rangle$

(ii) $\widetilde{A} - \widetilde{B} = \langle [a_1 - c_2, b_1 - b_2, c_1 - a_2]; \rangle$

(iii) $\widetilde{A} \cdot \widetilde{B} = \begin{cases} \langle [a_1 a_2, b_1 b_2, c_1 c_2]; \rangle, (c_1 > 0, c_2 > 0) \\ \langle [a_1 c_2, b_1 b_2, c_1 a_2]; \rangle, \text{otherwise} \end{cases}$

(iv) $\widetilde{A} / \widetilde{B} = \begin{cases} \langle [a_1/c_2, b_1/b_2, c_1/a_2]; \rangle, (c_1 > 0, c_2 > 0) \\ \langle [a_1/c_2, b_1/b_2, c_1/a_2]; \rangle, \text{otherwise} \end{cases}$

(v) $k\widetilde{A} = ([ka_1, kb_1, kc_1], 1 - \dfrac{1}{kt_1}, \dfrac{1}{ks_1})$

(vi) $\widetilde{A}^k = ([a_1^k, b_1^k, c_1^k], 1 - \dfrac{1}{t_1^k}, \dfrac{1}{s_1^k})$

Definition 5. Consider $\widetilde{A} = \langle ([a_1, b_1, c_1]; (t_1, t_2, t_3), (s_1, s_2, s_3) \rangle$ then

(i) $\widetilde{A}$ is called positive ITrFTD numbers if $a_1 > 0$

(ii) $\widetilde{A}$ is called negative ITrFTD numbers if $c_1 > 0$

(iii) $\widetilde{A}$ is called neither positive nor negative ITrFTD numbers if $a_1 > 0$ and $c_1 > 0$.

Theorem 6. *Let* $\widetilde{A} = \langle ([a_1, b_1, c_1]; (t_1, t_2, t_3), (s_1, s_2, s_3) \rangle$, $\widetilde{B} = \langle ([a_2, b_2,$

$c_2]; (t'_1, t'_2, t'_3), (s'_1, s'_2, s'_3))\rangle$, $\widetilde{C} = \langle([a_3, b_3, c_3]; (t''_1, t''_2, t''_3), (s''_1, s''_2, s''_3))\rangle$ *then the following properties hold*

(i) $\widetilde{A} + \widetilde{B} = \widetilde{B} + \widetilde{A}$

(ii) $(\widetilde{A} + \widetilde{B}) + \widetilde{C} = \widetilde{A} + (\widetilde{B} + \widetilde{C})$

(iii) $\widetilde{A} \cdot \widetilde{B} = \widetilde{B} \cdot \widetilde{A}$

(iv) $(\widetilde{A} \cdot \widetilde{B}) \cdot \widetilde{C} = \widetilde{A} \cdot (\widetilde{B} \cdot \widetilde{C})$

(v) $k_1\widetilde{A} + k_2\widetilde{A} = (k_1 + k_2)\widetilde{A}, (k_1 + k_2) \geq 0$

(vi) $k(\widetilde{A} + \widetilde{B}) = k\widetilde{A} + k\widetilde{B}, k \geq 0$

Proof. (i) $\widetilde{A} + \widetilde{B} = \widetilde{B} + \widetilde{A}$

$$\widetilde{A} + \widetilde{B} = \langle [a_1 + a_2, b_1 + b_2, c_1 + c_2], \frac{t_1 t'_2}{\max(t_1, t'_2, k)},$$

$$1 - \frac{(1 - s_1)(1 - s'_2)}{\max(1 - s_1, 1 - s'_2, k)} \rangle \; k \in [0, 1]$$

$$= \langle [a_1 + a_2, b_1 + b_2, c_1 + c_2], \frac{t_1 t'_2}{\max(t_1, t'_2, k)}, 1 - \frac{(1 - s_1)(1 - s'_2)}{\max(1 - s_1, 1 - s'_2, k)} \rangle$$

$\widetilde{A} + \widetilde{B} = \widetilde{B} + \widetilde{A}$

(ii) $\widetilde{A} \cdot \widetilde{B} = \widetilde{B} \cdot \widetilde{A}$

$$\widetilde{A} \cdot \widetilde{B} = \langle [a_1 a_2, b_1 b_2, c_1 c_2], \frac{t_1 t'_2}{\max(t_1, t'_2, k)}, 1 - \frac{(1 - s_1)(1 - s'_2)}{\max(1 - s_1, 1 - s'_2, k)} \rangle$$

$k \in [0, 1]$

$$= \langle [a_1 a_2, b_1 b_2, c_1 c_2], \frac{t_1 t_2}{\max(t_1, t_2, k)}, 1 - \frac{(1 - s_1)(1 - s_2)}{\max(1 - s_1, 1 - s_2, k)} \rangle$$

$\widetilde{A} \cdot \widetilde{B} = \widetilde{B} \cdot \widetilde{A}$

(iii) $k_1\widetilde{A} + k_2\widetilde{A} = (k_1 + k_2)\widetilde{A}, (k_1 + k_2) \geq 0$

By definition $k\widetilde{A} = ([ka_1,\ kb_1,\ kc_1],\ \frac{1}{kt_1},\ 1 - \frac{1}{ks_1})$

Consider $k_2\widetilde{A} = ([ka_1,\ kb_1,\ kc_1],\ \frac{1}{k_2t_1},\ 1 - \frac{1}{k_2s_1})$

Then

$$k_1\widetilde{A} + k_2\widetilde{A} = ([k_1a_1 + k_2a_1,\ k_1b_1 + k_2b_1,\ k_1c_1 + k_2c_1],\ \frac{1}{k_1t_1} + \frac{1}{k_2t_1},\ 1 - \frac{1}{k_1s_1} + \frac{1}{k_2s_1})$$

$$= ([(k_1 + k_2)a_1,\ (k_1 + k_2)b_1,\ (k_1 + k_2)c_1],\ \frac{1}{k_1t_1} + \frac{1}{k_2t_1},\ 2 - \frac{1}{k_1s_1} - \frac{1}{k_2s_1})$$

