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RADIO MEAN LABELING OF PATH UNION OF GRAPHS

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Abstract

Let $G = (V, E)$ be a simple graph with p vertices and q edges. For a connected graph G, a radio mean labeling is a one to one mapping f from $V(G)$ to N satisfying the condition, $d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 1 + diam(G)$ for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of G. The radio mean number of G, $rmn(G)$ is the lowest span taken over all radio mean labelings of the graph G. In this paper, we analyze radio mean labeling for some path union of graphs.

1. Introduction

The graph labeling problem is one of the recent developing area in graph theory. Alex Rosa first introduced this problem in 1967 [13]. Radio labeling is motivated by the channel assignment problem introduced by W. K. Hale in 1980 [2]. In 2001, Gary Chartrand defined the concept of radio labeling of G [1]. Liu and Zhu first determined the radio number in 2005 [3]. Ponraj et al.

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[8] introduced the notion of radio mean labeling of graphs and investigated radio mean number of some graphs [9].

Radio Labeling is used for X-ray, crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing ,cell biology etc.,

The following results are used in the subsequent section.

2. Preliminaries

Definition 2.1. Let $G = (V, E)$ be a simple graph with p vertices and q edges. For a connected graph G, a radio mean labeling is a one to one mapping f from $V(G)$ to N satisfying the condition $d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 1 + diam(G)$ for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of G. The radio mean number of G, $rmn(G)$ is the lowest span taken over all radio mean labelings of the graph G.

Definition 2.2 [5]**.** $P_m(G)$ is obtained by taking a path on m points and at each vertex of the path a copy of G is fused. The point of fusion on G is same and fixed for all copies of G.

Definition 2.3 [12]**.** *Subdivision* of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints $\{u, v\}$ yields a graph containing one vertex w, and with an edge set replacing e by two new edges $\{u, w\}$ and $\{w, v\}$.

Definition 2.4 [7]**.** *Globe* is a graph obtained from two isolated vertices joined by n paths of length 2.

Definition 2.5 [6]**.** *Fan* $(F_n = P_n + K_1)$ is obtained from the path P_n by joining each vertex of P_n to a vertex u.

Definition 2.6 [11]**.** The *double fan* DF_n consists of two fan graphs with a common path. In other words, $DF_n = P_n + \overline{K_2}$.

3. Main Results

Theorem 3.1. *The radio mean number of the path union of two copies of star graph is* $2n+2$ *for* $n \geq 2$.

Proof. Let G be the path union of two copies of star graph. Let $V(G) = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots, v_2, u, v\}$ and $E(G) = \{uu_i / 1 \leq i \leq n\}$ $\bigcup \{vv_i / 1 \leq i \leq n\} \bigcup \{uv\}$ Here, the diameter is 3.

Define $f : V(G) \to N$ by

$f(u_i) = i, 1 \leq i \leq n.$

$f(v_i) = n + i, 1 \leq i \leq n.$

$f(u) = 2n + 1$

$f(v) = 2n + 2$

Claim: *f* is a valid radio mean labeling.

The diameter here is 3. Hence it is enough to prove that

$$d(x, y) + \left\lceil \frac{f(x) + f(y)}{2} \right\rceil \geq 4 \ (1)$$

for every pair of vertices (x, y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum *f* values and minimum $d(x, y)$ values.

Hence, the proof involves the following cases.

Case a. Consider the pairs (u, u_i) with $d(u, u_i) = 1$.

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 1 + i}{2} \right\rceil = n + 2 \geq 4$$

Case b. Consider the pairs (v, v_i) with $d(v, v_i) = 1$

$$d(v, v_i) + \left\lceil \frac{f(v) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 2 + n + i}{2} \right\rceil = 2n + 3 > 4$$

Case c. Consider the pairs (u_i, u_j) with $d(u_i, u_j) = 1$

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{i + j}{2} \right\rceil \geq 4$$

Case d. Consider the pairs (v_i, v_j) with $d(v_i, v_j) = 1$

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + i + n + j}{2} \right\rceil = n + 4 > 4$$

Case e. Consider the pair (u, v) with $d(u, v) = 1$

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil = 1 + \left\lceil \frac{2n + 1 + 2n + 2}{2} \right\rceil = 2n + 3 > 4.$$

Hence, by all the above cases, the radio mean condition is satisfied by f.

