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SIMPLE APPROACH ON CAUCHY'S INTEGRAL THEOREM AND CAUCHY'S INTEGRAL FORMULA

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Abstract

Cauchy's theorem is one of the fundamental theorems in complex integral calculus. Cauchy's integral formula can be applied to that if the integrand has first order as well as the derivative of higher-order in the integration region. This approach is demonstrated for problems of the analytic function of first-order in and on an enclosed curve in a connected region.

Introduction

Complex integration is powerful and a useful tool for mathematicians, which is essential for the study of complex variables. As in the fundamental theorem of calculus is important, as it relates integration with differentiation and at the same time provides a method of evaluating integral so is the complex analog to develop integration along arcs and contours is complex integration (Azram and Elfaki [1]).

Let $f(x)$ be any real-valued function, the definite integral $\int_a^b f(x)dx$ can be calculated along the real axis $x = a$, to $x = b$, but for a complex function $f(x)$, the path of the definite integral $\int_a^b f(z)dz$ can be along any curve

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joining $z = a$ to $z = b$. The value of the integral depends upon the path of integration.

In the 19th century the modern analysis contains a research field known as complex analysis, the main founders were Augustin Cauchy (1789-1857), Bernhard Riemann (1826-1866), and Karl Weierstrass (1815-1897). The differential and integral theorems of complex variables were established by Cauchy. He was introduced by an integral theorem known as Cauchy's integral theorem (1826), Residue theorem (1831) and Cauchy's integral formula (1846) (Xuefeng Cao, Xingrong Sun, [9]). Later on, integral theorems of complex variables were extended by E. Goursat (1858-1936) without assuming the continuity of $f'(z)$. Azram [2] has given a simple proof of the Cauchy-Goursat integral theorem. The proof of Cauchy's integral theorem depends on Cauchy's-Riemann equation of complex variables; although Cauchy's theorem has no standard approach (Garcia and Ross, [5]). One of the important consequences of Cauchy's integral formula is the extended form of Cauchy's integral theorem. Cauchy's integral formula gives the value of the analytic function inside and on the boundary of the curve (Ricardo Estrada, [7]). This formula can also prove the residue theorem for meromorphic function and argument principle. In this section, we discuss Cauchy's theorem allows us to evaluate certain contour integrals.

Preliminaries

Connected region

A region is said to be connected in which any two points in it can be connected by a curve which lies entirely within the region.

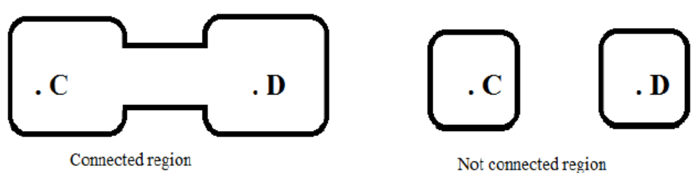

Figure 1.

Simple Connected Region

A simple curve does not cross itself, but the multiple curves cross itself.

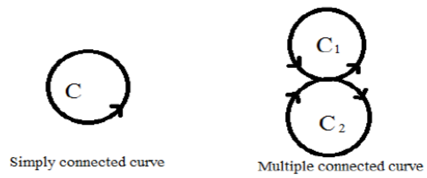

Figure 2.

A region is said to be simply connected if every closed curve in it encloses points of the region only, that is, simply connected region doesn't contains holes. (Joseph Bak and Popvassilev [6]). Otherwise it is said to be multiply connected region (Balaji [3]).

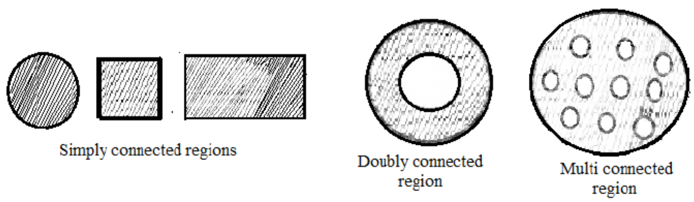

Figure 3.

Contour Integral

An integral along a simple closed curve is called a contour integral (Xin-She Yang [8]).

Note 1. Consider, closed curve C that rotates anticlockwise direction is positively oriented contour and closed curve that rotates clockwise direction is negatively oriented contour.

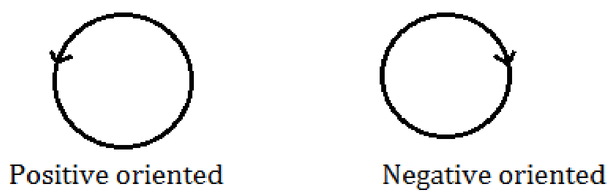

Figure 4.

Note 2. Let as C_1 and C_2 be two simple closed positively oriented contours such that C_1 lie inside C_1 in simply connected region R. If f is analytic in a connected region R that contains both C_1 and C_2 and the region between them, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ (Balaji [3]).

Note 3. Consider a contour is two parts as C_1 and C_2 then

$$\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

$$f(z) = u(x, y) + iv(x, y)$$

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i\int_C (udx + vdy).$$

Note 4. If C is a point on the arc joining a and b, then

$$\int_a^b f(z)dz = \int_a^c f(z)dz + \int_c^b f(z)dz.$$

Cauchy's integral theorem

Let $f(z)$ be an analytic function and its derivative $f'(z)$ is continuous at every point lying interior, exterior and on a closed curve c, then $\int_C f(z)dz = 0$.

Proof. Let the domain enclosed by 'C' be denoted by R.