LHS

$$(k_1 + k_2)\widetilde{A} = (k_1 + k_2)\langle[a_1,\ b_1,\ c_1];\ t_1,\ s_1\rangle$$

$$= ([(k_1 + k_2)a_1,\ (k_1 + k_2)b_1,\ (k_1 + k_2)c_1],\ \frac{1}{k_1t_1} + \frac{1}{k_2t_1},\ 2 - \frac{1}{k_1s_1} - \frac{1}{k_2s_1})$$

RHS

Therefore $k_1\widetilde{A} = k_2\widetilde{A} = (k_1 + k_2)\widetilde{A}$

(iv) $k(\widetilde{A} + \widetilde{B}) = k\widetilde{A} + k\widetilde{B},\ k \geq 0$

$$k(\widetilde{A} + \widetilde{B}) = k(\langle[a_1,\ b_1,\ c_1];\ t_1,\ s_1\rangle + \langle[a_2,\ b_2,\ c_2];\ t_2,\ s_2\rangle)$$

$$= ([ka_1,\ kb_1,\ kc_1];\ \frac{1}{kt_1},\ 1 - \frac{1}{ks_1}) + ([ka_2,\ kb_2,\ kc_2];\ \frac{1}{kt_2},1 - \frac{1}{ks_2})$$

$$k(\widetilde{A} + \widetilde{B}) = k\widetilde{A} + k\widetilde{B}$$

Definition 7. Let $\widetilde{A} = \langle[a_1,\ b_1,\ c_1];\ (t_1,\ t_2,\ t_3),\ (s_1,\ s_2,\ s_3)\rangle$. Then the normalized intuitionistic triangular fuzzy three dimensional (ITrFTD) numbers of $\mathbb{R}$ is given by

$$\widetilde{A} = \langle\left[\frac{a_1}{a_1 + b_1 + c_1},\ \frac{b_1}{a_1 + b_1 + c_1},\ \frac{c_1}{a_1 + b_1 + c_1}\right];\ (t_1,\ t_2,\ t_3),\ (s_1,\ s_2,\ s_3)\rangle$$

Example 8. Assume that $\widetilde{A} = \langle(3, 6, 7); (0.02, 0.37, 0.8), (0.23, 0.35, 0.1)\rangle$ then normalized (ITrFTD) number is

$$\widetilde{A} = \langle\left(\frac{3}{16}, \frac{6}{16}, \frac{7}{16}\right); (0.02, 0.37, 0.8), (0.23, 0.35, 0.1)\rangle$$

Definition 9. Consider $\widetilde{A} = \langle[a_1, b_1, c_1]; (t_1, t_2, t_3), (s_1, s_2, s_3)\rangle$

$$\widetilde{B} = \langle[a_1, b_1, c_1]; (t_1', t_2', t_3'), (s_1', s_2', s_3')\rangle,$$

then the normalized similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as

$$S(\widetilde{A}, \widetilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$$

$$\left.\left(\frac{\min\{P(A)^1, P(A)^2, P(A)^3, P(B)^1, P(B)^2, P(B)^3\} + \min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}}{\max\{P(A)^1, P(A)^2, P(A)^3, P(B)^1, P(B)^2, P(B)^3\} + \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}}\right)\right]$$

where $S(\widetilde{A}, \widetilde{B}) \in [0, 1]$; $P(\widetilde{A})^i$ and $P(\widetilde{B})^i$ for $i = 1, 2, 3$ are defined as

$$P(\widetilde{A})^i = \sqrt{(a_1 - b_1)^2 + (t^i_{\widetilde{A}})^2} + \sqrt{(c_1 - b_1)^2 + (s^i_{\widetilde{B}})^2} + (c_1 - a_1),$$

$$P(\widetilde{B})^i = \sqrt{(a_2 - b_2)^2 + (t^i_{\widetilde{B}})^2} + \sqrt{(c_2 - b_2)^2 + (s^i_{\widetilde{B}})^2} + (c_2 - a_2)$$

Theorem 10. *Consider* $\widetilde{A} = \langle([a_1, b_1, c_1]; (t_1, t_2, t_3), (s_1, s_2, s_3)\rangle$, $\widetilde{B} = \langle([a_2, b_2, c_2]; (t_1', t_2', t_3'), (s_1', s_2', s_3')\rangle$, $\widetilde{C} = \langle([a_3, b_3, c_3]; (t_1'', t_2'', t_3''), (s_1'', s_2'', s_3'')\rangle$. $S(\widetilde{A}, \widetilde{B})$ *satisfies the following properties.*

(i) *Normalized ITrFTD numbers* $\widetilde{A}$ *and* $\widetilde{B}$ *are identical if and only if* $S(\widetilde{A}, \widetilde{B}) = 1$

(ii) $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$

(iii) *Consider* $\widetilde{A}$ *and* $\widetilde{B}$ *be normalized ITrFTD numbers with the same shape, the same scale.*

That is $t_1 = t_1', t_2 = t_2', t_3 = t_3', s_1 = s_1', s_2 = s_2', s_3 = s_3'$ *and*

$d = a_2 - a_1 = b_2 - b_1 = c_2 - c_1$ *then* $S(\tilde{A}, \tilde{B}) = 1 - |d|$

(iv) *If* $\tilde{A}\tilde{B}\tilde{C}$, *then* $S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B})$ *and* $S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C})$

Proof (i). If $\tilde{A}$ and $\tilde{B}$ are identical, then $a_2 = a_1, b_2 = b_1, c_2 = c_1$ and $t_1 = t_1', t_2 = t_2', t_3 = t_3', s_1 = s_1', s_2 = s_2', s_3 = s_3'$. Thus $\min\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\} = \max\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\}$ and $\min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} = \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\} = \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}$ the degree of similarity between $\tilde{A}$ and $\tilde{B}$ is calculated as follows

$$S(\tilde{A}, \tilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$$

$$\left.\left(\frac{\begin{array}{l}\min\{P(A)^1, P(A)^2, P(A)^3, P(B)^1, P(B)^2, P(B)^3\} \\ + \min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}{\begin{array}{l}\max\{P(A)^1, P(A)^2, P(A)^3, P(B)^1, P(B)^2, P(B)^3\} \\ + \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}\right)\right] = 1$$

Conversely suppose $S(\tilde{A}, \tilde{B}) = 1$ then $a_2 = a_1, b_2 = b_1, c_2 = c_1$ and $t_1 = t_1', t_2 = t_2', t_3 = t_3', s_1 = s_1', s_2 = s_2', s_3 = s_3'$. Thus $\min\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\} = \max\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2,$ and $\min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} = \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\} = \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}$ Therefore, normalized ITrFTD numbers $\tilde{A}$ and $\tilde{B}$ are identical.