Further, f assigns integers starting from 1 consecutively up to $2n + 2$.

Therefore, $2n + 2$ is the minimum of the maximum integer that could be assigned to the vertices of G.

Hence, $rmn(G) = 2n + 2$ for $n \geq 2$.

Theorem 3.2. *The radio mean number of the path union of two copies of subdivision of star graph is* $4n + 2$ *for* $n > 2$.

Proof. Let G be the path union of two copies of subdivision of star graph. Let $V(G) = \{v_1, v_2, v_3, \ldots v_{2n}, u_1, u_2, u_3, \ldots, u_{2n}, u, v\}$ and $E(G) = \{uu_i/1 \leq i \leq n\} \cup \{vu_i/n + 1 \leq i \leq 2n\} \cup \{uv\} \cup \{u_iv_i/1 \leq i \leq 2n\}$

The diameter here is 5.

Define $f : V(G) \to N$ by

$f(v_i) = i, 1 \leq i \leq 2n.$

$f(u_i) = 2n + i + 2, 1 \leq i \leq 2n.$

$f(u) = 2n + 1$

$f(v) = 2n + 2$

Claim: f is a valid radio mean labeling.

The diameter here is 5.

Hence, it is enough to prove that

$$d(x, y) + \left\lceil \frac{f(x) + f(y)}{2} \right\rceil \geq 6 \ (1)$$

for every pair of vertices (x, y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x, y)$ values.

Hence, the proof involves the following cases.

Case a. Consider the pairs (u, u_i) with $d(u, u_i) = 1$

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 1 + 2n + i + 2}{2} \right\rceil = 2n + 3 > 6 (\text{Since } n > 2)$$

Case b. Consider the pairs (v, u_i). Here, $d(v, u_i) \geq 1$. Here, $i \geq n + 1$.

$$d(v, u_i) + \left\lceil \frac{f(v) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 2 + 2n + i + 2}{2} \right\rceil = 3n + 4 > 6$$

Case c. Consider the pairs (v_i, v_j) with $d(v_i, v_j) = 4$

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil = 4 + \left\lceil \frac{i + j}{2} \right\rceil = 6$$

Case d. Consider the pair (u, v) and $d(u, v) = 1$

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil = 1 + \left\lceil \frac{2n + 1 + 2n + 2}{2} \right\rceil = 2n + 3 > 6$$

Case e. Consider the pairs (u_i, u_j) with $d(u_i, u_j) = 2$

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil = 2 + \left\lceil \frac{2n + i + 2 + 2n + j + 2}{2} \right\rceil = 2n + 6 > 6$$

Case f. Consider the pairs (u_i, v_i). Here, $d(u_i, v_i) \geq 1$

$$d(u_i, v_i) + \left\lceil \frac{f(u_i) + f(v_i)}{2} \right\rceil = 1 + \left\lceil \frac{2n + i + 2 + i}{2} \right\rceil = n + 3 = 6$$

Hence, by all the above cases, the radio mean condition is satisfied by f.

Since G contains $4n+2$ vertices, $4n+2$ is the minimum of the maximum integer that could be assigned to the vertices of G.

Hence, $rmn(G) = 4n + 2$ for $n > 2$.

Theorem 3.3. *The radio mean number of the path union of two copies of globe graph is* $2n + 5$ *for* $n \geq 2$.