Let

$$f(z) = u(x, y) + iv(x, y)$$

$$Z = x + iy;\ dz = dx + idy$$

$$\int_C f(z)dz = \int_C [u(x, y) + iv(x, y)][dx + idy]$$

$$= \int_C u(x, y)dx + i\int_C u(x, y)dy + i\int_C v(x, y)dx - \int_C v(x, y)dy$$

$$= \int_C [u(x, y)dx - v(x, y)dy] + i\int_C [v(x, y)dx + u(x, y)dy] \tag{1}$$

Since $f'(z)$ is continuous, the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous in R and c. Hence, we can apply Green's theorem for a plane.

$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy,$$

$$\int_C f(z) dz = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy. \tag{2}$$

From 1 and 2 we get,

$$\int_C u(x, y) dx - v(x, y) dy = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

$$\int_C v(x, y) dx + u(x, y) dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy.$$

By Cauchy's-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\, dy = 0.$$

$$\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\, dy = 0.$$

This gives the equation $\int_C f(c) dz = 0.$

Cauchy's integral formula

If $\xi(z)$ is analytic inside and on a closed curve c of a simply connected region R and if a is any point within c then $\int \frac{\zeta(z)}{z - a} dz = 2\pi i \xi(a)$ the integration around c being taken in the positive direction.

Proof. Let $\zeta(z)$ is an analytic function within and on c, $\frac{\xi(z)}{z - a}$ is also analytic within and on c, except at a point $z = a$. Therefore, we consider a small circle γ with center at $z = a$ and the radius ρ lying entirely inside c. Then $\frac{\xi(z)}{z-a}$ is analytic in the region enclosed between c and γ.

Therefore, by Cauchy's extended theorem

$$\int_C \frac{\xi(Z)}{Z-a}\,dz = \int_\gamma \frac{\xi(Z)}{Z-a}\,dz. \tag{3}$$

On γ any point of z is given by

$$Z = a + \rho e^{i\theta},\ dz = \rho i e^{i\theta} d\theta$$

$$\int_C \frac{\xi(Z)}{Z-a}\,dz = \int_\gamma \frac{\xi(a+\rho e^{i\theta})}{\rho e^{i\theta}}\,\rho i e^{i\theta} d\theta$$

$$= \int_\gamma \xi(a+\rho e^{i\theta})\, i d\theta. \tag{4}$$

In the limit, as $\rho \to 0$ the circle γ tends to a point.

$$2 \Rightarrow \int_\gamma \frac{\xi(Z)}{Z-a}\,dz = \int_0^{2\pi} \xi(a) i d\theta = i\xi(a)\int_0^{2\pi} d\theta.$$

Therefore,

$$\int_C \frac{\xi(Z)}{Z-a}\,dz = i\xi(a)\,[\theta]_0^{2\pi}$$

$$= i\xi(a)\,[2\pi - \theta] = 2\pi i\xi(a)$$

$i\xi(a) = \frac{1}{2\pi i}\int_c \frac{\xi(Z)}{Z-a}\,dz,\ a$ [a lies inside the c.]

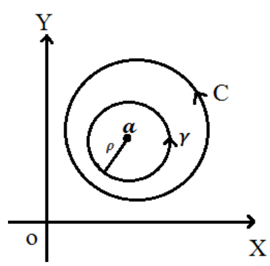

Figure 5.

Results (Explanation about the theorem)

Explanation 1. Evaluate $\int_C \frac{1}{2z-4}\,dz$ if c is the circle $|z| = 1$.

Solution: By Cauchy's integral formula is $\int_C \frac{\xi(z)}{z-a}\,dz = 2\pi i \xi(a)$, a lies inside $c = 0$, a lies outside c.

Given $\int_C \frac{1}{2z-4}\,dz = \frac{1}{2}\int_C \frac{1}{z-\frac{4}{2}}\,dz$. Here $\xi(z) = 1$ $a = \frac{4}{2} = 2$ is lies outside the circle $|z| = 1$. Hence by the Cauchy integral formula $\int_C \frac{1}{2z-4}\,dz = 2\pi i(0) = 0$.

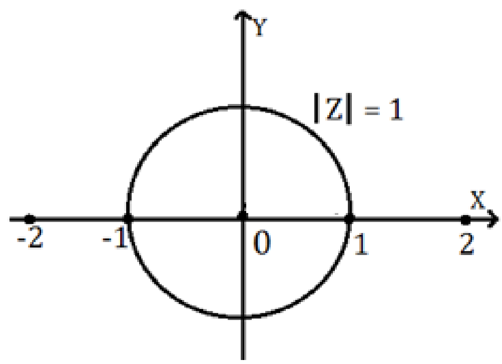

Figure 6.

Explanation 2. Evaluate $\int_C \frac{e^z}{z}\,dz$ where c is $|z| = 1$.

Solution. We know that Cauchy's integral formula is $\int_C \frac{\xi(z)}{z-a}dz = 2\pi i\xi(a)$. Given $\int_C \frac{e^z}{z}\,dz = \int_C \frac{e^z}{z-0}\,dz$, $\xi(z) = e^z$, $a = 0$, is lies inside $|z| = 1$.

Therefore by Cauchy's integral formula

$$\int_C \frac{e^z}{z}\, dz = 2\pi i \cdot \xi(0)$$

$$= 2\pi i \cdot e^0 = 2\pi i.$$

Conclusion

Cauchy's theorem is the basic fundamental theorem of complex integral calculus. The important consequences of Cauchy's integral formula are the extended form of Cauchy's theorem; and its formula uses the analyticity of a function in and on the boundary of a simple closed curve in a connected region. Hence, the complex integral could use Cauchy's integral theorem, Cauchy's integral formula, and analytic functions of the first order to calculate.

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