Proof (ii) $S(\tilde{A}, \tilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$

$$\left.\left(\frac{\begin{array}{l}\min\{P(\widetilde{A})^1, P(\widetilde{A})^2, P(\widetilde{A})^3, P(\widetilde{B})^1, P(\widetilde{B})^2, P(\widetilde{B})^3\}\\ + \min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}{\begin{array}{l}\max\{P(\widetilde{A})^1, P(\widetilde{A})^2, P(\widetilde{A})^3, P(\widetilde{B})^1, P(\widetilde{B})^2, P(\widetilde{B})^3\}\\ + \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}\right)\right]$$

$$= \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$$

$$\left.\left(\frac{\begin{array}{l}\min\{P(\widetilde{B})^1, P(\widetilde{B})^2, P(\widetilde{B})^3, P(\widetilde{A})^1, P(\widetilde{A})^2, P(\widetilde{A})^3\}\\ + \min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}{\begin{array}{l}\max\{P(\widetilde{B})^1, P(\widetilde{B})^2, P(\widetilde{B})^3, P(\widetilde{A})^1, P(\widetilde{A})^2, P(\widetilde{A})^3\}\\ + \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}\right)\right]$$

$S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$

Proof (iii) $S(\widetilde{A}, \widetilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$

$$\left.\left(\frac{\begin{array}{l}\min\{P(\widetilde{A})^1, P(\widetilde{A})^2, P(\widetilde{A})^3, P(\widetilde{B})^1, P(\widetilde{B})^2, P(\widetilde{B})^3\}\\ + \min\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \max\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}{\begin{array}{l}\max\{P(\widetilde{A})^1, P(\widetilde{A})^2, P(\widetilde{A})^3, P(\widetilde{B})^1, P(\widetilde{B})^2, P(\widetilde{B})^3\}\\ + \max\{(t_1, t_2, t_3), (t_1', t_2', t_3')\} + \min\{(s_1, s_2, s_3), (s_1', s_2', s_3')\}\end{array}}\right)\right]$$

$$= \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)(1)\right]$$

$S(\widetilde{A}, \widetilde{B}) = 1 - |d|$

Proof (iv). If $\widetilde{A}, \widetilde{B}, \widetilde{C}$ then $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C} \langle = \rangle t^i_{\widetilde{A}} \le t^i_{\widetilde{B}} \le t^i_{\widetilde{C}}$ and $s^i_{\widetilde{A}} \le s^i_{\widetilde{B}} \le s^i_{\widetilde{C}}$

Consider $S(\widetilde{A}, \widetilde{C}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$

$$\left.\left(\frac{\begin{array}{l}\min\{P(\tilde{A})^1,\, P(\tilde{A})^2,\, P(\tilde{A})^3,\, P(\tilde{C})^1,\, P(\tilde{C})^2,\, P(\tilde{C})^3\}\\ +\min\{(t_1,\, t_2,\, t_3),\, (t_1'',\, t_2'',\, t_3'')\}+\max\{(s_1,\, s_2,\, s_3),\, (s_1'',\, s_2'',\, s_3'')\}\end{array}}{\begin{array}{l}\max\{P(\tilde{A})^1,\, P(\tilde{A})^2,\, P(\tilde{A})^3,\, P(\tilde{C})^1,\, P(\tilde{C})^2,\, P(\tilde{C})^3\}\\ +\max\{(t_1,\, t_2,\, t_3),\, (t_1'',\, t_2'',\, t_3'')\}+\min\{(s_1,\, s_2,\, s_3),\, (s_1'',\, s_2'',\, s_3'')\}\end{array}}\right)\right]$$

$$\leq \frac{1}{3}\left[\left(1-\frac{|\,a_2-a_1\,|+|\,b_2-b_1\,|+|\,c_2-c_1\,|}{3}\right)(1)\right]$$

$$\left.\left(\frac{\begin{array}{l}\min\{P(\tilde{A})^1,\, P(\tilde{A})^2,\, P(\tilde{A})^3,\, P(\tilde{B})^1,\, P(\tilde{B})^2,\, P(\tilde{B})^3\}\\ +\min\{(t_1,\, t_2,\, t_3),\, (t_1',\, t_2',\, t_3')\}+\max\{(s_1,\, s_2,\, s_3),\, (s_1',\, s_2',\, s_3')\}\end{array}}{\begin{array}{l}\max\{P(\tilde{A})^1,\, P(\tilde{A})^2,\, P(\tilde{A})^3,\, P(\tilde{B})^1,\, P(\tilde{B})^2,\, P(\tilde{B})^3\}\\ +\max\{(t_1,\, t_2,\, t_3),\, (t_1',\, t_2',\, t_3')\}+\min\{(s_1,\, s_2,\, s_3),\, (s_1',\, s_2',\, s_3')\}\end{array}}\right)\right]$$

$$S(\tilde{A},\, \tilde{C}) = S(\tilde{A},\, \tilde{B})$$

Definition 11. Let $\tilde{A} = \langle([a_1,\, b_1,\, c_1];\, (t_1,\, t_2,\, t_3),\, (s_1,\, s_2,\, s_3)\rangle$, $\tilde{B} = \langle([a_2,\, b_2, c_2];\, (t_1',\, t_2',\, t_3'),\, (s_1',\, s_2',\, s_3')\rangle$ be ITrFTD. Define positive ideal solution and ITrFTD numbers negative ideal solution as