Proof. Let G be the path union of two copies of globe graph. (See Figure 3.1) Let $V(G) = \{v_1, v_2, v_3, \ldots v_{2n}, u, u', v'\}$ and $E(G) = \{uv_i/1 \leq i \leq n\}$ $\bigcup\{vv_i/n+1 \leq i \leq 2n\} \bigcup \{uv\} \bigcup \{u'v_i/1 \leq i \leq n-1\} \bigcup \{v'v_i/n+1 \leq i \leq 2n\}$

The diameter here is 5.

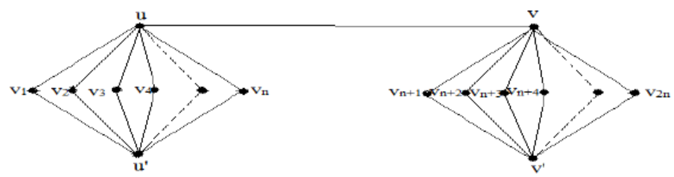

Figure 3.1.

Let f be a radio mean labeling of G.

If under f, 1 is assigned to any vertex of G, say 'a', then the vertex assigned 2 must be at a distance ≥ 4 from a.

Without loss of generality, assign 1 and 2 to v' and u' respectively. Then the vertices adjacent to u' must get values ≥ 7 and the vertices adjacent to v' must get values ≥ 8. If the integers ≥ 7 are assigned consecutively to the vertices adjacent to u' and v', then the maximum integer assigned is $2n + 6$.

We are left with two more vertices. Assigning 5 and 6 to u and v respectively, we get a radio mean labeling.

On the other hand, start labeling from 2. If u' and v' are assigned 3 and 2 respectively, then the vertices adjacent to u' must get values ≥ 6 and the vertices adjacent to v' must get values ≥ 7. If the integers ≥ 6 are assigned consecutively to the vertices adjacent to u' and v', then the maximum integer assigned is $2n + 5$.

Labeling 4 and 5 to u and v respectively, the radio mean condition is satisfied for every two vertices and we get a radio mean labeling. Also, here consecutive integers are assigned.

Hence $f : V(G) \to N$ is defined by

$f(v_i) = i + 5, 1 \leq i \leq 2n.$

$f(u) = 4, f(v) = 5, f(u') = 3, f(v') = 2$

is also a valid radio mean labeling.

Further, f attains its maximum corresponding to v_{2n} and is $2n + 5$ for $n \geq 2.$

Similarly, using integers $\geq 4,$ even if we get any radio mean labeling surely the span will be larger.

Therefore, comparing the above three radio mean labeling, the smallest span is $2n + 5.$

Hence, $rmn(G) = 2n + 5$ for $n \geq 2.$

Theorem 3.4. *The radio mean number of the path union of two copies of fan graph is* $2n + 2$ *for* $n \geq 2.$

Proof. Let G be the path union of two copies of fan graph. Let $V(G) = \{u_1, u_2, u_3, \ldots u_n, v_1, v_2, v_3, \ldots, v_n, u, v\}$ and $E(G) = \{uu_i/1 \leq i \leq n\}$ $\cup \{vv_i/n+1 \leq i \leq n\} \cup \{uv\} \cup \{u'v_{i+1}/1 \leq i \leq n-1\} \cup \{v'v_{i+1}/1 \leq i \leq n-1\}$

The diameter of here is 3.

Define $f : V(G) \to N$ by

$f(u_i) = i + 1, 1 \leq i \leq n.$

$f(v_i) = n + i + 1, 1 \leq i \leq n.$

$f(u) = 2n + n$

$f(v) = 1$

Claim: *f* is a valid radio mean labeling.

The diameter here is 3. Hence, it is enough to prove that

$$d(x, y) + \left\lceil \frac{f(x) + f(x)}{2} \right\rceil \geq 4 \ (1)$$

for every pair of vertices (x, y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $d(x, y)$ values.

Hence, the proof involves the following cases.

Case a. Consider the pairs (u, u_i) with $d(u, u_i) = 1$

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 2 + i + 1}{2} \right\rceil = n + 3 > 4$$

Case b. Consider the pairs (v, v_i) with $d(v, v_i) = 1$

$$d(v, v_i) + \left\lceil \frac{f(v) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{1 + n + i + 1}{2} \right\rceil = n + 3 > 4$$

Case c. Consider the pairs (u_i, u_j) with $d(u_i, u_j) = 1$

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil = 1 + \left\lceil \frac{n + i + 1 + n + j + 1}{2} \right\rceil = n + 4 > 4$$

Case d. Consider the pairs (v_i, v_j) with $d(v_i, v_j) = 1$

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil = 1 + \left\lceil \frac{n + i + 1 + n + j + 1}{2} \right\rceil = n + 4 > 4$$

Case e. Consider the pair (u, v) with $d(u, v) = 1$

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil = 1 + \left\lceil \frac{2n + 2 + 1}{2} \right\rceil = n + 3 > 4$$

Hence, by all the above cases, the radio mean condition is satisfied by f.

Since G contains $2n + 2$ vertices, $2n + 2$ is the minimum of the maximum integer that could be assigned to the vertices of G.

Hence, $rmm(G) = 2n + 2$ for $n \geq 2$.

Theorem 3.5. *The radio mean number of the path union of two copies of double fan graph is* $2n + 5$ *for* $n \geq 2$.

Proof. Let G be the path union of two copies of double fan graph. Let $V(G) = \{v_1, v_2, v_3, \ldots v_{2n}, u, u', v'\}$ and $E(G) = \{uv_i / 1 \leq i \leq n\} \cup \{vv_i / n+1 \leq i \leq 2n\} \cup \{uv\} \cup \{u'v_i / 1 \leq i \leq n\} \cup \{v'v_i / n+1 \leq i \leq 2n\} \cup \{v_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} / n+1 \leq i \leq 2n-1\}$ The graph looks as in figure 3.3

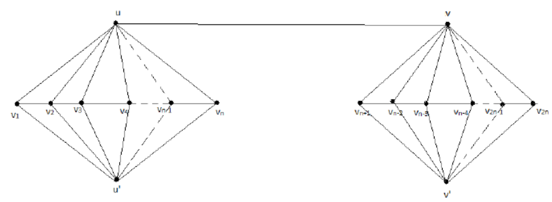

Figure 3.3.

The diameter here is 5.

Let f be a radio mean labeling of G.

If under f, 1 is assigned to any vertex of G, say 'a', then the vertex assigned 2 must be at a distance ≥ 4 from a.

Without loss of generality, assign 1 and 2 to v' and u' respectively. Then the vertices adjacent to u' must get values ≥ 7 and the vertices adjacent to v' must get values ≥ 8. If the integers ≥ 7 are assigned consecutively to the vertices adjacent to u' and v' and we get the maximum integer assigned is $2n + 6$.

We are left with two more vertices. Assigning 5 and 6 to u and v respectively, we get a radio mean labeling.

On the other hand, start labeling from 2. If u' and v' are assigned 3 and 2 respectively, then the vertices adjacent to u' must get values ≥ 6 and the vertices adjacent to v' mustget values ≥ 7. If the integers ≥ 6 are assigned consecutively to the vertices adjacent to u' and v', then the maximum integer assigned is $2n + 5$.

Labeling 4 and 5 to u and v respectively, the radio mean condition is satisfied for every two vertices and we get a radio mean labeling. Also, here consecutive integers are assigned.

Hence $f : V(G) \to N$ is defined by

$f(v_i) = i + 5, 1 \leq i \leq 2n.$

$f(u) = 4, f(v) = 5, f(u') = 3, f(v') = 2$

is also a valid radio mean labeling.

Further, f attains its maximum corresponding to v_{2n} and is $2n + 5$ for $n \geq 2$ in this labeling.

Similarly, using integers ≥ 4, even if we get any radio mean labeling surely the span will be larger.

Therefore, comparing the above three radio mean labelings, the smallest span is $2n + 5$.

Hence, $rmn(G) = 2n + 5$ for $n \geq 2$.

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