$$r_{\tilde{A}}^{+} = \langle a_1^{+},\, b_1^{+},\, c_1^{+};\, ((t_1)^{+},\, (t_2)^{+},\, (t_3)^{+}),\, ((s_1)^{+},\, (s_2)^{+},\, (s_3)^{+})\rangle$$

$$= \langle[1,\, 1,\, 1];\, (1,\, 1,\, 1),\, (0,\, 0,\, 0)\rangle$$

$$r_{\tilde{A}}^{-} = \langle a_1^{-},\, b_1^{-},\, c_1^{-};\, ((t_1)^{-},\, (t_2)^{-},\, (t_3)^{-}),\, ((s_1)^{-},\, (s_2)^{-},\, (s_3)^{-})\rangle$$

$$= \langle[0,\, 0,\, 0];\, (0,\, 0,\, 0),\, (1,\, 1,\, 1)\rangle$$

Definition 12. Let $\tilde{A} = \langle([a_1,\, b_1,\, c_1];\, (t_1,\, t_2,\, t_3),\, (s_1,\, s_2,\, s_3)\rangle$, $\tilde{B} = \langle([a_2,\, b_2, c_2];\, (t_1',\, t_2',\, t_3'),\, (s_1',\, s_2',\, s_3')\rangle$ be ITrFTD and r^{+} and r^{-} be an ITrFTD positive ideal solution and negative ideal solution, then

If $S(\tilde{A},\, r^{+}) > S(\tilde{B},\, r^{+})$, then $\tilde{B}$ is smaller than $\tilde{A}$

If $S(\tilde{A},\, r^{+}) = S(\tilde{B},\, r^{+}) \wedge S(\tilde{A},\, r^{-}) < S(\tilde{B},\, r^{-})$, then $\tilde{A}$ is smaller than $\tilde{B}$ denoted by $\tilde{A} < \tilde{B}$

If $S(\tilde{A},\, r^{+}) = S(\tilde{B},\, r^{+}) \wedge S(\tilde{A},\, r^{-}) = S(\tilde{B},\, r^{-})$, then $\tilde{A}$ is similar to $\tilde{B}$ denoted by $\tilde{A}\tilde{B}$.

4. Some Aggregation Operators on ITrFTD Numbers

Definition 1. Let $\widetilde{A}_j,\ j \in I_3$ be a collection of ITrFTD numbers. The Intuitionistic triangular fuzzy three dimensional weighted harmonic (ITrFTDWH) is defined as

$$(\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3) = (\frac{3}{\frac{1}{\widetilde{A}_1^w} + \frac{1}{\widetilde{A}_2^w} + \frac{1}{\widetilde{A}_3^w}})$$

where $w = (w_1, w_2, w_3)^T$ is the weight vector of $\widetilde{A}_j,\ j \in I_3$ with $w_1 \in [0, 1]$ and $\sum_{j=1}^{3} w_j = 1$.

Definition 2. Let $\widetilde{A}_j,\ j \in I_n$ be a collection of ITrFTD numbers. The Intuitionistic triangular fuzzy three dimensional weighted arithmetic (ITrFTDWA) is defined as

$$(\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3) = \frac{\sum_{j=1}^{3} w_j \widetilde{A}_j}{\sum_{j=1}^{3} w_j}$$

where $w = (w_1, w_2, w_3)^T$ is the weight vector of $\widetilde{A}_j,\ j \in I_3$ with $w_1 \in [0, 1]$ and $\sum_{j=1}^{3} w_j = 1$.

Theorem 3. *Let $\widetilde{A}_j,\ j \in I_3$ be a collection of ITrFTD numbers then their aggregated value defined by using ITrFTDWH operator is also an ITrFTD number and it is given as*

$$ITrFTDWH = \prod_{J=1}^{3} \widetilde{A}_j^{w_j} \langle [\prod_{j=1}^{3} \widetilde{a}_j^{w_j}, \prod_{j=1}^{3} \widetilde{b}_j^{w_j}, \prod_{j=1}^{3} \widetilde{c}_j^{w_j}];$$

$$\frac{\prod_{i=1}^{3} (t_i)}{\max(t_1, t_2, t_3, k)}, 1 = -\frac{\prod_{i=1}^{3} (1 - s_i)}{\max((1 - s_1), (1 - s_2), (1 - s_3), k)} \rangle \qquad (1)$$

Proof. We prove the result by using mathematical induction on n. Consider

$$(\widetilde{A}_1)^{w_1} = \langle (a_1^{w_1}, b_1^{w_1}, c_1^{w_1}); ((t_{\widetilde{A}_1}^1)^{w_1}, (t_{\widetilde{A}_1}^2)^{w_1}, (t_{\widetilde{A}_1}^3)^{w_1}, \ldots, (t_{\widetilde{A}_1}^p)^{w_1}),$$

$$((s_{\widetilde{A}_1}^1)^{w_1}, (s_{\widetilde{A}_1}^2)^{w_1}, (s_{\widetilde{A}_1}^3)^{w_1}, \ldots, (s_{\widetilde{A}_1}^p)^{w_1})\rangle$$

$$(\widetilde{A}_2)^{w_2} = \langle (a_2^{w_2}, b_2^{w_2}, c_2^{w_2}); ((t_{\widetilde{A}_2}^1)^{w_2}, (t_{\widetilde{A}_2}^2)^{w_2}, (t_{\widetilde{A}_2}^3)^{w_2}, \ldots, (t_{\widetilde{A}_2}^p)^{w_2}),$$

$$((s_{\widetilde{A}_2}^1)^{w_2}, (s_{\widetilde{A}_2}^2)^{w_2}, (s_{\widetilde{A}_2}^3)^{w_2}, \ldots, (s_{\widetilde{A}_2}^p)^{w_2})\rangle$$

$$\text{ITrFTDWH}(\widetilde{A}_1, \widetilde{A}_2) = \widetilde{A}_1 \times \widetilde{A}_2$$

$$= \langle (a_1^{w_1}a_2^{w_2}, b_1^{w_1}b_2^{w_2}, c_1^{w_1}c_2^{w_2}); \frac{(t_1t_2)}{\max(t_1t_2, \alpha)}, 1 - \frac{(1-s_1)(1-s_2)}{\max(1-s_1)(1-s_2), \alpha)} \rangle$$

Equation (1) holds for $n = 2$

If equation (1) holds for $n = k$ then

$$\text{ITrFTDWH} = \prod_{J=1}^{k} \widetilde{A}_j^{w_j} \langle [\prod_{j=1}^{k} \widetilde{a}_j^{w_j}, \prod_{j=1}^{k} \widetilde{b}_j^{w_j}, \prod_{j=1}^{k} \widetilde{c}_j^{w_j}];$$

$$\frac{\prod_{i=1}^{k}(t_i)}{\max(t_1, t_2, \ldots, t_i, \alpha)}, 1 - \frac{\prod_{i=1}^{k}(1-s_i)}{\max((1-s_1), (1-s_2), \ldots, (1-s_i), \alpha)} \rangle$$

Multiply by $\widetilde{A}_{k+1}$ on both sides and on applying operational laws

$$\text{ITrFTDWH} = \prod_{J=1}^{k+1} \widetilde{A}_j^{w_j} \langle [\prod_{j=1}^{k+1} \widetilde{a}_j^{w_j}, \prod_{j=1}^{k+1} \widetilde{b}_j^{w_j}, \prod_{j=1}^{k+1} \widetilde{c}_j^{w_j}];$$

$$\frac{\prod_{i=1}^{k+1}(t_i)}{\max(t_1, t_2, \ldots, t_i, \alpha)}, 1 - \frac{\prod_{i=1}^{k+1}(1-s_i)}{\max((1-s_1), (1-s_2), \ldots, (1-s_i), \alpha)} \rangle$$

Hence, equation (1) holds for $n = k + 1$.

5. An Approach to MCDM Problems with ITrFTD Numbers

In this section let us construct a multi criteria decision making called ITrFTD multi-criteria decision making problem, using the ITrFTDWH and ITrFTDWA operators is done as follows

Definition 1. Consider $Y = (y_1, y_2, y_3)$ a set of alternatives, $V = (v_1, v_2, v_3)$ the set of attributes and $[\widetilde{A}_{ij}] = \langle [a_{ij}, b_{ij}, c_{ij}]; (t^1_{\widetilde{A}_{ij}}, t^2_{\widetilde{A}_{ij}}, t^3_{\widetilde{A}_{ij}}), (s^1_{\widetilde{A}_{ij}}, s^2_{\widetilde{A}_{ij}}, s^3_{\widetilde{A}_{ij}}) \rangle$ be anITrFTD numbers for all $i, j \in I_3$

For a normalized ITrFTD numbers decision-making matrix

$$M = (r_{ij})_{m\times n} = \langle [a_{ij}, b_{ij}, c_{ij}]; (t^1_{\widetilde{A}_{ij}}, t^2_{\widetilde{A}_{ij}}, t^3_{\widetilde{A}_{ij}}), (s^1_{\widetilde{A}_{ij}}, s^2_{\widetilde{A}_{ij}}, s^3_{\widetilde{A}_{ij}}) \rangle_{m\times n}$$

where $0 \le a_{ij} \le b_{ij} \le c_{ij} \le 1$, $0 \le t^1_{\widetilde{A}_{ij}}, t^2_{\widetilde{A}_{ij}}, t^3_{\widetilde{A}_{ij}}, s^1_{\widetilde{A}_{ij}}, s^2_{\widetilde{A}_{ij}}, s^3_{\widetilde{A}_{ij}} \le 1$

$$[\widetilde{A}_{ij}]_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is called an ITrFTD multi-criteria decision-making matrix of the decision maker.

5.2 Algorithm to Solve ITrFTD MCDM Problem Using ITrFTDWH is Constructed

Step 1. The ITrFTD multi-criteria decision-making matrix $\widetilde{A} = (a_{ij})_{3\times 3}$

Step 2. Calculate overall values

$r_i = \text{ITrFTDWH}(a_{i1}, b_{i2}, c_{i3})$; $(i = 1, 2, 3)$ if r_i for all $i \in I_3$ is not normalized ITrFTD number, then calculate the normalized ITrFTD number.

Step 3. Calculate the distances between collective overall values

$r_i = \langle [a_{ij}, b_{ij}, c_{ij}]; (t^1_{\widetilde{A}_{ij}}, t^2_{\widetilde{A}_{ij}}, t^3_{\widetilde{A}_{ij}}), (s^1_{\widetilde{A}_{ij}}, s^2_{\widetilde{A}_{ij}}, s^3_{\widetilde{A}_{ij}}) \rangle$ and Positive ideal solution r_i^+ (or negative ideal solution r_i^-) using similarity measure

$$S(\widetilde{A}, \widetilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$$

$$\left.\left(\frac{\min\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\} + \min\{(t_1, t_2, t_3), (t'_1, t'_2, t'_3)\} + \max\{(s_1, s_2, s_3), (s'_1, s'_2, s'_3)\}}{\max\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\} + \max\{(t_1, t_2, t_3), (t'_1, t'_2, t'_3)\} + \min\{(s_1, s_2, s_3), (s'_1, s'_2, s'_3)\}}\right)\right]$$

Step 4. Rank all the alternatives $\tilde{A}_i (i = 1, 2, 3)$ and select the best one (*s*) in accordance with $S(r_i, r_i^+)$. Larger the distance $S(r_i, r_i^+)$, the good are the alternatives $\tilde{A}_i$, $i \in I_3$.

Step 5. Write the optimal solution.

5.3 Algorithm to Solve ITrFTD MCDM Problem using ITrFTDWH is Constructed

Step 1. Calculate the ITrFTD multi-criteria decision-making matrix $\tilde{A} = (a_{ij})_{3\times 3}$

Step 2. Calculate overall values

$r_i = \text{ITrFTDWH}(a_{i1}, b_{i2}, c_{i3})$; $(i = 1, 2, 3)$ if r_i for all $i \in I_3$ is not normalized ITrFTD numbers, then calculate the normalized ITrFTD numbers.

Step 3. Calculate the distances between collective overall values

$r_i = \langle [a_{ij}, b_{ij}, c_{ij}]; (t^1_{\tilde{A}_{ij}}, t^2_{\tilde{A}_{ij}}, t^3_{\tilde{A}_{ij}}), (s^1_{\tilde{A}_{ij}}, s^2_{\tilde{A}_{ij}}, s^3_{\tilde{A}_{ij}})\rangle$ and Positive ideal solution r_i^+ (or negative ideal solution r_i^-) using similarity measure

$$S(\tilde{A}, \tilde{B}) = \frac{1}{3}\left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1|}{3}\right)\right.$$

$$\left.\left(\frac{\min\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\} + \min\{(t_1, t_2, t_3), (t'_1, t'_2, t'_3)\} + \max\{(s_1, s_2, s_3), (s'_1, s'_2, s'_3)\}}{\max\{P(\tilde{A})^1, P(\tilde{A})^2, P(\tilde{A})^3, P(\tilde{B})^1, P(\tilde{B})^2, P(\tilde{B})^3\} + \max\{(t_1, t_2, t_3), (t'_1, t'_2, t'_3)\} + \min\{(s_1, s_2, s_3), (s'_1, s'_2, s'_3)\}}\right)\right]$$

Step 4. Rank all the alternatives $\tilde{A}_i (i = 1, 2, 3)$ and select the best one(s) in accordance with $S(r_i, r_i^+)$. Larger the distance $S(r_i, r_i^+)$, the good are the alternatives $\widetilde{A}_i$, $i \in I_m$.

Step 5. Write the optimal solution

6. Application

Many restaurants and cafes need quality chef to keep their customer happy. So, chefs will be interviewed by three experts and they have importance in this process. They will review the chef according to three criteria, experience and credentials, speed and cooking ability, organizational skills. Different weights are given to different criteria. That is $w = (0.2, 0.3, 0.1)$. After thorough investication, three alternatives are taken into consideration, that is $\{y_1, y_2, y_3\}$. There are many factors to affect the interview process and three factors are considered based on the experience of the restaurant personnel, including v_1 : Experience and credentials (such as aware of the importance of practical training in becoming successful chefs) v_2 : Speed and cooking ability(such as improving the uptake of fruit and vegetables and increased recognition of healthier foods) v_3 : organizational skills (keeping an organized workspace canmake eachtast faster, planning, efficiency), which in turn gives rise to different membership functions for each alternatives.

Solution:

Case (i) Using Algorithm 5.2 the problem is solved in the following manner

Step 1. The decision making matrix $\widetilde{A} = (a_{ij})_{3\times 3}$ for the above problem is constructed as

$$\begin{array}{cc} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} v_1 \\ \\ v_2 \\ \\ v_3 \\ \\ \end{array} & \left(\begin{array}{rrr} \langle [0.3,\ 0.5,\ 0.7];\ (0.5,\ 0.2,\ 0.4), & \langle [0.1,\ 0.3,\ 0.5];\ (0.5,\ 0.2,\ 0.4), & \langle [0.3,\ 0.5,\ 0.6];\ (0.5,\ 0.1,\ 0.3), \\ (0.2,\ 0.6,\ 0.4)\rangle & (0.4,\ 0.5,\ 0.3)\rangle & (0.3,\ 0.7,\ 0.4)\rangle \\ \langle [0.4,\ 0.6,\ 0.8];\ (0.4,\ 0.1,\ 0.5), & \langle [0.2,\ 0.4,\ 0.6];\ (0.1,\ 0.4,\ 0.3), & \langle [0.2,\ 0.4,\ 0.7];\ (0.2,\ 0.4,\ 0.7), \\ (0.4,\ 0.6,\ 0.2)\rangle & (0.6,\ 0.2,\ 0.5)\rangle & (0.6,\ 0.3,\ 0.1)\rangle \\ \langle [0.2,\ 0.4,\ 0.7];\ (0.2,\ 0.6,\ 0.5), & \langle [0.5,\ 0.6,\ 0.8];\ (0.6,\ 0.3,\ 0.8), & \langle [0.1,\ 0.2,\ 0.4];\ (0.3,\ 0.5,\ 0.7), \\ (0.6,\ 0.1,\ 0.3)\rangle & (0.3,\ 0.6,\ 0.1)\rangle & (0.5,\ 0.1,\ 0.2)\rangle \end{array} \right) \end{array}$$

Step 2. The collective overall preference intuitionistic triangular fuzzy three dimensional number is derived using ITrTDWH operator

$$r_1 = \langle [0.04,\ 0.074,\ 0.117];\ (0.044,\ 0.04,\ 0.077),\ (0.06,\ 0.026,\ 0.048)\rangle$$

$$r_2 = \langle [0.035,\ 0.072,\ 0.107];\ (0.05,\ 0.045,\ 0.083),\ (0.058,\ 0.0692,\ 0.024)\rangle$$

$$r_3 = \langle [0.023,\ 0.044,\ 0.079];\ (0.05,\ 0.038,\ 0.084),\ (0.071,\ 0.025,\ 0.031)\rangle$$

Step 3. The distances and similarity between r_i and r^+ are calculated as follows

$$P(r_1)^1 = \sqrt{(a_1 - b_1)^2 + (t_1)^2} + \sqrt{(c_1 - b_1)^2 + (t_1)^2} + (c_1 - a_1)$$

$$= \sqrt{0.001 + 0.002} + \sqrt{0.002 + 0.002} + 0.077$$

$$= 0.195$$

$$P(r_1)^2 = \sqrt{(a_1 - b_1)^2 + (t_2)^2} + \sqrt{(c_1 - b_1)^2 + (t_2)^2} + (c_1 - a_1)$$

$$= \sqrt{0.001 + 0.002} + \sqrt{0.002 + 0.002} + 0.077$$

$$= 0.195$$

$$P(r_1)^3 = \sqrt{(a_1 - b_1)^2 + (t_3)^2} + \sqrt{(c_1 - b_1)^2 + (t_3)^2} + (c_1 - a_1)$$

$$= \sqrt{0.001 + 0.006} + \sqrt{0.002 + 0.006} + 0.077$$

$$= 0.195$$

$$P(r_2)^1 = \sqrt{(a_1 - b_1)^2 + (t_1')^2} + \sqrt{(c_1 - b_1)^2 + (t_1')^2} + (c_1 - a_1)$$

$$= \sqrt{0.001 + 0.003} + \sqrt{0.002 + 0.003} + 0.072$$

$$= 0.199$$

$$P(r_2)^2 = \sqrt{(a_1 - b_1)^2 + (t'_2)^2} + \sqrt{(c_1 - b_1)^2 + (t'_2)^2} + (c_1 - a_1)$$

$$= \sqrt{0.001 + 0.002} + \sqrt{0.002 + 0.002} + 0.072$$

$$= 0.182$$

$$P(r_2)^3 = \sqrt{(a_1 - b_1)^2 + (t'_1)^2} + \sqrt{(c_1 - b_1)^2 + (t'_1)^2} + (c_1 - a_1)$$

$$= \sqrt{0.001 + 0.007} + \sqrt{0.002 + 0.007} + 0.072$$

$$= 0.251$$

$$P(r_3)^1 = \sqrt{(a_1 - b_1)^2 + (t''_1)^2} + \sqrt{(c_1 - b_1)^2 + (t''_1)^2} + (c_1 - a_1)$$

$$= \sqrt{0.0004 + 0.0025} + \sqrt{0.0012 + 0.0025} + 0.056$$

$$= 0.1707$$

$$P(r_3)^3 = \sqrt{(a_1 - b_1)^2 + (t''_1)^2} + \sqrt{(c_1 - b_1)^2 + (t''_1)^2} + (c_1 - a_1)$$

$$= \sqrt{0.0004 + 0.0014} + \sqrt{0.0012 + 0.0014} + 0.056$$

$$= 0.1494$$

$$P(r_3)^3 = \sqrt{(a_1 - b_1)^2 + (t''_1)^2} + \sqrt{(c_1 - b_1)^2 + (t''_1)^2} + (c_1 - a_1)$$

$$= \sqrt{0.0004 + 0.0071} + \sqrt{0.0012 + 0.0071} + 0.056$$

$$= 0.2337$$

$P(r^+)^1 = P(r^+)^1 = P(r^+)^1 = 2,$ hence $S(r_1, r^+) = 0.0063, S(r_2, r^+) = 0.0023, S(r_3, r^+) = 0.0015.$

Step 4. Rank all the alternatives in accordance with the ascending order of $S(r_i, r^+)$, hence

$$A_3 < A_2 < A_1$$

Step 5. The most best alternative is A_1

Case (ii) Using algorithm (5.3) the problem is solved in the following manner

Step 1. The decision making matrix $\widetilde{A} = (a_{ij})_{3\times 3}$ for the above problem is constructed

$$\begin{array}{cccc} & y_1 & y_2 & y_3 \\ v_1 & \langle [0.3, 0.5, 0.7]; (0.5, 0.2, 0.4), (0.2, 0.6, 0.4)\rangle & \langle [0.1, 0.3, 0.5]; (0.5, 0.2, 0.4), (0.4, 0.5, 0.3)\rangle & \langle [0.3, 0.5, 0.6]; (0.5, 0.1, 0.3), (0.3, 0.7, 0.4)\rangle \\ v_2 & \langle [0.4, 0.6, 0.8]; (0.4, 0.1, 0.5), (0.4, 0.6, 0.2)\rangle & \langle [0.2, 0.4, 0.6]; (0.1, 0.4, 0.3), (0.6, 0.2, 0.5)\rangle & \langle [0.2, 0.4, 0.7]; (0.2, 0.4, 0.7), (0.6, 0.3, 0.1)\rangle \\ v_3 & \langle [0.2, 0.4, 0.7]; (0.2, 0.6, 0.5), (0.6, 0.1, 0.3)\rangle & \langle [0.5, 0.6, 0.8]; (0.6, 0.3, 0.8), (0.3, 0.6, 0.1)\rangle & \langle [0.1, 0.2, 0.4]; (0.3, 0.5, 0.7), (0.5, 0.1, 0.2)\rangle \end{array}$$

Step 2. The collective overall preference intuitionistic triangular fuzzy three dimensional number is derived using ITrFTDWA operator

$$r_1 = \langle [0.333, 0.533, 0.75]; (0.4, 0.217, 0.467), (0.367, 0.517, 0.283)\rangle$$

$$r_2 = \langle [0.217, 0.4, 0.6]; (0.317, 0.317, 0.417), (0.483, 0.367, 0.367)\rangle$$

$$r_3 = \langle [0.217, 0.4, 0.617]; (0.317, 0.317, 0.567), (0.483, 0.4, 0.217)\rangle$$

Step 3. The distances and similarity between r_i and r^+ are calculated as follows

$$P(r_1)^1 = \sqrt{(a_1 - b_1)^2 + (t_1)^2} + \sqrt{(c_1 - b_1)^2 + (t_1)^2} + (c_1 - a_1)$$

$$= \sqrt{0.04 + 0.16} + \sqrt{0.047 + 0.16} + 0.417$$

$$= 1.1152$$

$$P(r_1)^2 = \sqrt{(a_1 - b_1)^2 + (t_2)^2} + \sqrt{(c_1 - b_1)^2 + (t_2)^2} + (c_1 - a_1)$$

$$= \sqrt{0.04 + 0.047} + \sqrt{0.047 + 0.047} + 0.417$$

$$= 1.019$$

$$P(r_1)^3 = \sqrt{(a_1 - b_1)^2 + (t_3)^2} + \sqrt{(c_1 - b_1)^2 + (t_3)^2} + (c_1 - a_1)$$

$$= \sqrt{0.04 + 0.218} + \sqrt{0.047 + 0.218} + 0.417$$

$$= 1.44$$

$$P(r_2)^1 = \sqrt{(a_1 - b_1)^2 + (t_1')^2} + \sqrt{(c_1 - b_1)^2 + (t_1')^2} + (c_1 - a_1)$$

$$= \sqrt{0.034 + 0.101} + \sqrt{0.04 + 0.101} + 0.383$$

$$= 1.126$$

$$P(r_2)^2 = \sqrt{(a_1 - b_1)^2 + (t_2')^2} + \sqrt{(c_1 - b_1)^2 + (t_2')^2} + (c_1 - a_1)$$

$$= \sqrt{0.034 + 0.101} + \sqrt{0.04 + 0.101} + 0.383$$

$$= 1.126$$

$$P(r_2)^3 = \sqrt{(a_1 - b_1)^2 + (t_1')^2} + \sqrt{(c_1 - b_1)^2 + (t_1')^2} + (c_1 - a_1)$$

$$= \sqrt{0.034 + 0.174} + \sqrt{0.04 + 0.174} + 0.383$$

$$= 1.302$$

$$P(r_3)^1 = \sqrt{(a_1 - b_1)^2 + (t_1'')^2} + \sqrt{(c_1 - b_1)^2 + (t_1'')^2} + (c_1 - a_1)$$

$$= \sqrt{0.034 + 0.101} + \sqrt{0.047 + 0.101} + 0.4$$

$$= 1.152$$

$$P(r_3)^3 = \sqrt{(a_1 - b_1)^2 + (t_1'')^2} + \sqrt{(c_1 - b_1)^2 + (t_1'')^2} + (c_1 - a_1)$$

$$= \sqrt{0.034 + 0.101} + \sqrt{0.047 + 0.101} + 0.4$$

$$= 1.152$$

$$P(r_3)^3 = \sqrt{(a_1 - b_1)^2 + (t_1'')^2} + \sqrt{(c_1 - b_1)^2 + (t_1'')^2} + (c_1 - a_1)$$

$$= \sqrt{0.034 + 0.322} + \sqrt{0.047 + 0.322} + 0.4$$

$$= 1.604$$

$P(r^+)^1 = P(r^+)^1 = P(r^+)^1 = 2$, hence $S(r_1, r^+) = 0.1049$, $S(r_2, r^+) = 0.0868$, $S(r_3, r^+) = 0.0892$.

Step 4. Rank all the alternatives in accordance with the ascending order of $S(r_i, r^+)$, hence

$$A_3 < A_3 < A_1$$

Step 5. The most best alternative is A_1.

References

[1] K. V. Atanassov, Intuitionistic fuzzy sets, Pysica-Verlag a Springer-Verlag Company, New York, (1999).

[2] S. K. De, R. Biswas and A. R. Roy, Some operations on intuitionistic fuzzy sets, Fuzzy Sets Syst. 114(3) (2000), 477-484.

[3] P. A. Ejegwa and J. A. Awolola, Intuitionistic fuzzy multiset (IFMS) in binomial distributions, Int. J. Sci. Technol. Res. 3(4) (2014), 335-337.

[4] M. Esmailzadeh and M. Esmailzadeh, New distance between triangular Intuitionistic fuzzy numbers, Adv. Comput. Math. Appl. 2(3) (2013), 310-314.

[5] P. Liu and Y. Liu, An approach to multiple attribute group decision making based on intuitionistic trapezoidal fuzzy power generalized aggregation operator, Int. J. Comput. Intell. Syst. 7(2) (2014), 291-304.

[6] P. Rajarajeswari and N. Uma N, Correlation measure for intuitionistic fuzzy multi sets, Int. J. Res. Eng. Techno. 3(1) (2014), 611-617.

[7] T. K. Shinoj and S. J. John, Intuitionistic fuzzy multisets and its application in medical diagnosis, World Acad. Sci. Eng. Technol. 6(1) (2012), 1418-1421.

[8] T. K. Shinoj and S. J. John, Intuitionistic fuzzy multisets, Int. J. Eng. Sci. Innov. Technol. (IJESIT) 2(6) (2012), 1-24.

[9] T. K. Shinoj and J. J. Sunil, Accuracy in collaborative robotics: an intuitionistic fuzzy Multiset approach. Global. J. Sci. Front. Res. Math. Decis. Sci. 13(10) (2013), 21-28.

[10] J. Wang, R. Nie, H. Zhang and X. Chen, New operators on triangular intuitionistic fuzzy numbers and their applications in system fault analysis, Inf. Sci. 251 (2013), 79-95.

[11] J. Wang and Z. Zhang, Multi-criteria decision-making method with incomplete certain information based on intuitionistic fuzzy number, Control Decis. 24(2) (2009), 226-230.

[12] S. P. Wan and J. Y. Dong, Power geometric perator soft rapezoidal intuitionistic fuzzy numbers and application to multi-attribute group decision making, Appl. Soft. Comput. 29 (2015), 153-168.

[13] R. R. Yager, On the theory of bags, Int. J. Gen. Syst. 13 (1986), 23-37.

[14] D. Yu, Intuitionistic fuzzy information aggregation under confidence levels, Appl. Soft. Comput. 19 (2014), 147-160.

[15] S. Zhao, C. Liang and J. Zhang, Some intuitionistic trapezoidal fuzzy aggregation operators based on Einstein operations and their Application in multiple attribute group decision making, Int. J. Mach Learn Cybern. 8(2) (2015), 547-569.